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LINEABILITY, SPACEABILITY, AND ALGEBRABILITY OF CERTAIN SUBSETS OF FUNCTION SPACES

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Abstract. We construct infinite-dimensional Banach spaces and infinitely generated Banach algebras of functions that, except for 0, satisfy some kind of *special* or *pathological* property. Three of these structures are: a Banach algebra of everywhere continuous bounded functions which are not Riemann-integrable ; a Banach space of Lebesgue-integrable functions that are not Riemann-integrable; an algebra of continuous unbounded functions defined on an arbitrary non-compact metric space.

1. Preliminaries

In mathematical analysis, many examples of functions with some sort of pathological behavior or enjoying certain special properties have been studied. Moreover, large structures (dense manifolds, Banach spaces, algebras, etc.) of functions enjoying such properties have been constructed. Given a property, we say that the subset L of functions which satisfies it is *spaceable* if $L \cup \{0\}$ contains a *closed* infinite dimensional subspace. The set L will be called *lineable* if $L \cup \{0\}$ contains an infinite dimensional vector space. This terminology of lineable and spaceable was first introduced in [6] and, later, in [1, 2, 13].

One of the first results in this direction was proved by Gurariy (see [11, 12]), who proved that the set of nowhere differentiable functions on $[0, 1]$ is lineable. Later, Fonf, Gurariy, and Kadec ([7]) showed that this set is also spaceable. This last result was, later, improved ([14]) when Hencl showed that any separable Banach space

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is isometrically isomorphic to a subspace of $\mathcal{C}[0,1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Hölder. On the other hand, the set of everywhere differentiable functions on $[0,1]$ is linear and, therefore, lineable, but it is not spaceable ([11]). Recently, Enflo and Gurariy have shown ([6]) that for any infinite dimensional subspace $X \subset \mathcal{C}[0,1]$, the set of functions in X having infinitely many zeros in $[0,1]$ is spaceable in X. Recently, Aron, Gurariy and the third author have shown that the set of everywhere surjective functions contains a vector subspace of the largest possible dimension, 2^c, and that the set $\mathcal{DNM}(\mathbb{R})$ of differentiable functions on $\mathbb{R}$ which are nowhere monotone is lineable in $\mathcal{C}(\mathbb{R})$ ([2]).

Aron, Pérez-García, and the third author showed in [3] that, given any set $E \subset \mathbb{T}$ of Lebesgue measure zero, there exists an infinitely generated and dense algebra every non-zero element of which is a continuous function whose Fourier series expansion is divergent at any point $t \in E$, introducing the new concept of *algebrability*: We say that a set L is *algebrable* if $L \cup \{0\}$ contains an infinitely generated algebra. The algebrability of certain subsets of functions has been studied, lately, by several authors (see [4, 5, 9]).

In this paper we continue the search for *large* vector spaces and algebras of functions enjoying these special or pathological properties. This paper is divided in several sections. In each of them we focus on a particular property of a function. These properties are: almost everywhere continuous functions that are not Riemann-integrable; Riemann integrable functions that are not Lebesgue-integrable and viceversa; and continuous unbounded functions on any arbitrary non-compact metric space.

Let us finish this introduction by fixing some notation. For any set I, $\mathcal{B}(I)$ will denote, as usual, the Banach space of all real bounded functions on I, endowed with the supremum norm. This space is also a Banach algebra with the usual product defined pointwise. When $I = \mathbb{N}$, we write, $\mathcal{B}(\mathbb{N}) = \ell_\infty$. Also, c_0 and $\mathcal{C}(X)$ denote, respectively, the set of null sequences and the set of continuous functions on X. Given an interval I (bounded or not) we denote by $\mathcal{R}(I)$ to the set of Riemann-integrable functions on I, and by $\mathcal{L}(I)$ the set of Lebesgue-integrable functions on I.

