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GLOBAL CONVERGENCE FOR THE XOR BOOLEAN NETWORKS

Juei-Ling Ho

Abstract. Shih and Ho have proved a global convergent theorem for boolean network: if a map from $\{0,1\}^n$ to itself defines a boolean network has the conditions: (1) each column of the discrete Jacobian matrix of each element of $\{0,1\}^n$ is either a unit vector or a zero vector; (2) all the boolean eigenvalues of the discrete Jacobian matrix of this map evaluated at each element of $\{0,1\}^n$ are zero, then it has a unique fixed point and this boolean network is global convergent to the fixed point. The purpose of this paper is to give a global convergent theorem for XOR boolean network, it is a counterpart of the global convergent theorem for boolean network.

1. Introduction

In 1999, Shih and Ho proved a global convergent theorem for boolean network as the following. We propose in this paper a proof of a counterpart of it for the XOR boolean networks.

Theorem 1.1. [6, Theorem 3.1]. *Suppose the map F from $\{0,1\}^n$ to itself defines a boolean network has the conditions:*

(1) $F(V_x) \subset V_{F(x)}$ *for all* $x \in \{0,1\}^n$*;*
(2) $\rho(F'(x)) = 0$ *for all* $x \in \{0,1\}^n$*.*

Then F has a unique fixed point and the boolean network is global convergent to this fixed point.

Here $\{0,1\}^n$ denotes the set of all 01-strings of length n and is equipped with a boolean structure, and $\rho(A)$ denotes the boolean spectral radius of a boolean matrix

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A. For $x \in \{0,1\}^n$, V_x and $V_{F(x)}$ denote the von Neumann neighborhood of x and $F(x)$ respectively. For $x \in \{0,1\}^n$, $F'(x)$ stands for the discrete Jacobian matrix of F evaluated at x. We will explain these notion concerning the von Neumann neighborhood, the discrete Jacobian matrix and the boolean spectral radius in next section.

Let us remark that in Theorem 1.1 the boolean network $F : \{0,1\}^n \to \{0,1\}^n$ is global convergent to a fixed point ξ if ξ is a global attractor for the boolean network, that is, the trajectory $x^{t+1} = F(x^t)$ tends forward to ξ for any starting at x^0 of $\{0,1\}^n$; i.e., there exists a positive integer $p(\leq 2^n)$ such that $F^p(x^0) = x^p = \xi$ for any starting $x^0 \in \{0,1\}^n$(see[8, p.66]). Concerning Theorem 1.1, remark also that the condition "$F(V_x) \subset V_{F(x)}$" is equivalent to the condition "each column of $F'(x)$ is either a unit vector or a zero vector"(see[8, Lemma 4.1]). For the boolean network, the condition"$\rho(F'(x)) = 0$" is equivalent to the condition "all the boolean eigenvalues of $F'(x)$ are zero", because $\sigma(F'(x)) \neq \phi$ for the boolean network $F : \{0,1\}^n \to \{0,1\}^n$(see[6, p.48]). We will explain $\sigma(F'(x))$ may be empty for the XOR boolean network $F : \{0,1\}^n \to \{0,1\}^n$ in the section 3.

2. Preliminaries

In order to provide the main theorem, we state some notations, notions and results concerning the discrete Jacobian matrix and the boolean eigenvalues. The material can be found in the fundamental paper by Robert[1], [2] and [3], and also in the book by Robert[4], [5] and [6].

Let $\{0,1\}$ be with operations $+$, $\oplus$, and $\cdot$ defined as follows,

$$0+0=0\oplus 0=1\oplus 1=1\cdot 0=0\cdot 1=0\cdot 0=0,$$

$$1+1=1+0=0+1=1\oplus 0=0\oplus 1=1\cdot 1=1,\ \bar{0}=1, \text{ and } \bar{1}=0.$$

For $a, b \in \{0,1\}$, ab is the abbreviation of $a \cdot b$. For each positive integer n, let $\{0,1\}^n$ be the set of ordered n-tuples,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

with components $x_i \in \{0,1\}$ $(i = 1, \ldots, n)$. We may think of x as a *bit string* of length n, thus we may write $x = x_1 x_2 \cdots x_n$. We also write $x = (x_1, x_2, \cdots, x_n)$. The zero element of $\{0,1\}^n$ is the n-tuple $\mathbf{0} = (0, 0, \cdots, 0)$. For $j \in \{1, \ldots, n\}$, the j-th unit vector $\mathbf{e}_j$ is the element of $\{0,1\}^n$, all of whose coordinates are 0 except for the j-th component is 1. The order "$\leq$" in $\{0,1\}$ is given by $0 \leq 0 \leq 1 \leq 1$.

