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MODIFIED EXTRAGRADIENT METHODS FOR STRICT PSEUDO-CONTRACTIONS AND MONOTONE MAPPINGS

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Abstract. In this paper we introduce an iterative process to find a common element of the set of fixed points of a strict pseudo-contraction and the set of solutions of the variational inequality problem for a monotone and Lipschitz continuous mapping. The iterative process is based on the so-called modified extragradient method. We obtain a weak convergence theorem for two sequences generated by this process. Using this theorem, we also construct an iterative process to find a common element of the set of fixed points of a strict pseudo-contraction and the set of zeroes of a monotone and Lipschitz continuous mapping.

1. INTRODUCTION

Throughout this paper, H denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

Definition 1.1. Let C be a nonempty subset of H. A mapping $A : C \to H$ is said to be (see [1, 9]):

(i) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \text{for all } x, y \in C;$$

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(ii) α-inverse strongly monotone if there exists a positive number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \text{for all } x, y \in C;$$

(iii) β-strongly monotone if there exists a positive number β such that

$$\langle Ax - Ay, x - y\rangle \geq \beta\|x - y\|^2, \quad \text{for all } x, y \in C;$$

(iv) k-Lipschitz continuous if there exists a positive number k such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \text{for all } x, y \in C.$$

It is obvious that every α-inverse strongly monotone mapping A is monotone and Lipschitz continuous.

Definition 1.2. Let C be a nonempty subset of H. A self-mapping $S : C \to C$ is said to be (see [10, 14]):

(i) nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \text{for all } x, y \in C;$$

(ii) a strict pseudo-contraction if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \text{for all } x, y \in C;$$

(iii) a quasi-strict pseudo-contraction if the set of fixed points of S, $F(S) = \{z \in C : Sz = z\}$, is nonempty and if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + \kappa\|x - Sx\|^2, \quad \text{for all } x \in C,\ p \in F(S).$$

We also say that S is a κ-strict pseudo-contraction if condition (ii) holds, and respectively, S is a κ-quasi-strict pseudo-contraction if condition (iii) holds.

Let C be a nonempty subset of H. Given a mapping $A : C \to H$, the variational inequality problem, denoted VI(A, C), is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \text{for all } v \in C.$$

The set of solutions of the variational inequality problem VI(A, C) will be denoted by $\Omega(A, C)$. The variational inequality problem was first discussed by Lions [8]. Subsequently, this problem has been widely studied since it covers diverse disciplines such as partial differential equations, optimal control, optimization theory, mathematical programming, mechanics and mathematical economics. It is well known that if A is a strongly monotone and Lipschitz continuous mapping on C, then $\mathrm{VI}(A, C)$ has a unique solution. The mapping $A : C \to H$ is said to be *pseudomonotone* if for any $x, y \in C$, $\langle x - y, Ay \rangle \geq 0$ implies $\langle x - y, Ax \rangle \geq 0$. Every monotone mapping is pseudomonotone. In particular, if C is a nonempty closed convex subset of H and $A : C \to H$ is pseudomonotone and continuous on finite-dimensional subspaces, then the set of solutions $\Omega(A, C)$ of $\mathrm{VI}(A, C)$ is closed and convex [17, Lemma 3.1]. The various approaches and interesting results to this problem in finite-dimensional and infinite-dimensional spaces have been intensively developed; see [2]-[5], [7, 9, 12, 15, 16], and [18]-[20].

It is remarkable that, in 1976, Korpelevich [7] found a solution of the nonconstrained variational inequality problem in the finite-dimensional Euclidean space $\mathbf{R^n}$ under the assumption that $C \subset \mathbf{R^n}$ is closed and convex and $A : C \to \mathbf{R^n}$ is monotone and k-Lipschitz continuous by introducing the following so-called extragradient method:

$$\text{(1)} \qquad \begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A \bar{x}_n), \quad n \geq 0, \end{cases}$$

where $\lambda \in (0, 1/k)$. He showed that if $\Omega(A, C)$ is nonempty, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$ generated by (1) converge to the same point $z \in \Omega(A, C)$.

Recently, motivated by the idea of Korpelevich's extragradient method, a variety of iterative schemes were introduced to find a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone and k-Lipschitz continuous mapping. In 2006, Nadezhkina and Takahashi [12] provided an iterative process and proved the following weak convergence result.