2. Riemann-integrable Functions, Almost Everywhere Continuos Functions, and Subalgebras of ℓ_∞

It is well known the theorem by Lebesgue about Riemann-integrability that states that if I is a bounded interval and $f : I \longrightarrow \mathbb{R}$ is a bounded function, then f is Riemann-integrable if and only if f is almost everywhere continuous (see e.g [19, Theorem 11.33]). The proof can be easily adapted to show that a Riemann-integrable function on an arbitrary interval (bounded or not) is always almost everywhere continuous. Obviously, the converse to this assertion is not true, since one can consider any non-zero constant function on any unbounded interval. Here, our purpose is, given any unbounded interval I, to construct an infinite-dimensional and infinitely generated closed subalgebra of $\mathcal{B}(I)$ every non-zero element of which is almost everywhere continuous but not Riemann-integrable. In order to do that, we need a similar result for $\ell_\infty \setminus c_0$.

Proposition 2.1. $\left(\ell_\infty \setminus c_0\right) \cup \{0\}$ *contains a closed infinitely generated subalgebra. In particular,* $\ell_\infty \setminus c_0$ *is spaceable and algebrable.*

Proof. Let us denote by P the set of all prime numbers. For every $p \in P$, let us consider the bounded sequence x_p given by

$$x_p\,(j) = \begin{cases} 1 & j = p^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us relabel the elements of P by writing $P = \{p_1, p_2, \ldots\}$ assuming that the sequence $p_1, p_2, \ldots$ is increasing. Next, let us take the subspace of ℓ_∞ given by

$$V = \left\{ \sum_{i=1}^{\infty} \lambda_i x_{p_i} : (\lambda_i)_{i\in\mathbb{N}} \in \ell_\infty \right\}$$

and write $W = \overline{V}$. Now, our aim is to prove that $W \cap c_0 = \{0\}$ and we will be done. For this, let us pick $a \in W \cap c_0$. Then, for every $n \in \mathbb{N}$ we can take $z_n \in V$ so that $\|z_n - a\| < 1/n$, i.e.

$$|z_n(j) - a(j)| \leqslant \|z_n - a\| < \frac{1}{n} \text{ for every } j \in \mathbb{N}.$$

Firstly, we can write $z_n = \sum_{i=1}^{\infty} \lambda_{i,n} x_{p_i}$ for every $n \in \mathbb{N}$. We will now prove that $|\lambda_{i,n}| < 1/n$ for every $i, n \in \mathbb{N}$. Fix $i, n \in \mathbb{N}$. We have that, for every $k \in \mathbb{N}$,

$$\left|\lambda_{i,n} - a\left(p_i^k\right)\right| = \left|z_n\left(p_i^k\right) - a\left(p_i^k\right)\right| \leqslant \|z_n - a\| < \frac{1}{n}.$$

Since $a \in c_0$, taking limit as k goes to ∞, we obtain

$$|\lambda_{i,n}| < \frac{1}{n}. \tag{1}$$

Finally, let us see that $a\,(j) = 0$ for every $j \in \mathbb{N}$. If $j \neq p_i^k$ for every $i, k \in \mathbb{N}$, then (by construction) $z_n\,(j) = 0$ for every $n \in \mathbb{N}$, and thus $a\,(j) = 0$. If $j = p_i^k$ for some $i, k \in \mathbb{N}$, then $z_n(j) = \lambda_{i,n}$ for every $n \in \mathbb{N}$, and by equation (1), $a(j) = 0$. Thus, $a = 0$ and $W \cap c_0 = \{0\}$, as desired.

To finish the proof, since ℓ_∞ with the pointwise product is a Banach algebra, we observe that V (and thus W) is a subalgebra with an infinite number of generators. Indeed, we first observe that for $p, q \in P$, $p \neq q$, the supports of x_p and x_q are disjoint. This implies that, on the one hand, the product of two elements of V remains in V and, on the other hand, that $\{x_p : p \in P\}$ is a minimal system of generators of V. ∎

Remark 2.2.

(a) *It is worth mentioning that the spaceability of $\ell_\infty \backslash c_0$ was known to H. P. Rosenthal from the sixties (see [17, 18]).* Indeed, it was shown in [17] that c_0 is *quasi-complemented* in ℓ_∞ (a closed subspace Y of a Banach space X is *quasi-complemented* if there is a closed subspace Z of X such that $Y \cap Z = \{0\}$ and $Y + Z$ is dense in X); this clearly implies that $\ell_\infty \setminus c_0$ is spaceable. The algebrability of this set seems to be, to the authors' knowledge, a new result.

