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EXPLICIT NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF NONNEGATIVE SOLUTIONS OF A p-LAPLACIAN BLOW-UP PROBLEM

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Abstract. We establish explicit necessary and sufficient conditions for the existence of nonnegative solutions of the p-Laplacian boundary blow-up problem

$$\begin{cases} (\varphi_p(u'(x)))' = \lambda f(u(x)),\ 0 < x < 1, \\ \lim\limits_{x \to 0^+} u(x) = \infty = \lim\limits_{x \to 1^-} u(x), \end{cases}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2} y$ and $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, λ is a positive bifurcation parameter and f is a locally Lipschitz continuous function on $[0, \infty)$. The gap is extremely small between the explicit necessary condition and the explicit sufficient condition for the existence of nonnegative solutions. Our results improve and extend some main results of Anuradha, Brown and Shivaji [2] and of Wang [30] from $p = 2$ to any $p > 1$.

1. Introduction

In this paper we investigate the necessary and sufficient conditions for the existence of (classical) nonnegative solutions of the p-Laplacian boundary blow-up problem

$$\begin{cases} (\varphi_p(u'(x)))' = \lambda f(u(x)),\ 0 < x < 1, \\ \lim\limits_{x \to 0^+} u(x) = \infty = \lim\limits_{x \to 1^-} u(x), \end{cases} \tag{1.1}$$

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where $p > 1$, $\varphi_p(y) = |y|^{p-2}\, y$ and $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, λ is a positive bifurcation parameter and f is a locally Lipschitz continuous function on $[0, \infty)$.

From 1990s, many authors have extensively studied the problems of existence, uniqueness and asymptotic behavior of solutions of the p-Laplacian boundary blow-up problem

$$\begin{cases} \Delta_p u = f(u) & \text{in } \Omega, \\ u \to \infty & \text{as } x \to \partial\Omega, \end{cases} \tag{1.2}$$

where $p > 1$, Δ_p is the p-Laplacian div$(|\nabla u|^{p-2}\, \nabla u)$, and Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary. See, for examples, [1–32]. A problem of this type was first considered by Bieberbach [6] in 1916, where

$$p = 2,\ f(u) = e^u, \text{ and } N = 2.$$

Bieberbach showed that if Ω is a bounded domain in $\mathbb{R}^2$ such that $\partial\Omega$ is a C^2 submanifold of $\mathbb{R}^2$, then there exists a unique $u \in C^2(\Omega)$ such that $\Delta u(x) = e^u$ in Ω and $\left|u(x) - \ln(d(x))^{-2}\right|$ is bounded on Ω. Using the ideas of Bieberbach, Rademacher [28] extended this result to smooth bounded domains in $\mathbb{R}^3$. This problem plays an important role, when $N = 2$, in the theory of Riemann surfaces of constant negative curvature and in the theory of automorphic functions, and when $N = 3$, according to [28], in the study of the electric potential in a glowing hollow metal body.

Others pioneer contributions on existence of solutions of (1.2) are due to J. B. Keller [17] and R. Osserman [27]. In 1957, Keller [17] (cf. also Osserman [27]) first showed existence of positive solutions of (1.2) when $p = 2$ under the assumptions that f is locally Lipschitz continuous and *nondecreasing* on $[0, \infty)$, $f(0) = 0$ and

$$\int_\rho^\infty \frac{du}{F(u)^{1/2}} < \infty \quad \text{for all } \rho > 0, \tag{1.3}$$

where

$$F(s) := \int_0^s f(t)dt.$$

He obtained the result by the method of superpositions and subsolutions together with the uniform estimates of Keller [17, pp. 505–507]. He also showed that positive solutions of (1.2) when $p = 2$ exist if and only if there exists no entire positive solution of $\Delta u = f(u)$; i.e., no positive solution in the whole space. We point out that the nondecreasing nonlinearity f is called an absorption term; see Véron [29, p. 46]. Condition (1.3) is generally referred as the Keller-Osserman condition to (1.2) when $p = 2$; see e.g. [22, 25]. It is plausible that a boundary blow-up solution can only exist if $f(u)$ grows sufficiently fast at infinity.