Theorem 1.3. (Nadezhkina and Takahashi [12]). *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone and k-Lipschitz continuous mapping and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap \Omega(A, C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\text{(2)} \qquad \begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \quad n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega(A, C)$, where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega(A,C)} x_n$.

Inspired by Nadezhkina and Takahashi's iterative scheme (2), Zeng and Yao [18] gave an iterative process and asserted the following strong convergence theorem.

Theorem 1.4. (Zeng and Yao [18]). *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone and k-Lipschitz continuous mapping and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap \Omega(A, C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \quad n \geq 0, \end{cases}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:

(i) $\{\lambda_n k\} \subset (0, 1 - \delta)$, *for some* $\delta \in (0, 1)$;

(ii) $\{\alpha_n\} \subset (0, 1)$, $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the same element $P_{F(S)\cap\Omega(A,C)}(x_0)$ provided $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Furthermore, Ceng and Yao [2] introduced an iterative extragradient-like approximation method and established another strong convergence theorem.

Theorem 1.5. (Ceng and Yao [2]). *Let C be a nonempty closed convex subset of a real Hilbert space H, $f : C \to C$ a contraction, $A : C \to H$ a monotone and Lipschitz continuous mapping and $S : C \to C$ a nonexpansive mapping such that $F(S) \cap \Omega(A, C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$(3) \quad \begin{cases} x_0 = x \in C, \\ y_n = (1 - \gamma_n) x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \alpha_n - \beta_n) x_n + \alpha_n f(y_n) + \beta_n S P_C(x_n - \lambda_n A y_n), \quad n \geq 0, \end{cases}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \lambda_n < \infty$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$, *for all* $n \geq 0$;

(ii) $\displaystyle\lim_{n\to\infty} \alpha_n = 0$, $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \displaystyle\liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to the same point $q = P_{F(S)\cap\Omega(A,C)} f(q)$ if and only if $\{A x_n\}$ is bounded and $\liminf_{n\to\infty} \langle A x_n, y - x_n \rangle \geq 0$, for all $y \in C$.

Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $A : C \to H$ is a monotone and k-Lipschitz continuous mapping and $S : C \to C$ is a κ-strict pseudo-contraction for some $0 \le \kappa < 1$ such that $F(S) \cap \Omega(A, C) \ne \emptyset$. Based on the extragradient method (3) and Mann's iterative method [10], this paper is devoted to introduce a modified extragradient method as follows:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) S t_n, \quad n \ge 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (\kappa, 1)$. It is shown that the sequences $\{x_n\}$ and $\{y_n\}$ generated by this iterative scheme converge weakly to the same point $z \in F(S) \cap \Omega(A, C)$, where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega(A,C)} x_n$. We also apply this result to construct an iterative process to find a common element of the set of fixed points of a strict pseudo-contraction and the set of zeroes of a monotone and Lipschitz continuous mapping.

2. Preliminaries

Suppose that C is a nonempty closed convex subset of a real Hilbert space H. For each point $x \in H$ there exists a unique nearest point in C, denoted $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. Then P_C is a nonexpansive mapping from H onto C characterized by the following properties [14]: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x \rangle \le 0, \quad \text{for all } x \in H,\ y \in C, \tag{4}$$

$$\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \text{for all } x \in H,\ y \in C. \tag{5}$$

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, the inequality (5) implies that

$$u \in \Omega(A, C) \iff u = P_C(u - \lambda A u), \quad \text{for all } \lambda > 0.$$

We will use the notations $\to$ for strong convergence and $\rightharpoonup$ for weak convergence. The following facts will be used in the sequel.

Lemma 2.1. ([10, Lemma 1.1]). *Let H be a real Hilbert space. Then we have*

(*i*) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H$;

(*ii*) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2 - t(1-t)\|x-y\|^2$, *for all* $t \in [0,1]$ *and* $x, y \in H$;

(*iii*) *If* $\{x_n\} \rightharpoonup z$, *then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2, \quad \textit{for all } y \in H.$$

A set-valued mapping $T : H \to 2^H$ is said to be *monotone* if

$$\langle x-y, f-g\rangle \geq 0, \quad \text{for all } (x,f), (y,g) \in G(T),$$

where $G(T)$ denotes the graph of T. A monotone mapping $T : H \to 2^H$ is said to be *maximal* if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. A monotone mapping T is maximal if and only if whenever $(x,f) \in H \times H$ and $\langle x-y, f-g\rangle \geq 0$ for all $(y,g) \in G(T)$, then $f \in Tx$; see [6]. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \text{ for all } u \in C\}.$$

Define a set-valued mapping $T : H \to 2^H$ by

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega(A,C)$; see [13].