(b) The proof of Proposition 2.1 can be easily adapted to show that $\ell_\infty(\Gamma) \setminus c_0(\Gamma)$ *is spaceable and algebrable for every infinite set* Γ. Let us comment that it was proved by J. Lindenstrauss [15] that if Γ is uncountable, $c_0(\Gamma)$ in not quasi-complemented in $\ell_\infty(\Gamma)$.

Now, we go back to our original set M of almost everywhere continuous functions on I. Let us start by assuming, without loss of generality, that the interval I contains the interval $[1, \infty)$. Now, we consider the function

$$\begin{array}{rccl} \phi: & \ell_\infty & \longrightarrow & \mathcal{B}(I) \\ & x & \longmapsto & \phi(x) = \phi_x = \displaystyle\sum_{n=1}^{\infty} x(n)\, \chi_{(n,n+1)}. \end{array}$$

Some properties enjoyed by this function ϕ are the following:

1. ϕ is a linear isometry and an algebra homomorphism.
2. ϕ_x is almost everywhere continuous for all $x \in \ell_\infty$.
3. ϕ_x is Riemann-integrable if and only if $\sum_{n=1}^{\infty} x(n)$ converges. In that situation,
$$\int_I \phi_x(t)\, dt = \sum_{n=1}^{\infty} x(n).$$

Next, by Proposition 2.1, there exists an infinite-dimensional and infinitely generated closed subalgebra W of ℓ_∞ such that $c_0 \cap W = \{0\}$. Then, according to the third property above, $\phi(W) \subseteq M \cup \{0\}$, and by the first property above, $\phi(W)$ satisfies the same properties than W. Therefore, we can state the following theorem.

Theorem 2.3. *Given an arbitrary unbounded interval I, the set of all almost everywhere continuous bounded functions on I which are not Riemann-integrable contains an infinitely generated closed subalgebra. In particular, this set is spaceable and algebrable.*

With a very similar argument to the one above, it is also possible to prove the spaceability of the set of bounded continuous functions which are not Riemann-integrable. Let us start by assuming, without loss of generality, that the interval I contains the interval $[1, \infty)$, and for every $n \in \mathbb{N}$, we consider the bounded continuous function $\alpha_n : [1, +\infty) \longrightarrow \mathbb{R}$ given by

$$\alpha_n(t) = \begin{cases} 2(t-n) & \text{if } n \leqslant t \leqslant n + 1/2 \\ 2 - 2(t-n) & \text{if } n + 1/2 < t \leqslant n+1 \\ 0 & \text{otherwise} \end{cases}$$

i.e. α_n is zero outside the interval $(n, n+1)$, its value at $n + 1/2$ is 1 and it is affine in $[n, n+1/2]$ and $[n+1/2, n+1]$. Now, we consider the linear isometry

$$\begin{array}{rccl} \psi : & \ell_\infty & \longrightarrow & \mathcal{B}(I) \\ & x & \longmapsto & \psi(x) = \psi_x = \displaystyle\sum_{n=1}^{\infty} x(n)\, \alpha_n, \end{array}$$

and, arguing as above, we obtain that the non-zero elements of $\psi(\ell_\infty \setminus c_0)$ are continuous functions which are not Riemann-integrable. Unfortunately, the function ψ is not an algebra homomorphism, so the argument does not give algebrability.

Theorem 2.4. *Given an arbitrary unbounded interval I, the set of all continuous bounded functions on I which are not Riemann-integrable is spaceable.*

3. Riemann-integrable Functions Versus Lebesgue-integrable Functions

It is well known that for a bounded interval I, $\mathcal{R}(I) \subset \mathcal{L}(I)$ (see, e.g. [19]). If I is unbounded, it is also well known that $\mathcal{R}(I) \not\subseteq \mathcal{L}(I)$; a representative and classical example is given by the function $f(x) = \frac{\sin x}{x}$ for every $x \in \mathbb{R}$. This function verifies that

$$\int_{\mathbb{R}} f(x)\, dx = \pi \quad \text{and} \quad \int_{\mathbb{R}} |f(x)|\, dx = \infty.$$

Conversely, on any interval I (bounded or unbounded), there is a bounded Lebesgue-integrable function which is not equivalent to any Riemann-integrable function. An easy example of this type can be found in [10, Example 8.31]. Let us present the details for the sake of completeness.