For $f(u) = u^a$ with $a > 1$, problem (1.2) with $p = 2$ arises in the study of the high speed diffusion problem. For the special case where $f(u) = u^{(N+2)/(N-2)}$ and $N > 2$, which appears in geometrical problems, Loewner and Nirenberg [20] studied uniqueness and asymptotic behavior of positive solution of (1.2). Then, Bandle and Marcus [3, 4, 5] and Marcus and Véron [21] extended the results of Loewner and Nirenberg [20] to a much large class of nonlinearities including $f(u) = u^a$, $a > 1$. Díaz and Letelier [8] proved existence and uniqueness of positive solution of (1.2) for $f(u) = u^a$ with $a > p - 1$ and $p \neq 2$. Then, for (1.2), assuming that $f(u)$ satisfies the following generalized Keller-Osserman condition:

$$\Psi_p(\rho) := \int_\rho^\infty \frac{du}{F(u)^{1/p}} < \infty \text{ for all } \rho > 0, \tag{1.4}$$

Matero [23] extended the results of [17] to a much larger class of positive, *nondecreasing* nonlinearities f by some techniques originally due to Keller [17].

Problem (1.2) has also found new applications, for example, in understanding pattern formation for population models in environment. See [11, p. 740].

Cheng [7] studied the bifurcation curve $\lambda(\rho)$ with $\rho = \min_{x\in(0,1)} u(x)$ of (sign-changing and nonnegative) solutions of (1.1) mainly for

$$f = \tilde{f}(u) = \begin{cases} \varepsilon u^s + u^q, & u \geq 0, \\ \alpha |u|^r + \delta |u|^\tau, & u < 0, \end{cases}$$

satisfying

$$q > p - 1 > r > s > 0,\ \varepsilon, \alpha, \delta > 0, \text{ and } (\tau > p - 1 \text{ or } \tau = 0). \tag{1.5}$$

Hence he is able to prove the existence and multiplicity of (sign-changing and nonnegative) solutions of (1.1) for $f = \tilde{f}(u)$. We note that the continuous nonlinearity $\tilde{f}(u)$ satisfying (1.5) does *not* satisfy a Lipschitz condition of order $p - 1$ at 0.

When $p = 2$, assuming that nonlinearity f is a locally Lipschitz continuous function on $[0, \infty)$, Anuradha *et al.* [2] and Wang [30] studied necessary and sufficient conditions for the existence of nonnegative solutions of (1.1) basing on building a quadrature method.

Define, for any $p > 1$,

$$I_p = \{s \in [0, \infty) : f(s) > 0,\ F(u) > F(s) \text{ for all } u > s\}.$$

The next lemma on the quadrature method extends Anuradha *et al.* [2, Lemma 2.1] from $p = 2$ to any $p > 1$. Denote by $p' = p/(p - 1)$ the conjugate exponent of p.

Lemma 1.1. *Let $p > 1$. Suppose that f is a locally Lipschitz continuous function on $[0, \infty)$. Then, given $\lambda > 0$, there exists a unique (classical) nonnegative solution u to (1.1) with $\min_{x \in (0,1)} u(x) = \rho$ if and only if*

$$G(\rho) := 2\left(p'\right)^{-1/p} \int_{\rho}^{\infty} \frac{du}{(F(u) - F(\rho))^{1/p}} = \lambda^{1/p} < \infty \quad \textit{for } \rho \in I_p. \tag{1.6}$$

For $G(\rho)$ in (1.6), to make it more clear for the dependence on the nonlinearity f, we sometimes write $G_f(\rho)$ instead of $G(\rho)$.