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For $x, y \in \{0,1\}^n$, $x \leq y$ is meant that.$x_i \leq y_i \, (i = 1, \ldots, n)$. For $x, y \in \{0,1\}^n$, $\lambda, \gamma \in \{0,1\}$, define

$$x + y = \begin{pmatrix} \max\{x_1, y_1\} \\ \vdots \\ \max\{x_n, y_n\} \end{pmatrix}, \lambda x = \begin{pmatrix} \max\{\lambda, x_1\} \\ \vdots \\ \max\{\lambda, x_n\} \end{pmatrix}, \text{ and } x \oplus y = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix},$$

where $\gamma_i = 0$ if $x_1 = y_1$; otherwise, $\gamma_i = 1 \, (i = 1, \ldots, n)$. Hence

$$x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, cx = \begin{pmatrix} c + x_1 \\ \vdots \\ c + x_n \end{pmatrix}, \text{ and } x \oplus y = \begin{pmatrix} x_1 \oplus y_1 \\ \vdots \\ x_n \oplus y_n \end{pmatrix}.$$

Boolean network of n elements is a mapping $F : \{0,1\}^n \rightarrow \{0,1\}^n$(see [9, p.20]). XOR Boolean network is a boolean network that replace the operation $+$ with $\oplus$. Throughout this paper, a *boolean matrix* is meant to be a matrix over $\{0,1\}$. The set of $n \times n$ boolean matrix is denoted by M_n. For any $A \in M_n$, denote the j-th column of A by A_j. The symbol I stands for the identity matrix in M_n. Let α be a nonempty subset of $\{1, 2, \cdots, n\}$. For any $A \in M_n$, $A(\alpha)$ stands for the principal submatrix of A that lies in rows and columns indexed by α. Boolean matrix multiplication is the same as in the case of complex matrices but the concerned products of entries are boolean. For a boolean network, boolean matrix addition is the operation $+$, it is the same as in the case of complex matrices but the concerned sums of entries are boolean. For an XOR boolean network, boolean matrix addition is the operation $\oplus$ instead of the operation $+$. Let $\sum\limits_{j \in \alpha} A_j$ denote the summation of columns of A with the operation $+$. The summation of columns of A with the operation $\oplus$ is denoted by $\bigoplus\limits_{j \in \alpha} A_j$ and we denote $\bigoplus\limits_{j=1}^{n} A_j = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. A non-zero element $u \in \{0,1\}^n$ is called a (*boolean*) *eigenvector* of $A \in M_n$ if there exists λ in $\{0,1\}$ such that $Au = \lambda u$; λ is called the *(boolean) eigenvalue* associated with eigenvector. For $A \in M_n$, the symbol $\sigma(A)$ denote the *(boolean) spectrum* of A, it is the set of all eigenvalues of A, so that $\sigma(A) \subset \{0,1\}$. The *(boolean) spectral radius* of A, which is denoted by $\rho(A)$, is defined to be the largest eigenvalue of A. For $A \in M_n$ in boolean network, since $\sigma(A) \neq \phi$(see[6, p.48]), $\rho(A) = 0$ if and only if $1 \notin \sigma(A)$. But for $A \in M_n$ in XOR boolean network, we will show $\sigma(A)$ may be empty in the next section. For an element x of $\{0,1\}^n$, the *von Neumann neighborhood* of x is the set $V_x = \{x, \tilde{x}^1, \cdots, \tilde{x}^n\}$. Here $\tilde{x}^j \, (i = 1, \ldots, n)$ is the *j-th neighbor* of x, which is defined to be the element $(x_1, \cdots, \bar{x}_j, \cdots, x_n)$. According to Robert(see[6, p.97]), the *boolean Jacobian matrix* of the map F from $\{0,1\}^n$ to itself evaluated at x is defined by $F'(x) = (f_{ij}(x))$, where

For $x, y \in \{0,1\}^n$, $x \leq y$ is meant that.$x_i \leq y_i\,(i = 1, \ldots, n)$. For $x, y \in \{0,1\}^n$, $\lambda, \gamma \in \{0,1\}$, define

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$$f_{ij}(x) = \begin{cases} 1 & \text{if} \quad f_i(x) \neq f_i(\tilde{x}^j) \\ 0 & \text{otherwise} \end{cases}$$

Robert usually called the boolean Jacobian matrix of a map as the *boolean derivative* of a map.