Example 3.1. *In any interval I, we consider a Cantor set $A \subset I$ with positive and finite measure (see, e.g. [10, Example 8.4]); then, the function $f = \chi_A$ is bounded, Lebesgue-integrable, but it is not equivalent to any Riemann-integrable function.* Indeed, everything is clear but the last statement. To prove it, just observe that $f = 0$ in $I \setminus A$, which is dense in I and, moreover, if we change f in a null-set B of I, then $f = 0$ in the still dense subset $I \setminus (A \cup B)$, and $f = 1$ in the set of positive measure $A \setminus B$.

Next we will study the lineability of the sets $\mathcal{R}(I) \setminus \mathcal{L}(I)$ when I is unbounded and the spaceability of $\mathcal{L}(I) \setminus \mathcal{R}(I)$, where I is both bounded and unbounded.

3.1. Lineability of $\mathcal{R}(I) \setminus \mathcal{L}(I)$ for an unbounded interval I

Here, and without loss of generality, we can consider $I = \mathbb{R}$. We will construct an infinite dimensional vector space of functions, E, with $E \subseteq (\mathcal{R}(\mathbb{R}) \setminus \mathcal{L}(\mathbb{R})) \cup \{0\}$. In order to do that, let us define the following double sequences of natural numbers:

$$\begin{aligned} a_{n,m} &= \frac{n(n-1)}{2} + 2m - 3 \quad \text{if} \quad m > 1 \text{ and } n \geqslant 1, \\ b_{n,m} &= \frac{n(n+1)}{2} + m - 2 \quad \text{if} \quad m > 1 \text{ and } n \geqslant m - 1 \end{aligned}$$

and the sequence of functions $(f_m)_{m>1}$, where $f_m : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$f_m(x) := \begin{cases} \dfrac{\sin(x - a_{n,m}\pi)}{x - a_{n,m}\pi} & \text{if} \quad x \in [b_{n,m}\pi, (b_{n,m}+1)\pi] \text{ for some } n \geqslant m-1, \\ 0 & \text{otherwise.} \end{cases}$$

By construction we have that, for $1 < m \in \mathbb{N}$:

(P1) $\operatorname{supp}(f_m) = \bigcup_{n \geqslant m-1} (b_{n,m}\pi, (b_{n,m}+1)\pi)$.

(P2) $\operatorname{supp}(f_i) \bigcap \operatorname{supp}(f_j) \neq \varnothing$ if and only if $i = j$.

Property (P1) is clear by definition. Let us see that (P2) also holds. For that, suppose that $\operatorname{supp}(f_i) \bigcap \operatorname{supp}(f_j) \neq \varnothing$ for some $i, j > 1$. Then, there exist $p, q \in \mathbb{N}$ with $p \geqslant i - 1$ and $q \geqslant j - 1$, such that

$$(b_{p,i}\pi, (b_{p,i}+1)\pi) \bigcap (b_{q,j}\pi, (b_{q,j}+1)\pi) \neq \varnothing.$$

Since both of the above interval are open, have length π, and their extremes are integers multiples of π, it follows that

$$b_{p,i} = b_{q,j}, \tag{2}$$

which is equivalent to

$$p(p+1) + 2i = q(q+1) + 2j. \tag{3}$$

Suppose that $i < j$. From (3) and the above conditions, it follows that

$$p > q \geqslant j - 1 > i - 1. \tag{4}$$

Now we will see that for every $p > q$ it is $b_{p,i} > b_{q,j}$, which will lead to a contradiction. Indeed, if $p > q$, then $p = q + k$ for some $1 \leqslant k \in \mathbb{N}$. For our purpose, it suffices to show that $b_{q+1,i} > b_{q,j}$, since $(b_{q+k,i})_k$ is an increasing sequence. Thus, for $p = q + 1$ we obtain:

$$\begin{aligned}(q+1)(q+2) + 2i &= q(q+1) + 2q + 2 + 2i = q(q+1) + 2(q+1) + 2i \\ &\overset{(4)}{\geqslant} q(q+1) + 2j + 2i > q(q+1) + 2j,\end{aligned}$$

and, thus, $b_{p,i} > b_{q,j}$ for every $p > q$, which contradicts (2). A similar contradiction is reached if we suppose that $i > j$. Therefore, it must be $i = j$, and we are done.