Remark 1.2. Suppose that f is positive on $(0, \infty)$. Then

(i) Condition (1.6) $G(\rho) < \infty$ for all $\rho > 0$ implies condition (1.4) $\Psi_p(\rho) < \infty$ for all $\rho > 0$ since $\frac{1}{(F(u)-F(\rho))^{1/p}} > \frac{1}{F(u)^{1/p}}$ for $0 < \rho < u < \infty$.

(ii) Suppose that f is nondecreasing on $[0, \infty)$ and satisfies (2.4) stated below. Then it follows by [26, Theorem 1.1] that condition (1.4) $\Psi_p(\rho) < \infty$ for all $\rho > 0$ implies that $G(\rho)$ is a decreasing function on $(0, \infty)$. In addition,

$$\lim_{\rho \to 0^+} G(\rho) = 2\left(p'\right)^{-1/p} \int_0^{\infty} \frac{du}{F(u)^{1/p}}$$

under the *monotonicity* assumption on f on $[0, \infty)$ by applying the monotone convergence theorem.

2. Main Results

The main results in this paper are next Theorems 2.1 and 2.2 in which we establish *explicit* necessary and sufficient conditions for the existence of (classical) nonnegative solutions of (1.1) for $p > 1$. The gap is extremely small between the explicit necessary condition and the explicit sufficient condition for the existence of nonnegative solutions to (1.1). Note that Theorem 2.1 improves and extends Anuradha *et al.* [2, Theorem 3.1] and Wang [30, Theorem 2.1] from $p = 2$ to any $p > 1$. Theorem 2.2 improves and extends Anuradha *et al.* [2, Theorem 3.2] and Wang [30, Theorem 2.2] from $p = 2$ to any $p > 1$.

Let $e_0 = 1$. For $n \in \mathbb{N}$, let constants e_n denote the n-th iterate of $\exp(1)$ so that

$$\begin{cases} e_1 = \exp(1) \quad \text{for } n = 1, \\ e_n = \underbrace{\exp(\exp \cdots (\exp}_{n-\text{times}}(1))) \quad \text{for } n \geq 2. \end{cases}$$

Let

$$e_n^a = (e_n)^a \quad \text{for } a > 0.$$

Note that $\{e_n\}$ is a nonnegative strictly increasing sequence and $\lim_{n\to\infty} e_n = \infty$.

For $n \in \mathbb{N}$, let functions $L_n(u)$ denote the n-th iterate of $\ln u$ so that

$$\begin{cases} L_1 = L_1(u) = \ln u > 0 \quad \text{for } u > e_0 = 1,\ n = 1, \\ L_n = L_n(u) = \underbrace{\ln(\ln(\cdots(\ln}_{n\text{-times}} u))) > 0 \quad \text{for } u > e_{n-1},\ n \geq 2. \end{cases} \tag{2.1}$$

Let

$$L_n^a = (L_n(u))^a \quad \text{for } a > 0.$$

Note that, for $n \in \mathbb{N}$,

$$L_n = L_n(u) > 1 \quad \text{for } u > e_n. \tag{2.2}$$

Theorem 2.1. *Let $p > 1$. Suppose that f is a locally Lipschitz continuous function on $[0, \infty)$. If there exists any nonnegative solution to (1.1) for any $\lambda > 0$, then*

$$\limsup_{u\to\infty} \frac{f(u)}{u^{p-1} L_1^p L_2^p \cdots L_{n-1}^p L_n^p} = \infty \quad \textit{for any } n \in \mathbb{N}. \tag{2.3}$$

Theorem 2.2. *Let $p > 1$. Suppose that f is a locally Lipschitz continuous function on $[0, \infty)$. If f satisfies*

$$\begin{cases} \exists n \in \mathbb{N},\ a > p, \textit{ and } 0 < \beta \leq \infty \textit{ such that either} \\ \qquad n = 1,\ \liminf\limits_{u\to\infty} \dfrac{f(u)}{u^{p-1} L_1^a} = \beta, \\ \textit{or } n \geq 2,\ \liminf\limits_{u\to\infty} \dfrac{f(u)}{u^{p-1} L_1^p L_2^p \cdots L_{n-1}^p L_n^a} = \beta, \end{cases} \tag{2.4}$$

then there exist nonnegative solutions to (1.1) for some $\lambda > 0$ and $G(\rho)$ is well defined and continuous for all $\rho \in I_p$.

