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COMMON FIXED POINTS FROM BEST SIMULTANEOUS APPROXIMATIONS

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Abstract. We obtain some results on common fixed points from the set of best simultaneous approximations for a map T which is asymptotically (f, g)-nonexpansive where (T, f) and (T, g) are not necessarily commuting pairs. Our results extend and generalize recent results of Chen and Li [1], Jungck and Sessa [8], Sahab et al. [13], Sahney and Singh [14], Singh [15, 16] and Vijayaraju [17] and many others.

1. Introduction and Preliminaries

We first review needed definitions. Let M be a subset of a normed space $(X, \|.\|)$. The set $P_M(u) = \{x \in M : \|x - u\| = dist(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $dist(u, M) = inf\{\|y-u\| : y \in M\}$. Suppose that A and G are bounded subsets of X. Then we write

$$r_G(A) = inf_{g\in G} sup_{a\in A} \parallel a - g \parallel$$

$$cent_G(A) = \{g_0 \in G : sup_{a\in A} \parallel a - g_0 \parallel = r_G(A)\}.$$

The number $r_G(A)$ is called the *Chebyshev radius* of A w.r.t. G and an element $y_0 \in cent_G(A)$ is called a *best simultaneous approximation* of A w.r.t. G. If $A = \{u\}$, then $r_G(A) = dist(u, G)$ and $cent_G(A)$ is the set of all best approximations, $P_G(u)$, of u out of G. We also refer the reader to Milman [12] and Vijayaraju [17] for further details. We denote by IN, $cl(M)$ and $wcl(M)$ the set of positive integers, closure of M and weak closure of M, respectively. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called an (f, g)*-contraction* if there exists $0 \leq k < 1$ such that $\|Tx - Ty\| \leq k\|fx - gy\|$ for any $x, y \in M$. If $k = 1$, then T is called

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(f,g)-nonexpansive. The map T is called *asymptotically* (f,g)*-nonexpansive* if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $lim_n k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|fx - gy\|$ for all $x, y \in M$ and $n = 1, 2, 3, ...$; if $g = f$, then T is called *asymptotically* f*-nonexpansive* [17]. The map T is called *uniformly asymptotically regular* [17] on M, if for each $\eta > 0$, there exists $N(\eta) = N$ such that $\|T^n x - T^{n+1} x\| < \eta$ for all $n \geq N$ and all $x \in M$. The set of fixed points of T is denoted by $F(T)$. A point $x \in M$ is a coincidence point (common fixed point) of f and T if $fx = Tx$ $(x = fx = Tx)$. The set of coincidence points of f and T is denoted by $C(f,T)$. The pair $\{f, T\}$ is called: (1) *commuting* if $Tfx = fTx$ for all $x \in M$, (2) *compatible* (see [7]) if $\lim_n \|Tfx_n - fTx_n\| = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M; (3) *weakly compatible* if they commute at their coincidence points, i.e.,if $fTx = Tfx$ whenever $fx = Tx$. The set M is called q*-starshaped* with $q \in M$, if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining q to x is contained in M for all $x \in M$. The map f defined on a q-starshaped set M is called *affine* if

$$f((1-k)q + kx) = (1-k)fq + kfx, \quad for \quad all \quad x \in M.$$

A Banach space X satisfies *Opial's condition* if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \neq x$. Every Hilbert space and the space $l_p (1 < p < \infty)$ satisfy Opial's condition. The map $T : M \to X$ is said to be *demiclosed at* 0 if for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ convergent to $0 \in X$, then we have $0 = Tx$.

The class of asymptotically nonexpansive mappings was introduced by Goeble and Kirk [2] and further studied by various authors (see [17] and references therein). Recently, Chen and Li [1] introduced the class of Banach operator pairs, as a new class of noncommuting maps which is further investigated by Hussain [3]. In this paper, we improve and extend invariant approximation results of Chen and Li [1] and Vijayaraju [17] to the class of asymptotically (f,g)-nonexpansive map T where (T,f) and (T,g) are Banach operator pairs without the condition of linearity or affinity of f and g which is a key assumption in the results obtained in of [4-8, 11, 13, 16, 17].

2. Main Results

The ordered pair (T,f) of two selfmaps of a metric space (X,d) is called a *Banach operator pair*, if the set $F(f)$ is T-invariant, namely, $T(F(f)) \subseteq F(f)$.

Obviously, commuting pair (T, f) is a Banach operator pair but not conversely, in general; see [1, 3] and Example 2.8 below. If (T, f) is a Banach operator pair, then (f, T) need not be Banach operator pair(cf. Example 1[1]). If the selfmaps T and f of X satisfy

$$d(fTx, Tx) \leq kd(fx, x),$$

for all $x \in X$ and $k \geq 0$, then (T, f) is a Banach operator pair; in particular , when $f = T$ and X is a normed space, the above inequality can be rewritten as

$$\|T^2x - Tx\| \leq k\|Tx - x\| \quad for \quad all \quad x \in X.$$

The following recent result will be needed.