3. Weak Convergence Theorem

In this section, we establish a weak convergence theorem for strict pseudo-contractions and monotone mappings. The following two lemmas are required.

Lemma 3.1. (Marino and Xu [10, Proposition 2.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a mapping.*

(*i*) *If S is a strict pseudo-contraction, then the mapping $I - S$ is demiclosed (at* 0*), i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-S)x_n \to 0$, then $(I-S)\tilde{x} = 0$.*

(*ii*) *If S is a quasi-strict pseudo-contraction, then the fixed set $F(S)$ of S is closed and convex so that the projection $P_{F(S)}$ is well defined.*

Lemma 3.2. (Takahashi and Toyoda [15, Lemma 3.2]) *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that for each $u \in C$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad n \geq 0.$$

Then the sequence $\{P_C x_n\}$ converges strongly to some $z \in C$.

Our weak convergence theorem is obtained as follows.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ a monotone and k-Lipschitz continuous mapping and $S : C \to C$ a κ-strict pseudo-contraction for some $0 \leq \kappa < 1$ such that $F(S) \cap \Omega(A, C) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\text{(6)} \qquad \begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) S t_n, \quad n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\kappa, 1)$. Then $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega(A, C)$, where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega(A,C)} x_n$.

Proof. The proof is divided into five steps. First, note that $F(S) \cap \Omega(A, C)$ is closed and convex by Lemma 3.1(ii) and [17, Lemma 3.1].

Step 1. Let $u \in F(S) \cap \Omega(A, C)$ so that $\langle Au, v - u\rangle \geq 0$, for all $v \in C$. We claim that

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \text{for } n \geq 0,$$

which implies that

$$\lim_{n\to\infty} \|x_n - u\|$$

exists, for all $u \in F(S) \cap \Omega(A, C)$.

By (5), (6) and the monotonicity of A, we obtain

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n - Au, u - y_n\rangle \\ &\quad + 2\lambda_n \langle Au, u - y_n\rangle + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \end{aligned}$$

$$
\begin{aligned}
&= \|x_n - u\|^2 - (\|x_n - y_n\|^2 + 2\langle x_n - y_n, y_n - t_n\rangle + \|y_n - t_n\|^2) \\
&\quad + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{aligned}
$$

Since $y_n = P_C(x_n - \lambda_n Ax_n)$ and A is k-Lipschitz continuous, it follows from (4) that

$$
\begin{aligned}
\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle &= \langle x_n - \lambda_n Ax_n - y_n, t_n - y_n\rangle \\
&\quad + \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\le \langle \lambda_n Ax_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\le \lambda_n k\|x_n - y_n\|\|t_n - y_n\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|t_n - u\|^2 \\
&\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|x_n - y_n\|\|t_n - y_n\| \\
&\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad +\lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\
&= \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\le \|x_n - u\|^2.
\end{aligned}
\tag{7}
$$

Note that $u = Su$ and $\{\alpha_n\} \subset [\kappa, 1]$. Since S a κ-strict pseudo-contraction, we have by (6), (7) and Lemma 2.1(ii) that

$$
\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&= \|\alpha_n(t_n - u) + (1-\alpha_n)(St_n - u)\|^2 \\
&= \alpha_n\|t_n - u\|^2 + (1-\alpha_n)\|St_n - u\|^2 - \alpha_n(1-\alpha_n)\|t_n - St_n\|^2 \\
&\le \alpha_n\|t_n - u\|^2 + (1-\alpha_n)(\|t_n - u\|^2 + \kappa\|t_n - St_n\|^2) \\
&\quad -\alpha_n(1-\alpha_n)\|t_n - St_n\|^2 \\
&= \|t_n - u\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|t_n - St_n\|^2 \\
&\le \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\le \|x_n - u\|^2.
\end{aligned}
\tag{8}
$$

Our claim is proved. Hence $\lim_{n\to\infty}\|x_n - u\|$ exists and so the sequences $\{x_n\}$ and $\{t_n\}$ are bounded.