Remark 2. For any $p \geq 2$, condition (2.3) is necessary, but not sufficient for existence of nonnegative solutions. For example, let

$$f = f_1(u) := \begin{cases} u^{p-1} \quad \text{for } 0 \leq u < e_1, \\ u^{p-1}(\ln u)^p = u^{p-1} L_1^p \quad \text{for } e_1 \leq u < e_2, \\ u^{p-1} L_1^p L_2^p \cdots L_n^p \quad \text{for } e_n \leq u < e_{n+1},\ n \geq 2. \end{cases}$$

Then f_1 is a locally Lipschitz continuous function on $[0, \infty)$ and it satisfies (2.3). It can be proved that $G_{f_1}(\rho) = \infty$ for all $\rho \in (0, \infty)$, and hence (1.1) has no positive solution for any $\lambda > 0$. We omit the proof.

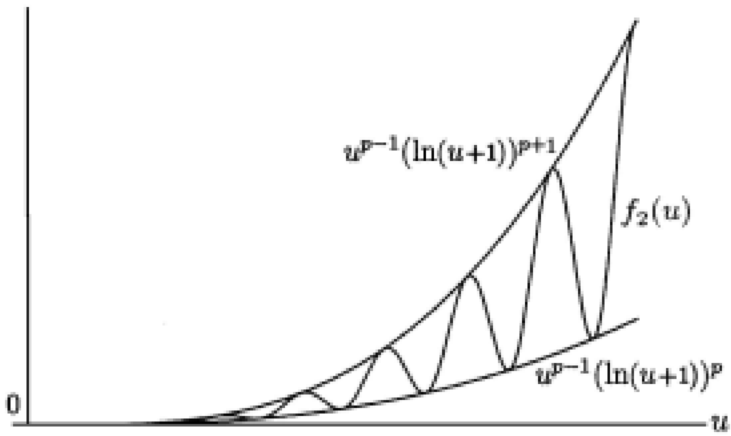

Fig. 1. Graph of function $f = f_2(u)$ for $u > 0$, $p \geq 2$. $u^{p-1}(\ln(u+1))^p \leq f_2(u) \leq u^{p-1}(\ln(u+1))^{p+1}$ for $u > 0$.

Remark 3. (See Fig. 1). For any $p \geq 2$, condition (2.4) is sufficient, but not necessary for existence of nonnegative solutions. For example, let

$$f = f_2(u) := \begin{cases} u^{p-1}(\ln(u+1))^p + \left[u^{p-1}(\ln(u+1))^{p+1} - u^{p-1}(\ln(u+1))^p\right] \\ \quad \cdot \dfrac{1+\sin u}{2} \quad \text{for } u > 0, \\ 0 \quad \text{for } u = 0. \end{cases}$$

Then f_2 is a locally Lipschitz continuous function on $[0, \infty)$ and it satisfies (2.3) but not (2.4). Function f_2 "oscillates" between functions $u^{p-1}(\ln(u+1))^p$ and $u^{p-1}(\ln(u+1))^{p+1}$ for $u > 1$. It can be proved that $G_{f_2}(\rho)$ exists for all $\rho > 1$ and $\lim_{\rho\to\infty} G_{f_2}(\rho) = 0$. Hence (1.1) has a positive solution for $\lambda > 0$ small enough. We omit the proof.

The next theorem improves and extends Anuradha *et al.* [2, Theorem 3.3] and Wang [30, Theorems 2.3] from $p = 2$ to any $p > 1$.