Lemma 2.1. ([3], Lemma 2.10). *Let C be a nonempty subset of a metric space (X, d), and (T, f) and (T, g) be Banach operator pairs on C. Assume that $cl(T(C))$ is complete, and T, f and g satisfy for all $x, y \in C$ and $0 \leq h < 1$,*

$$(2.1) \quad d(Tx, Ty) \leq h \max\{d(fx, gy), d(Tx, fx), d(Ty, gy), d(Tx, gy), d(Ty, fx)\}$$

If f and g are continuous, $F(f) \cap F(g)$ is nonempty, then there is a unique common fixed point of T, f and g.

The following result extends Theorem 2.3 due to Vijayaraju [17] and approximation results in [13, 14, 15, 16] to noncommuting pairs.

Theorem 2.2. *Let K be a nonempty subset of a normed space X and $y_1, y_2 \in X$. Suppose that T, f and g are selfmaps of K such that T is asymptotically $(f, g)-$nonexpansive. Suppose that the set $F(f) \cap F(g)$ is nonempty. Let the set D, of best simultaneous K-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element q in $F(f) \cap F(g)$ and D is invariant under T, f and g. Assume further that (T, f) and (T, g) are Banach operator pairs on D, $F(f)$ and $F(g)$ are q-starshaped with $q \in F(f) \cap F(g)$, f and g are continuous and T is uniformly asymptotically regular on D. Then D contains a T-, f- and g-invariant point.*

Proof. For each $n \geq 1$, define a mapping T_n from D to D by

$$T_nx = (1 - \mu_n)q + \mu_nT^nx,$$

where $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of numbers in $(0, 1)$ such that $lim_n\lambda_n = 1$. Since $T(D) \subset D$ and D is q-starshaped, it follows that T_n maps D into D. As (T, f) is a Banach operator pair, $T(F(f)) \subseteq F(f)$ implies that $T^n(F(f)) \subseteq F(f)$ for each $n \geq 1$. On utilizing q-starshapedness of $F(f)$ we see that for each $x \in$

$F(f)$, $T_nx = (1-\mu_n)q + \mu_nT^nx \in F(f)$, since $T^nx \in F(f)$ for each $x \in F(f)$. Thus (T_n, f) is a Banach operator pair on D for each $n \geq 1$. Similarly, (T_n, g) is a Banach operator pair on D for each $n \geq 1$. For each $x, y \in D$, we have

$$\begin{aligned} \|T_nx - T_ny\| &= \mu_n\|T^nx - T^ny\| \\ &\leq \lambda_n \|fx - gy\|. \end{aligned}$$

By Lemma 2.1, for each $n \geq 1$, there exists $x_n \in D$ such that $x_n = fx_n = gx_n = T_nx_n$. As $T(D)$ is bounded, so $\|x_n - T^nx_n\| = (1-\mu_n)\|T^nx_n - q\| \to 0$ as $n \to \infty$. Since (T, f) is a Banach operator pair and $fx_n = x_n$, so $fT^nx_n = T^nfx_n = T^nx_n$. Thus we have

$$\begin{aligned} \|x_n - Tx_n\| &= \|x_n - T^nx_n\| + \|T^nx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\leq \|x_n - T^nx_n\| + \|T^nx_n - T^{n+1}x_n\| + k_1\|fT^nx_n - gx_n\| \\ &= \|x_n - T^nx_n\| + \|T^nx_n - T^{n+1}x_n\| + k_1\|T^nx_n - x_n\| \end{aligned}$$

Since T is uniformly asymptotically regular on D, it follows that

$$T^nx_n - T^{n+1}x_n \to 0 \quad as \quad n \to \infty.$$

Thus we have

$$\|x_n - Tx_n\| \leq \|x_n - T^nx_n\| + \|T^nx_n - T^{n+1}x_n\| + k_1\|T^nx_n - x_n\| \to 0 \;\; as \;\; n \to \infty$$

. Since D is compact, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ as $m \to \infty$. By the continuity of $I - T$, we have $(I-T)x_m \to (I-T)y$. But $(I-T)x_m \to 0$, so we have $(I-T)y = 0$. Since f and g are continuous, it follows that

$$fy = f(lim_mx_m) = lim_mfx_m = lim_mx_m = y$$

$$and \;\; gy = g(lim_mx_m) = lim_mgx_m = lim_mx_m = y.$$

This completes the proof.

The following corollary follows from Theorem 2.2 as condition (i) implies that D is T-invariant.

Corollary 2.3. *Let X, K, y_1, y_2, f, g and T be as in Theorem 2.2. Assume that T satisfies the following condition:*
(i) $\|Tx - y_i\| \leq \|x - y_i\|$ for all $x \in X$ and $i = 1, 2$.
Suppose that the set D, of best simultaneous K-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element q in $F(f) \cap F(g)$. Then D contains a T-, f- and g-invariant point.