We now state some basic results concerning the characterizations of the eigenvalues in boolean networks.

Proposition 2.1. [6]. *Let $A \in M_n$. Then $0 \in \sigma(A)$ if and only if A has one or more zero columns.*

Proposition 2.2. [6]. *Let $A \in M_n$. Then $1 \in \sigma(A)$ if and only if A contains a principal sub-matrix having no zero rows.*

Proposition 2.3. [8]. *Let $A \in M_n$. Then $1 \in \sigma(A)$ if and only if A contains a principal sub-matrix having no zero columns.*

3. Characterizations of Eigenvalues in XOR Boolean Networks

In this section, we characterize the eigenvalues and the boolean Jacobian matrix in the XOR boolean networks needed to formulate the main result. Here we state and prove these lemmas with the boolean matrix addition $\oplus$.

Lemma 3.1. *Let $A \in M_n$. Then $0 \in \sigma(A)$ if and only if there exists a nonempty subset $\alpha \subset \{1, 2, \cdots, n\}$ such that $\bigoplus_{j\in\alpha} A_j = \mathbf{0}$.*

Proof. Let u be the eigenvector of A associated with the eigenvalue 0, and let $u = \bigoplus_{j\in\alpha} e_j$ for a nonempty subset $\alpha \subset \{1, 2, \cdots, n\}$; then

$$\bigoplus_{j\in\alpha} A_j = \bigoplus_{j\in\alpha} (Ae_j) = A\left(\bigoplus_{j\in\alpha} e_j\right) = Au = 0 \cdot u = \mathbf{0}.$$

Conversely, suppose α is a nonempty subset of $\{1, 2, \cdots, n\}$ such that $\bigoplus_{j\in\alpha} A_j = \mathbf{0}$. Put $u = \bigoplus_{j\in\alpha} e_j$. Then $u \neq \mathbf{0}$ and it follows that

$$Au = A\left(\bigoplus_{j\in\alpha} e_j\right) = \bigoplus_{j\in\alpha} Ae_j = \bigoplus_{j\in\alpha} A_j = \mathbf{0} = 0 \cdot u.$$

Thus $0 \in \sigma(A)$ ■

$$f_{ij}(x) = \begin{cases} 1 & \text{if} \quad f_i(x) \neq f_i(\tilde{x}^j) \\ 0 & \text{otherwise} \end{cases}$$

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Proof. Let u be the eigenvector of A associated with the eigenvalue 0, and let $u = \underset{j \in \alpha}{\oplus} e_j$ for a nonempty subset $\alpha \subset \{1, 2, \cdots, n\}$; then

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Conversely, suppose α is a nonempty subset of $\{1, 2, \cdots, n\}$ such that $\underset{j \in \alpha}{\oplus} A_j = \mathbf{0}$. Put $u = \underset{j \in \alpha}{\oplus} e_j$. Then $u \neq \mathbf{0}$ and it follows that

$$Au = A \left(\underset{j \in \alpha}{\oplus} e_j \right) = \underset{j \in \alpha}{\oplus} Ae_j = \underset{j \in \alpha}{\oplus} A_j = \mathbf{0} = 0 \cdot u.$$

Thus $0 \in \sigma(A)$ ∎

Lemma 3.2. *Let $A \in M_n$. Then $1 \in \sigma(A)$ if and only if there exists a nonempty subset $\alpha \subset \{1, 2, \cdots, n\}$ such that $\underset{j\in\alpha}{\oplus} (A \oplus I)_j = \mathbf{0}$.*

Proof. Let u be the eigenvector of A associated with the eigenvalue 1. Then $Au = 1 \cdot u = u$; hence $(A \oplus I)u = Au \oplus u = \mathbf{0} = 0 \cdot u$; hence $0 \in \sigma(A \oplus I)$. By Lemma 3.1, there exists a nonempty set $\alpha \subset \{1, 2, \cdots, n\}$, such that $\underset{j\in\alpha}{\oplus} (A \oplus I)_j = \mathbf{0}$.