Step 2. Observe that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0$$

and

$$\lim_{n\to\infty} \|y_n - t_n\| = 0.$$

In fact, we have by (8) that

$$(1-\lambda_n^2 k^2)\|x_n - y_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2$$

which shows that

$$\|x_n - y_n\|^2 \le \frac{1}{1-\lambda_n^2 k^2}(\|x_n - u\|^2 - \|x_{n+1} - u\|^2).$$

Hence $\lim_{n\to\infty} \|x_n - y_n\| = 0.$ On the other hand, since P_C is nonexpansive,

$$\begin{aligned}
\|t_n - y_n\|^2 &= \|P_C(x_n - \lambda_n A y_n) - P_C(x_n - \lambda_n A x_n)\|^2 \\
&\le \lambda_n^2 k^2 \|x_n - y_n\|^2 \\
&\le \frac{\lambda_n^2 k^2}{1-\lambda_n^2 k^2}(\|x_n - u\|^2 - \|x_{n+1} - u\|^2).
\end{aligned}$$

Thus $\lim_{n\to\infty} \|y_n - t_n\| = 0$ by Step 1. Since $\|x_n - t_n\| \le \|x_n - y_n\| + \|y_n - t_n\|$, it implies that $\lim_{n\to\infty} \|x_n - t_n\| = 0$. By Lipschitz continuity of A, we have $\lim_{n\to\infty} \|Ay_n - At_n\| = 0$.

Step 3. We claim that the following hold:

(i) $\lim\limits_{n\to\infty} \|t_n - St_n\| = 0$;

(ii) $\lim\limits_{n\to\infty} \|x_n - Sx_n\| = 0$.

Indeed, since $\kappa < c \le \alpha_n \le d < 1$ for all $n \ge 0$, it follows from (7) and the fourth inequality in (8) that

$$\begin{aligned}
(c-\kappa)(1-d)\|t_n - St_n\|^2 &\le (\alpha_n - \kappa)(1-\alpha_n)\|t_n - St_n\|^2 \\
&\le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.
\end{aligned}$$

By Step 1, $\lim_{n\to\infty} \|t_n - St_n\| = 0$.

By Lemma 2.1(ii),

$$\begin{aligned}
\|x_{n+1} - Sx_{n+1}\|^2 &= \|\alpha_n(t_n - Sx_{n+1}) + (1-\alpha_n)(St_n - Sx_{n+1})\|^2 \\
&= \alpha_n\|t_n - Sx_{n+1}\|^2 + (1-\alpha_n)\|St_n - Sx_{n+1}\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|t_n - St_n\|^2.
\end{aligned} \tag{9}$$

Note that $t_n - x_{n+1} = (1-\alpha_n)(t_n - St_n)$. The first term on the right-hand side of (9) can be written as

$$\begin{aligned} \|t_n - Sx_{n+1}\|^2 &= \|(t_n - x_{n+1}) + (x_{n+1} - Sx_{n+1})\|^2 \\ &= \|t_n - x_{n+1}\|^2 + \|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad +2\langle t_n - x_{n+1}, x_{n+1} - Sx_{n+1}\rangle \\ &= (1-\alpha_n)^2\|t_n - St_n\|^2 + \|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad +2\langle t_n - x_{n+1}, x_{n+1} - Sx_{n+1}\rangle. \end{aligned} \tag{10}$$

To estimate the second term on the right-hand side of (9), since S is a κ-strict pseudo-contraction, it follows that

$$\begin{aligned} \|St_n - Sx_{n+1}\|^2 &\le \|t_n - x_{n+1}\|^2 + \kappa\|(t_n - St_n) - (x_{n+1} - Sx_{n+1})\|^2 \\ &= \|t_n - x_{n+1}\|^2 + \kappa\|t_n - St_n\|^2 + \kappa\|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad -2\kappa\langle t_n - St_n, x_{n+1} - Sx_{n+1}\rangle \\ &= [\,(1-\alpha_n)^2 + \kappa\,]\|t_n - St_n\|^2 + \kappa\|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad -2\kappa\langle t_n - St_n, x_{n+1} - Sx_{n+1}\rangle. \end{aligned} \tag{11}$$