Take $g = f$ in Theorem 2.2 to get:

Corollary 2.4. *Let K be a nonempty subset of a normed space X and $y_1, y_2 \in X$. Suppose that T and f are selfmaps of K such that T is asymptotically $f-$nonexpansive. Suppose that the set $F(f)$ is nonempty. Let the set D, of best simultaneous K-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element q in $F(f)$ and D is invariant under T and f. Assume further that (T, f) is a Banach operator pair on D, $F(f)$ is q-starshaped with $q \in F(f)$, f is continuous and T is uniformly asymptotically regular on D. Then D contains a $T-$ and $f-$invariant point.*

A commuting pair (T, f) is a Banach operator pair and affineness of f implies that $F(f)$ is q-starshaped; hence we get the following from Corollary 2.4.

Corollary 2.5. ([17], Theorem 2.3). *Let K be a nonempty subset of a normed space X and $y_1, y_2 \in X$. Suppose that T and f are selfmaps of K such that T is asymptotically $f-$nonexpansive. Suppose that the set $F(f)$ is nonempty. Let the set D, of best simultaneous K-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element q in $F(f)$ and D is invariant under T and f. Assume further that T and f are commuting, T is uniformly asymptotically regular on D and f is affine. Then D contains a $T-$ and $f-$invariant point.*

Remark 2.6. Note that the condition "$f(D) = D$" in Theorem 2.3 of Vijayaraju [17] is not needed in our work.

Theorem 2.7. *Let K be a nonempty subset of a Banach space X and $y_1, y_2 \in X$. Suppose that T, f and g are selfmaps of K such that T is asymptotically $(f, g)-$nonexpansive. Suppose that the set $F(f) \cap F(g)$ is nonempty. Let the set D, of best simultaneous K-approximants to y_1 and y_2, is nonempty weakly compact and starshaped with respect to an element q of $F(f) \cap F(g)$ and D is invariant under T, f and g. Assume further that (T, f) and (T, g) are Banach operator pairs on D, $F(f)$ and $F(g)$ are q-starshaped with $q \in F(f) \cap F(g)$, f and g are continuous under weak and strong topologies and T is uniformly asymptotically regular on D. Then D contains a $T-$ $f-$ and $g-$invariant point provided $f - T$ is demiclosed at 0.*

Proof. Let $\{T_n\}$ be defined as in the proof of Theorem 2.2. The weak compactness of $wclT(D)$ implies that $wclT_n(D)$ is weakly compact and hence complete by the completeness of X (see [3, 7]). The analysis in Theorem 2.2, guarantees that there exists an $x_n \in D$ such that $x_n = fx_n = gx_n = T_n x_n$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. The weak compactness of $wclT(D)$ implies that there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to $z \in D$ as

$m \to \infty$. Weak continuity of f and g implies that $fz = z = gz$. Also, we have, $fx_m - Tx_m = x_m - Tx_m \to 0$ as $m \to \infty$. As so $f - T$ is demiclosed at 0, then $fz = Tz$. Thus $D \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$. This completes the proof.

Theorem 2.7 extends and improves the results due to Jungck and Sessa [8], Latif [11], Sahab et al. [13], Sahney and Singh [14], Singh [15, 16] and Vijayaraju [17].

Following example exhibits an important fact: $F(f)$ may be q-starshaped without the affineness of f.

Example 2.8. Consider $X = \mathbb{R}^2$ with the norm $\|(x, y)\| = |x| + |y|, (x, y) \in \mathbb{R}^2$. Define T and f on X as follows:

$$T(x,y) = \Big(\frac{1}{2}(x-2), \frac{1}{2}(x^2+y-4)\Big)$$

$$f(x,y) = \Big(\frac{1}{2}(x-2), x^2+y-4\Big).$$

Obviously, T being f-nonexpansive is asymptotically f-nonexpansive but f is not affine. Moreover, $F(T) = \{-2, 0\}$, $F(f) = \{(-2, y) : y \in R\}$ and $C(f, T) = \{(x, y) : y = 4 - x^2, x \in R\}$. Thus (T, f) is a continuous Banach operator pair which is not a compatible pair [1, 3], $F(f)$ is q-starshaped for any $q \in F(f)$ and $(-2, 0)$ is a common fixed point of f and T.

Definition 2.9. A subset M of a linear space X is said to have the property (N) with respect to T [5, 6] if,

(i) $T : M \to M$,

(ii) $(1-k_n)q + k_nTx \in M$, for some $q \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1 and for each $x \in M$.

Hussain et al. [5] noted that each q-starshaped set M has the property (N) but converse does not hold, in general. A mapping f is said to be affine on a set M with property (N) if $f((1-k_n)q + k_nTx) = (1-k_n)fq + k_nfTx$ for each $x \in M$ and $n \in \mathbb{N}$.

Remark 2.10. The results (2.2-2.5 and 2.7) of this paper remain valid, provided the q-starshapedness of the set D, $F(f)$ and $F(g)$ is replaced by the property (N). Consequently, recent results due to Hussain, O'Regan and Agarwal [5], Hussain and Rhoades [6], Khan et al. [9] and Khan and Khan [10] are extended to asymptotically $(f, g)-$nonexpansive map T where (T, f) and (T, g) are Banach operator pairs which are different from C_q-commuting and R-subweakly commuting maps (see Remark 2.15(ii) [3]).

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