Conversely, if α is a nonempty subset of $\{1, 2, \cdots, n\}$ such that $\underset{j\in\alpha}{\oplus} (A \oplus I)_j = \mathbf{0}$, then Lemma 3.1 shows that $0 \in \sigma(A \oplus I)$; hence there exists $u \neq \mathbf{0}$ such that $Au \oplus u = (A \oplus I)u = 0 \cdot u = \mathbf{0}$. Hence $Au = u = 1 \cdot u$, that is, $1 \in \sigma(A)$. ∎

Finally, the following example shows $\sigma(A) = \phi$ for some $A \in M_n$ in the XOR boolean networks. Hence we can not define $\rho(F'(x))$ for the XOR boolean network $F : \{0, 1\}^n \to \{0, 1\}^n$, so that we will use the condition "$1 \notin \sigma(F'(x))$" to instead of the condition"$\rho(F'(x)) = 0$" in our main result.

Example 3.1. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and let $\alpha = \{1, 2\}$; then $A \in M_2$ and

$$\underset{j\in\alpha}{\oplus} A_j = A_1 \oplus A_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \mathbf{0}$$

and

$$\underset{j\in\alpha}{\oplus} (A \oplus I)_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \mathbf{0}.$$

By Lemma 3.1 and Lemma 3.2, $\sigma(A) = \phi$.

4. Main Result

Let the map G from $\{0, 1\}^n$ to itself defines a XOR boolean network and let F be a map from $\{0, 1\}^n$ to itself defines a boolean network with $F(x) = G(x)$ for all $x \in \{0, 1\}^n$. Note that the spectrum of the boolean Jacobian matrix $F'(x)$ evaluated at x may not equal to the spectrum of $G'(x)$. For example, let G and F be the maps from $\{0, 1\}^2$ to itself, defined by

$$G(x) = \begin{pmatrix} \overline{x_1 \oplus x_2} \\ \overline{x}_1 \overline{x}_2 \end{pmatrix}$$

Lemma 3.2. *Let $A \in M_n$. Then $1 \in \sigma(A)$ if and only if there exists a nonempty subset $\alpha \subset \{1, 2, \cdots, n\}$ such that* $\underset{j\in\alpha}{\oplus}(A \oplus I)_j = \mathbf{0}$.

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Conversely, if α is a nonempty subset of $\{1, 2, \cdots, n\}$ such that $\underset{j\in\alpha}{\oplus}(A \oplus I)_j = \mathbf{0}$, then Lemma 3.1 shows that $0 \in \sigma(A \oplus I)$; hence there exists $u \neq \mathbf{0}$ such that $Au \oplus u = (A \oplus I)u = 0 \cdot u = \mathbf{0}$. Hence $Au = u = 1 \cdot u$, that is, $1 \in \sigma(A)$. ■

Finally, the following example shows $\sigma(A) = \phi$ for some $A \in M_n$ in the XOR boolean networks. Hence we can not define $\rho(F'(x))$ for the XOR boolean network $F : \{0, 1\}^n \to \{0, 1\}^n$, so that we will use the condition "$1 \notin \sigma(F'(x))$" to instead of the condition"$\rho(F'(x)) = 0$" in our main result.

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$$G(x) = \begin{pmatrix} \overline{x_1 \oplus x_2} \\ \overline{x}_1 \overline{x}_2 \end{pmatrix}$$

and

$$F(x) = \begin{pmatrix} x_1x_2 + \overline{x}_1\overline{x}_2 \\ \overline{x}_1\overline{x}_2 \end{pmatrix}.$$

Then $F(x) = G(x)$ for all $x \in \{0,1\}^2$, and they are given by Table 4.1. Put $x_0 = (0,0)$. Then

$$F'(x_0) = G'(x_0) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

By Proposition 2.1 and Proposition 2.3, we have $\sigma(G'(x_0)) = \{0\}$. But by Lemma 3.1 and Lemma 3.2, $\sigma(F'(x_0)) = \{1\}$.

Table 4.1.