Now, we deduce from (9), (10) and (11) that

$$\begin{aligned} \|x_{n+1} - Sx_{n+1}\|^2 &\le (1-\alpha_n)(1+\kappa-2\alpha_n)\|t_n - St_n\|^2 \\ &\quad +[\,\alpha_n + \kappa(1-\alpha_n)\,]\|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad +2(1-\alpha_n)(\alpha_n - \kappa)\langle t_n - St_n, x_{n+1} - Sx_{n+1}\rangle \\ &\le [\alpha_n + \kappa(1-\alpha_n)]\|x_{n+1} - Sx_{n+1}\|^2 \\ &\quad +(1-\alpha_n)(1+\kappa-2\alpha_n)\|t_n - St_n\|^2 \\ &\quad +2(1-\alpha_n)(\alpha_n - \kappa)\|t_n - St_n\|\|x_{n+1} - Sx_{n+1}\|. \end{aligned} \tag{12}$$

Putting $a_n = \|x_{n+1} - Sx_{n+1}\|$ and $b_n = \|t_n - St_n\|$ for $n \ge 0$, and then dividing (12) by $1-\alpha_n$ yields

$$(1-\kappa)a_n^2 \le (1+\kappa-2\alpha_n)b_n^2 + 2(\alpha_n - \kappa)a_n b_n.$$

If $b_n = 0$, then $a_n = 0$. If $b_n > 0$, divide the last inequality by b_n^2 and set $\gamma_n = a_n/b_n$ to get the quadratic inequality of γ_n

$$(1-\kappa)\gamma_n^2 - 2(\alpha_n - \kappa)\gamma_n - (1+\kappa-2\alpha_n) \le 0$$

which implies that

$$\gamma_n \leq \frac{\alpha_n - \kappa + \sqrt{(\alpha_n - \kappa)^2 + (1-\kappa)(1+\kappa-2\alpha_n)}}{1-\kappa} = 1.$$

Therefore $a_n \leq b_n$. It follows that $\|x_{n+1} - Sx_{n+1}\| \leq \|t_n - St_n\|$, for all $n \geq 0$. Since $\lim_{n\to\infty} \|t_n - St_n\| = 0$, we have $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Step 4. Denote the weak ω-limit set of $\{x_n\}$ by

$$\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x, \text{ for some subsequence } \{x_{n_i}\}\}.$$

We claim that

$$\omega_w(x_n) \subset F(S) \cap \Omega(A, C).$$

Since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $z \in \omega_w(x_n)$ so that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to z. We will show that $z \in \Omega(A, C)$. Since $x_n - t_n \to 0$ and $y_n - t_n \to 0$, we have $t_{n_i} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$. Define a set-valued mapping $T : H \to 2^H$ by

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Then $w \in Tv = Av + N_C v$. Hence $w - Av \in N_C v$ and so

$$\langle v - u, w - Av \rangle \geq 0, \quad \text{for all } u \in C.$$

On the other hand, since $t_n = P_C(x_n - \lambda_n Ay_n)$ and $v \in C$, we have by (4) that

$$\langle x_n - \lambda_n Ay_n - t_n, t_n - v \rangle \geq 0$$

and so

$$\langle v - t_n, Ay_n + \frac{t_n - x_n}{\lambda_n} \rangle \geq 0. \tag{13}$$

Since $w - Av \in N_C v$, $t_{n_i} \in C$ and A is monotone, (13) shows that

$$\begin{aligned}
&\langle v - t_{n_i}, w \rangle \\
&\geq \langle v - t_{n_i}, Av \rangle \\
&\geq \langle v - t_{n_i}, Av \rangle - \langle v - t_{n_i}, Ay_{n_i} + \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle \\
&= \langle v - t_{n_i}, Av - At_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle \\
&\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \rangle.
\end{aligned} \tag{14}$$

As i approaching ∞,

$$\langle v - z, w\rangle \geq 0.$$

Since T is maximal monotone, $z \in T^{-1}0$ and hence $z \in \Omega(A, C)$; see [13].

To prove that $z \in F(S)$, note that S is a κ-strict pseudo-contraction and so by Lemma 3.1 (i) the mapping $I - S$ is demiclosed at zero. Since $\|x_n - Sx_n\| \to 0$ and $x_{n_i} \rightharpoonup z$, we have $z \in F(S)$. Consequently, $z \in F(S) \cap \Omega(A, C)$ and hence $\omega_w(x_n) \subset F(S) \cap \Omega(A, C)$.