Next, let us see that $f_m \in \mathcal{R}(\mathbb{R})$ for every $m > 1$. Indeed, let us fix any $1 < m \in \mathbb{N}$, then

$$\int_{\mathbb{R}} f_m(x)dx = \int_{\mathrm{supp}(f_m)} f_m(x)dx = \sum_{n \geqslant m-1} \int_{\pi b_{n,m}}^{\pi b_{n,m}+\pi} \frac{\sin(x - a_{n,m}\pi)}{x - a_{n,m}\pi} dx.$$

Next, making the substitution $t = x - a_{n,m}\pi$, and noticing that

$$\pi b_{n,m} - \pi a_{n,m} = (n - m + 1)\pi,$$

we obtain that

$$\begin{aligned}\int_{\mathbb{R}} f_m(x)dx &= \sum_{n \geqslant m-1} \int_{\pi b_{n,m}-\pi a_{n,m}}^{\pi b_{n,m}-\pi a_{n,m}+\pi} \frac{\sin t}{t}\, dt \\ &= \sum_{n \geqslant m-1} \int_{(n-m+1)\pi}^{(n-m+2)\pi} \frac{\sin t}{t}\, dt \\ &= \sum_{n \geqslant 0} \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t}\, dt \\ &= \int_0^{+\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}.\end{aligned}$$

Similarly, one can see that for every $m > 1$, $f_m \notin \mathcal{L}(\mathbb{R})$. Indeed, repeating the previous calculations, we obtain that:

$$\int_{\mathbb{R}} |f_m(x)|dx = \int_0^{+\infty} \left|\frac{\sin t}{t}\right| dt = \infty.$$

We claim that

$$E = \text{span}\{f_m : 1 < m \in \mathbb{N}\}$$

is an infinite dimensional vector space such that $E \subseteq \big(\mathcal{R}(\mathbb{R}) \setminus \mathcal{L}(\mathbb{R})\big) \cup \{0\}$. First of all, by (P2), it is clear that $\dim(E) = \aleph_0$. Since $\mathcal{R}(\mathbb{R})$ is a vector space, it is clear than $E \subseteq \mathcal{R}(\mathbb{R})$.

Let us see that no $f \in E \setminus \{0\}$ is Lebesgue-integrable on $\mathbb{R}$. We write

$$f = \alpha_1 f_{m_1} + \alpha_2 f_{m_2} + \cdots + \alpha_k f_{m_k},$$

where $0 \neq \alpha_i \in \mathbb{R}$ for every $i \in \{1, 2, \ldots, k\}$, and $1 < m_i \in \mathbb{N}$ for every $i \in \{1, 2, \ldots, k\}$. Also, denote $S_i = \text{supp}(f_{m_i})$ for every $i \in \{1, 2, \ldots, k\}$. Then,

$$\begin{aligned} \int_{\mathbb{R}} |f(x)| dx &= \int_{\bigcup_{i=1}^k S_i} |\alpha_1 f_{m_1}(x) + \alpha_2 f_{m_2}(x) + \cdots + \alpha_k f_{m_k}(x)| dx \\ &\overset{(P2)}{=} \sum_{i=1}^{k} |\alpha_i| \cdot \int_{S_i} |f_{m_i}(x)| dx = \sum_{i=1}^{k} |\alpha_i| \cdot \int_0^{+\infty} \left| \frac{\sin t}{t} \right| dt = \infty, \end{aligned}$$

and, therefore $f \notin \mathcal{L}(\mathbb{R})$.

We have proved the following result.