Bit string x	00	01	10	11
Bit string $F(x)$	11	00	00	10

In order to prove the main result we shall employ the following lemma.

Lemma 4.1. *Suppose $x \in \{0,1\}^n$, and suppose the map F from $\{0,1\}^n$ to itself defines a XOR boolean network has the conditions:*

(1) $F(V_x) \subset V_{F(x)}$;

(2) $1 \notin \sigma(F'(x))$.

Then

(a) *Each entry in the diagonal of boolean Jacobian matrix $F'(x)$ is 0;*

(b) *Each principal submatrix of $F'(x)$ contains a zero row;*

(c) 0 *is a boolean eigenvalue of $F'(x)$.*

Proof. By (1), each column of $F'(x)$ is either a unit vector or a zero vector(see[8, Lemma 4.1]). Hence for each i in $\{1,2,\cdots,n\}$ there exists k in $\{1,2,\cdots,n\}$ such that either $(F'(x))_i = e_k$ or $(F'(x))_i = \mathbf{0}$.

We claim that $k \neq i$.

If not, then the i-th column of $F'(x) \oplus I$ equals zero. Put $\alpha = \{i\}$. Then $\bigoplus_{j\in\alpha} (F'(x) \oplus I)_j = (F'(x) \oplus I)_i = \mathbf{0}$. By Lemma 3.2, $1 \in \sigma(F'(x))$, a contradiction with (2).

and

$$F(x) = \begin{pmatrix} x_1x_2 + \overline{x}_1\overline{x}_2 \\ \overline{x}_1\overline{x}_2 \end{pmatrix}.$$

Then $F(x) = G(x)$ for all $x \in \{0,1\}^2$, and they are given by Table 4.1. Put $x_0 = (0,0)$. Then

$$F'(x_0) = G'(x_0) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

By Proposition 2.1 and Proposition 2.3, we have $\sigma(G'(x_0)) = \{0\}$. But by Lemma 3.1 and Lemma 3.2, $\sigma(F'(x_0)) = \{1\}$.

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Thus $k \neq i$, that is, each entry in the diagonal of $F'(x)$ is $0.$ which proves (a)

Let $F'(x) = (f_{ij}(x))_{n\times n}$. Assume, on the contrary, that α is a nonempty subset of $\{1, 2, \cdots, n\}$ such that the principal submatrix

$$F'(x)(\alpha) \text{ has no zero rows.} \tag{4.1}$$

Let ω be the number of 1 in $F'(x)(\alpha)$, and let c be the cardinal number of α. Then $\omega \geq c$, since $F'(x)(\alpha)$ has no zero rows. By (1), each column of $F'(x)(\alpha)$ is either a unit vector or a zero vector; hence the inequality is reversed, and the equality is proved, that is,

$$\omega = c. \tag{4.2}$$

Combining (a), (4.1) and (4.2), we see that for any i in α, $f_{ii}(x) = 0$ and there exists a unique k in α with $k \neq i$ such that $f_{ik}(x) = 1.$ Hence

$$\underset{j\in\alpha}{\oplus} \left(F'(x)(\alpha) \oplus I(\alpha)\right)_j = \mathbf{0}. \tag{4.3}$$

Similarly, combining (a) and (4.2), we obtain that for any i in α, there exists a unique k in α with $k \neq i$ such that $f_{ki}(x) = 1.$ It follows that

$$F'(x)(\alpha) \text{has no zero columns.} \tag{4.4}$$

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$$\underset{j\in\alpha}{\oplus} (F'(x) \oplus I)_j = \mathbf{0}. \tag{4.5}$$

By Lemma 3.2 and (4.5), $1 \in \sigma(F'(x))$. But this contradicts (2). Thus (b) follows.

Next, we claim that $0 \in \sigma(F'(x))$. Otherwise, Lemma 3.1 shows that $F'(x)$ has no zero column and every column is different from the other column. Thus by (1), we obtain that all columns of $F'(x)$ are unit vectors and they are not the same each other. Now (a) shows

$$\overset{n}{\underset{j=1}{\oplus}} (F'(x) \oplus I)_j = \mathbf{0}. \tag{4.6}$$

Put $\alpha = \{1, 2, \cdots, n\}$. Hence, by Lemma 3.2 and (4.6) we obtain $1 \in \sigma(F'(x))$, a contradiction with (2).