Step 5. The sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega(A, C)$, where $z = \lim_{n\to\infty} P_{F(S)\cap\Omega(A,C)}x_n$. To see this, we first show that $\omega_w(x_n)$ is a single-point set. Take $z_1, z_2 \in \omega_w(x_n)$ arbitrarily and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup z_1$ and $x_{m_j} \rightharpoonup z_2$, respectively. It follows from Step 1 and Step 4 that $\lim_{n\to\infty} \|x_n - z_1\|$ and $\lim_{n\to\infty} \|x_n - z_2\|$ exist. By Lemma 2.1 (iii) we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\|^2 &= \lim_{j\to\infty} \|x_{m_j} - z_1\|^2 \\
&= \lim_{j\to\infty} \|x_{m_j} - z_2\|^2 + \|z_2 - z_1\|^2 \\
&= \lim_{i\to\infty} \|x_{k_i} - z_2\|^2 + \|z_2 - z_1\|^2 \\
&= \lim_{i\to\infty} \|x_{k_i} - z_1\|^2 + 2\|z_2 - z_1\|^2 \\
&= \lim_{n\to\infty} \|x_n - z_1\|^2 + 2\|z_2 - z_1\|^2
\end{aligned}$$

which asserts that $z_1 = z_2$. Therefore $\omega_w(x_n)$ is a single-point set, say $\{z\}$, and so $x_n \rightharpoonup z$. Since $x_n - y_n \to 0$, we also have $y_n \rightharpoonup z$. For $n \geq 0$, let $u_n = P_{F(S)\cap\Omega(A,C)}x_n$ so that by (4),

$$\langle z - u_n, x_n - u_n\rangle \leq 0.$$

Then Step 1 and Lemma 3.2 assure that $\{u_n\}$ converges strongly to some $z_0 \in F(S) \cap \Omega(A, C)$. Hence $\langle z - z_0, z - z_0\rangle \leq 0$ which shows that $z = z_0$. This completes the proof. ∎

4. Applications

In this section we apply Theorem 3.1 to construct an iterative process to find a common element of the set of fixed points of a strict pseudo-contraction and the set of zeroes of a monotone and Lipschitz continuous mapping.

Theorem 4.1. *Let H be a real Hilbert space, $A : H \to H$ a monotone and k-Lipschitz continuous mapping and $S : H \to H$ a κ-strict pseudo-contraction for* *some* $0 \leq \kappa < 1$ *such that* $F(S) \cap A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n A x_n, \\ t_n = x_n - \lambda_n A y_n, \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) S t_n, \quad \text{for } n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$ *and* $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (\kappa, 1)$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to the same point* $z \in F(S) \cap A^{-1}0$, *where*

$$z = \lim_{n \to \infty} P_{F(S) \cap A^{-1}0} x_n.$$

Proof. This is exactly the case when $C = H$ in Theorem 3.1. Since $\Omega(A, H) = A^{-1}0$ and $P_H = I$, the desired result follows. ∎

Remark. Notice that $F(S) \cap A^{-1}0 \subset \Omega(A, F(S))$. See also [16, Yamada] for the case when $A : H \to H$ is a strongly monotone and Lipschitz continuous mapping and $S : H \to H$ is a nonexpansive mapping.

It is well known (see [11]) that if $A : H \to H$ is a maximal monotone mapping, then for each $u \in H$ and $\lambda > 0$ there is a unique $z \in H$ such that

$$u \in (I + \lambda A)(z).$$

The (single-valued) function $J_\lambda^A := (I + \lambda A)^{-1}$ thus defined is called the *resolvent* of A of parameter λ, and it is also known as the proximal mapping. The mapping $J_\lambda^A : H \to H$ is nonexpansive and $J_\lambda^A(z) = z$ if and only if $0 \in A(z)$.

Theorem 4.2. *Let* H *be a real Hilbert space,* $A : H \to H$ *a monotone and* k*-Lipschitz continuous mapping and* $B : H \to 2^H$ *a maximal monotone mapping such that* $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. *Let* J_r^B *be the resolvent of* B, *for each* $r > 0$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n A x_n, \\ t_n = x_n - \lambda_n A y_n, \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) J_r^B t_n, \quad \text{for } n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$ *and* $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$. *Then* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to the same point* $z \in A^{-1}0 \cap B^{-1}0$, *where*

$$z = \lim_{n \to \infty} P_{A^{-1}0 \cap B^{-1}0} x_n.$$

Proof. We have $\Omega(A, H) = A^{-1}0$, $F(J_r^B) = B^{-1}0$, $P_H = I$ and $\kappa = 0$ in Theorem 3.1. The desired result obtains. ■

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