Theorem 3.2. *Given any unbounded interval I, the set of Riemann-integrable functions on I that are not Lebesgue-integrable is lineable.*

Remark 3.3. *It is not possible to obtain any kind of algebrability of $\mathcal{R}(I)\setminus\mathcal{L}(I)$.* Indeed, for every $f \in \mathcal{R}(I)$, either $f^2 \notin \mathcal{R}(I)$ or $f^2 = |f^2| \in \mathcal{R}(I)$ and, therefore, $f^2 \in \mathcal{L}(I)$. Therefore,

Proposition 3.4. *The set $\mathcal{R}(I) \setminus \mathcal{L}(I)$ is not algebrable.*

3.2. Spaceability of $\mathcal{L}(I) \setminus \mathcal{R}(I)$ for any interval I

The spaceability of $\mathcal{L}(I) \setminus \mathcal{R}(I)$ is a consequence of a stronger result from [8, §4]. For the sake of completeness of this paper, we include here a direct (and different) proof of this result. In order to do that, let us fix an arbitrary non-trivial interval I. As we already mentioned, there is a function $f \in \mathcal{L}(I)$ which is not equivalent to any Riemann-integrable function. This family of functions will give us the spaceability. To do so, we write $I = \bigcup_{n=1}^{\infty} I_n$, where the I_n are non-trivial disjoint intervals. As we did in Example 3.1, for every $n \in \mathbb{N}$, we take a Cantor subset A_n of I_n with $\lambda(A_n) > 0$, and notice that the function χ_{A_n} belongs to $\mathcal{L}(I)$ but it is not equivalent to any Riemann-integrable function. By the disjointness of their supports, the same is true for any linear combination of the χ_{A_n}. This proves the lineability, but the spaceability is also easy to establish. Let us consider the function

$$\begin{array}{rccl} \phi: & \ell_1 & \longrightarrow & \mathcal{L}\,(I) \\ & x & \longmapsto & \phi\,(x) = \phi_x = \displaystyle\sum_{n=1}^{\infty} \frac{x\,(n)}{\lambda(A_n)}\,\chi_{A_n}. \end{array}$$

Then, ϕ is an isometric embedding and each of the the functions in $\phi(\ell_1) \setminus \{0\}$ is not equivalent to a Riemann-integrable function. Summarizing:

Theorem 3.5. *Given any interval I, the set of Lebesgue-integrable functions that are not Riemann-integrable is spaceable.*

Remark 3.6.

(a) In the case when I is unbounded, in the above construction we may choose all the Cantor sets A_n with the same positive measure, and then, all the functions in $\phi(\ell_1)$ are bounded.

(b) When I is bounded, it is of course not possible to choose the Cantor sets with the same positive measure and, therefore, there are unbounded functions in $\phi(\ell_1)$. Even so, the linear span of the functions χ_{A_n} consists only of bounded functions. Therefore, the set of bounded Lebesgue-integrable functions that are not Riemann-integrable is lineable.

4. Continuous Unbounded Functions on an Arbitrary Non-compact Metric Space

In [10, Example 2.3] it is shown that for every arbitrary non-compact subset A of $\mathbb{R}$ there exists a continuous unbounded function having the subset as domain. Such an example is given in [10] as follows. If the set A is unbounded, then it suffices to consider the identity function on A. On the other hand, if A is bounded and not close it is enough to consider c a limit point of A, and let $f(x) = (x-c)^{-1}$ for $x \in A$. Moreover, we can consider a more general example: If A is not bounded, then each non-null polynomial is unbounded on A; if A is bounded and non-compact, then there is $c \in \overline{A} \setminus A$ and the function $x \longmapsto \dfrac{1}{|x-c|^n}$ is unbounded for every $n \in \mathbb{N}$. It follows easily that the set of all continuous unbounded functions on any non-compact subset A of $\mathbb{R}$ is lineable and, with a little more of effort, it can be proved that it is algebrable (just change the potentials by exponentials to get infinitely many algebraic independent functions).

Our purpose in this section is to simplify and extend the above arguments to show that on any non-compact metric space X one may construct an infinite dimensional vector space every non-zero element of which is a continuous unbounded function defined on X.