Thus $0 \in \sigma(F'(x))$, which proves (c) and the proof is complete. ∎

The aim of this paper is to prove the following theorem.

Theorem 4.1. *Suppose the map F from $\{0, 1\}^n$ to itself defines a XOR boolean network has the conditions:*

Thus $k \neq i$, that is, each entry in the diagonal of $F'(x)$ is $0.$ which proves (a)

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The aim of this paper is to prove the following theorem.

Theorem 4.1. *Suppose the map F from $\{0, 1\}^n$ to itself defines a XOR boolean network has the conditions:*

(1) $F(V_x) \subset V_{F(x)}$ *for all* $x \in \{0,1\}^n$;
(2) $1 \notin \sigma(F'(x))$ *for all* $x \in \{0,1\}^n$.

Then F has a unique fixed point and F is global convergent to this fixed point.

Proof. Let G be a map from $\{0,1\}^n$ to itself defines a boolean network with $G(x) = F(x)$ for all $x \in \{0,1\}^n$. Then by (1), we can obtain the following result directly

$$G(V_x) = F(V_x) \subset V_{F(x)} = V_{G(x)} \text{ for all } x \in \{0,1\}^n. \tag{4.7}$$

By the construction of G, the boolean Jacobian matrices $G'(x) = F'(x)$ for all $x \in \{0,1\}^n$; hence now follows from Lemma 4.1(b) that each principal submatrix of $G'(x)$ contains a zero row. By Proposition 2.2, $1 \notin \sigma(G'(x))$ for all $x \in \{0,1\}^n$. Since $\sigma(G'(x)) \neq \phi$(see[6, p. 48]), we have

$$\rho(G'(x)) = 0 \text{ for all } x \in \{0,1\}^n. \tag{4.8}$$

Apply Theorem 1.1 with (4.7) and (4.8), we obtain that G has a unique fixed point ξ and G is global convergent to ξ. Since F and G have the same orbits, then ξ is the unique fixed point of F and F is global convergent to ξ, which is the desired conclusion. ∎

Remark 4.1. The conditions of main theorem are essential.

The next two examples will show the Theorem 4.1 fails to hold if just possesses with one condition.

Example 4.1. Let $F : \{0,1\}^2 \longrightarrow \{0,1\}^2$ be defined by

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Hence each column of the boolean Jacobian matrix of each element of $\{0,1\}^n$ is a unit vector, we obtain $F(V_x) \subset V_{F(x)}$ for all x in $\{0,1\}^2$. Hence F possesses with the condition (1) of Theorem 4.1. Since there is no fixed point, Theorem 4.1 fails to hold. That is, condition (2) is essential for the result.

(1) $F(V_x) \subset V_{F(x)}$ *for all* $x \in \{0,1\}^n$*;*
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Then F is given by Table 4.3. and the boolean Jacobian matrices of F are the following:

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Hence for any nonempty subset $\alpha \subset \{1,2\}$, we obtain $\sum\limits_{j\in\alpha}(F'(x) \oplus I)_j \neq \mathbf{0}$ for all x in $\{0,1\}^2$. By Lemma 3.2, $1 \notin \sigma(F'(x))$ for all x in $\{0,1\}^2$; hence F just possesses with the condition(2) of Theorem 4.1. Note that now $(0,0)$ is the fixed point of F. Put $x^0 = (0,1)$. Then the trajectory $x^{t+1} = F(x^t)$ doesn't tend forward to $(0,0)$. Thus Theorem 4.1.fails in this case. In other words, the condition(1) is necessary for Theorem 4.1.

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Bit string x	00	01	10	11
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5. Concluding Remarks

Shih and Dong proved the combinatorial fixed point theorem over $\{0,1\}^n$[7]: if the map G from $\{0,1\}^n$ to itself defines a boolean network is such that $\rho(G'(x)) = 0$ for all $x \in \{0,1\}^n$, then G has a fixed point. It is nature to ask the following question: if the map F from $\{0,1\}^n$ to itself defines a XOR boolean network is such that $1 \notin \sigma(F'(x))$ for all $x \in \{0,1\}^n$, does it has a fixed point? That is false as the following example shows.