Theorem 2.3. *Let $p > 1$. Suppose that f is a locally Lipschitz continuous function on $[0, \infty)$. If f satisfies (2.4), then*

$$G(\rho) \to 0 \quad \textit{as } \rho \to \infty.$$

The next theorem improves and extends Anuradha *et al.* [2, Lemma 4.2] and Wang [30, Theorems 2.4] from $p = 2$ to any $p > 1$. Recall that, f is said to satisfy a *Lipschitz condition of order* $p-1$ at s if there exist constants $M > 0$, $\delta > 0$ such that

$$|f(u) - f(s)| < M\,|u-s|^{p-1} \quad \text{for } s-\delta < u < s+\delta,\ u \neq s. \tag{2.5}$$

When $p = 2$, f is said to be *locally Lipschitz continuous* at s if f satisfies a Lipschitz condition of order 1 at s.

Theorem 2.4. *Let $p > 1$. Suppose that f is a locally Lipschitz continuous function on $[0, \infty)$. Let $f(u)$ satisfy (2.4). Assume that there exits $s \in [0, \infty)$ such that $f(s) = 0$ and f satisfies a Lipschitz condition of order $p - 1$ at s. If there exits $\epsilon > 0$ such that $(s, s + \epsilon) \subset I$, then $G(\rho) \to \infty$ as $\rho \to s^+$. Furthermore, if there exits $\epsilon > 0$ such that $(s - \epsilon, s) \subset I$, then $G(\rho) \to \infty$ as $\rho \to s^-$.*

The next theorem improves and extends Wang [30, Theorems 2.5] from $p = 2$ to any $p > 1$.

Theorem 2.5. *Let $p > 1$. Suppose that f is a locally Lipschitz continuous function on $[0, \infty)$. If f satisfies (2.4) and $0 \in I_p$, then*

$$G(0) < \infty.$$

We finally give a remark to Theorem 2.4 and we may assume $s = 0$ without loss of generality. For p-Laplacian problem (1.1) with $p > 1$, suppose that f satisfies all assumptions in Theorem 2.4 except Lipschitz conditions of order $p-1$ and of order 1 at $s = 0$, it is possible that $\lim_{\rho\to 0^+} G(\rho)$ exists and is finite. Hence problem (1.1) may *not* admit (classical) nonnegative solutions for any $\lambda > 0$ large enough. We give the next explicit example as follows.

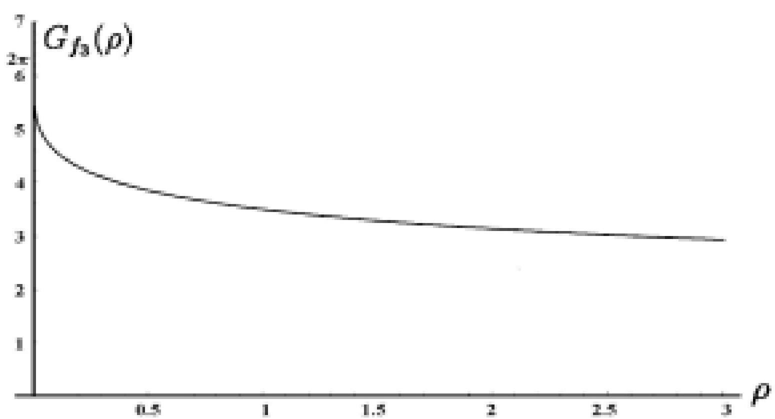

Fig. 2. $p = 2$. Numerical simulation of $G_{f_3}(\rho)$, $\rho > 0$. $G_{f_3}(0) := \lim_{\rho\to 0^+} G_{f_3}(\rho) = 2\pi \approx 6.283$ and $\lim_{\rho\to\infty} G_{f_3}(\rho) = 0$.