Indeed, taking into account that X is not compact we can find a sequence $(x_n)_{n\in\mathbb{N}}$ with no convergent subsequences. Then, the set $A = \{x_n : n \in \mathbb{N}\}$ is closed and has the discrete topology. Obviously, we can assume without loss of generality that $x_n \neq x_m$ if $n \neq m$. Next, for every $m \in \mathbb{N}$ let us consider the function

$$\begin{array}{rccl} g_m : & A & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & g_m(x) = m^n \text{ if } x = x_n. \end{array}$$

Since A is closed and has the discrete topology then, for every $m \in \mathbb{N}$, there exists a continuous function $f_m : X \longrightarrow \mathbb{R}$ such that $f_m|_A = g_m$ (Tietze Extension Theorem, see e.g. [16, Theorem 35.1]).

Let us see that the family $\{f_m : m \in \mathbb{N}\}$ is linearly independent. Take any linear combination identically 0, i.e.

$$\lambda_1 f_{m_1} + \cdots + \lambda_k f_{m_k} = 0, \tag{5}$$

where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Then, by evaluating equation (5) at $x_1, \ldots, x_k$ we obtain the following linear system of equations:

$$\begin{pmatrix} m_1 & m_2 & m_3 & \cdots & m_k \\ m_1^2 & m_2^2 & m_3^2 & \cdots & m_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1^k & m_2^k & m_3^k & \cdots & m_k^k \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The matrix of the previous system is non-singular (it is a Van der Monde-type matrix). Therefore we have that, necessarily, $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$.

Finally, let us see that every non-zero element in $\operatorname{span}\{f_m : m \in \mathbb{N}\}$ is a continuous unbounded function. Take a linear combination

$$\lambda_1 f_{m_1} + \cdots + \lambda_k f_{m_k}$$

where $k \geqslant 2$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R} \setminus \{0\}$. It is clear that this linear combination is a continuous function. Let us see that it is an unbounded function. We can assume that $m_1 > m_h$ for $2 \leqslant h \leqslant k$. For every $j \geqslant 1$ we have

$$\begin{aligned} & \left|\left(\lambda_1 f_{m_1} + \cdots + \lambda_k f_{m_k}\right)(x_j)\right| \\ = & \left|\lambda_1 m_1^j + \cdots + \lambda_k m_k^j\right| \\ \geqslant & \left|\lambda_1 m_1^j\right| - \left|\lambda_2 m_2^j\right| - \cdots - \left|\lambda_k m_k^j\right| \\ = & \left(\frac{|\lambda_1|}{k-1} m_1^j - |\lambda_2|\, m_2^j\right) + \cdots + \left(\frac{|\lambda_1|}{k-1} m_1^j - |\lambda_k|\, m_k^j\right). \end{aligned}$$

Since $m_1 > m_h$ for $2 \leqslant h \leqslant k$, we obtain that

$$\left(\frac{|\lambda_1|}{k-1} m_1^j - |\lambda_h|\, m_h^j\right) \longrightarrow \infty \text{ as } j \to \infty,$$

therefore

$$|(\lambda_1 f_{m_1} + \cdots + \lambda_k f_{m_k})\,(x_j)| \longrightarrow \infty \text{ as } j \to \infty,$$

thus $\lambda_1 f_{m_1} + \cdots + \lambda_k f_{m_k}$ is unbounded.

Let us observe that what we have actually proved is that any non-zero linear combination of g_m's is unbounded, and this fact allows us to obtain algebrability of the set of unbounded continuous functions on X. Indeed, let us consider the usual structure of algebra on $\mathcal{C}(X)$ given by the pointwise product and let $\mathcal{A}$ be the subalgebra generated by the family $\{f_m : m \in \mathbb{N}\}$, i.e. the elements of $\mathcal{A}$ are products of linear combinations of f_m's. Since, clearly,

$$g_{m_1} g_{m_2} = g_{m_1 m_2}$$

for every $m_1, m_2 \in \mathbb{N}$, we obtain that $\mathcal{A}$ is infinitely generated (the functions f_p for $p \in \mathbb{N}$ prime are algebraically independent since the functions $f_p|_A = g_p$ are) and, on the other hand, that every element of $\mathcal{A}$ is unbounded.

As a consequence of all of this we can state the following theorem.

Theorem 4.1. *In every non-compact metric space, the set of all continuous unbounded functions defined on it is algebrable.*

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