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Example 5.1. Let $F:\{0,1\}^3 \longrightarrow \{0,1\}^3$ be defined by

$$F(x)=\left(\begin{array}{c} x_2\oplus x_1\bar{x}_3\oplus x_2x_1\bar{x}_3 \\ x_1\bar{x}_2\oplus x_3(\overline{x_1\oplus x_2})\oplus x_1\bar{x}_2x_3(\overline{x_1\oplus x_2}) \\ \overline{x_2x_3}(x_1\oplus x_2\oplus x_3) \end{array}\right)$$

Then F is given by Table 5.1. The the boolean Jacobian matrices of F are the following:

$$F'(0,0,0)=\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix},\qquad F'(1,0,0)=\begin{pmatrix}1&0&1\\1&1&0\\1&1&1\end{pmatrix},$$

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It follows that for any nonempty subset $\alpha\subset\{1,2,3\}$, $\sum\limits_{j\in\alpha}(F'(x)\oplus I)_j\neq\mathbf{0}$ for all x in $\{0,1\}^3$; hence by Lemma 3.2, $1\notin\sigma(F'(x))$ for all x in $\{0,1\}^3$. But F has no fixed point.

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Bit string x	000	001	010	011	100	101	110	111
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Actually, the spectrum of the map F in Example 5.1 is empty. It is shown by Lemma 3.1 and for any nonempty subset $\alpha\subset\{1,2,3\}$, $\sum\limits_{j\in\alpha}(F'(x))_j\neq\mathbf{0}$ for all x in $\{0,1\}^3$. If we want to avoid the empty spectrum situation, then we can replace the condition"$1\notin\sigma(F'(x))$" by "$\sigma(F'(x))=\{0\}$". Hence we will ask the following question: if the map F from $\{0,1\}^n$ to itself defines a XOR boolean network is such that $\sigma(F'(x))=\{0\}$ for all $x\in\{0,1\}^n$, does it has a fixed point? But it still fails to be true as shown by the following example.

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Given a nonempty subset $\alpha \subset \{1,2,3\}$ Then $\sum\limits_{j\in\alpha}(F'(x) \oplus I)_j \neq \mathbf{0}$ for all x in $\{0,1\}^3$. Since α was arbitrary and by Lemma 3.2, we conclude that $1 \notin \sigma(F'(x))$ for all x in $\{0,1\}^3$. Now, we set a rule to choose a subset β of $\{1,2,3\}$ as following:

$$\text{Put } \beta = \begin{cases} \{1,2,3\} & \text{if } x \in \{(0,0,0),(0,0,1),(1,0,0),(0,1,0)\} \\ \{2,3\} & \text{if } x \in \{(1,0,1),(1,1,0)\} \\ \{2\} & \text{if } x = (0,1,1) \\ \{3\} & \text{if } x = (1,1,1) \end{cases}$$

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$$F'(0,1,1) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad F'(1,1,1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Given a nonempty subset $\alpha \subset \{1,2,3\}$ Then $\sum_{j\in\alpha} (F'(x) \oplus I)_j \neq \mathbf{0}$ for all x in $\{0,1\}^3$. Since α was arbitrary and by Lemma 3.2, we conclude that $1 \notin \sigma(F'(x))$ for all x in $\{0,1\}^3$. Now, we set a rule to choose a subset β of $\{1,2,3\}$ as following:

$$\text{Put } \beta = \begin{cases} \{1,2,3\} & \text{if } x \in \{(0,0,0),(0,0,1),(1,0,0),(0,1,0)\} \\ \{2,3\} & \text{if } x \in \{(1,0,1),(1,1,0)\} \\ \{2\} & \text{if } x = (0,1,1) \\ \{3\} & \text{if } x = (1,1,1) \end{cases}$$

Table 5.2.

Bit string x	000	001	010	011	100	101	110	111
Bit string $F(x)$	100	110	011	110	001	100	001	001

Then, for any x in $\{0,1\}^3$, there is a nonempty subset β of $\{1,2,3\}$ such that $\sum_{j\in\beta}(F'(x))_j = \mathbf{0}$; hence by Lemma3.1, $0 \in \sigma(F'(x))$ for all x in $\{0,1\}^3$; hence $\sigma(F'(x)) = \{0\}$ for all x in $\{0,1\}^3$. But F has no fixed point.

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