Example 2.6. (See Fig. 2) Let $p = 2$ and

$$f = f_3(u) := 3u^{1/2} + 8u + 5u^{3/2} \quad \text{for } u \geq 0.$$

It is obvious that $f_3(0) = 0$, $f(u) > 0$ on $(0, \infty)$, and f_3 is locally Lipschitz continuous at all points on $[0, \infty)$ except at 0. In addition, f_3 satisfies (2.4) with $n = 1$, $a = 3$ and $\beta = \infty$. So $G_{f_3}(\rho)$ exists and is continuous for *all* $\rho \in I_2 = (0, \infty)$. It is interesting to notice that, for this particular nonlinearity $f = f_3(u)$, it can be checked that the trigonometric function

$$u = \cot^4(\pi x),\ 0 < x < 1$$

satisfying $\rho = \min_{x\in(0,1)} u(x) = u(1/2) = 0$ is a nonnegative solution of (1.1) corresponding to $\lambda = 4\pi^2$ (= $\left(\lim_{\rho\to 0^+} G_{f_3}(\rho)\right)^2$). In addition, it can be proved that

(i) $\lim\limits_{\rho\to 0^+} G_{f_3}(\rho) = 2\pi \approx 6.283$. (We omit the proof.)

(ii) $\lim\limits_{\rho\to\infty} G_{f_3}(\rho) = 0$ by [2, Theorem 3.3].

(iii) $G_{f_3}(\rho)$ is a strictly decreasing function of $\rho > 0$ by [2, Theorem 3.4] since f_3 is strictly decreasing on $(0, \infty)$.

Thus (1.1) has exactly one positive solution for $0 < \lambda < 4\pi^2$, exactly one nonnegative solution $u = \cot^4(\pi x)$ with $\min_{x\in(0,1)} u(x) = u(1/2) = 0$ for $\lambda = 4\pi^2$, and no nonnegative solution for $\lambda > 4\pi^2$.

3. Lemmas

The next Lemmas 3.1-3.3 are needed in the proofs of Theorems 2.1-2.4. For $n \in \mathbb{N}$, $a \geq p$, and L_n in (2.1) for $u > e_{n-1}$. For convenience of notations, we define

$$A_{p,a}(u,n) = \begin{cases} u^{p-1}L_1^a = u^{p-1}(\ln u)^a > 0 & \text{if } n = 1, \\ u^{p-1}L_1^pL_2^p\cdots L_{n-1}^pL_n^a > 0 & \text{if } n \geq 2, \end{cases} \tag{3.1}$$

$$B_{p,a}(u,n) = \frac{d}{du}\left(uA_{p,a}(u,n)\right), \tag{3.2}$$

$$C_p(u,n) = u^pL_1^pL_2^p\cdots L_n^p, \tag{3.3}$$

$$D_p(u,n) = \frac{d}{du}C_p(u,n) = \frac{d}{du}(u^pL_1^pL_2^p\cdots L_n^p). \tag{3.4}$$

Lemma 3.1. *Let $p > 1$. For $n \in \mathbb{N}$ and $a \geq p$,*

$$(i) \quad D_p(u,n) \ \left(= \frac{d}{du}C_p(u,n) = \frac{d}{du}(u^p L_1^p L_2^p \cdots L_n^p)\right)$$
$$= pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_n^{p-1}(1 + L_n + L_{n-1}L_n + \ \cdots + L_1L_2\cdots L_{n-1}L_n).$$

$$(ii) \quad B_{p,a}(u,n) \ \left(= \frac{d}{du}\left(uA_{p,a}(u,n)\right)\right)$$
$$= \begin{cases} pu^{p-1}L_1^{a-1}(\frac{a}{p} + L_1) \ \ \textit{if } n = 1, \\ pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^{a-1} \\ \quad \cdot \left(\frac{a}{p} + L_n + L_{n-1}L_n + \cdots + L_1L_2\cdots L_{n-1}L_n\right) \ \ \textit{if } n \geq 2. \end{cases}$$

Lemma 3.2. *Let* $p > 1$. *For* $n \in \mathbb{N}$ *and* $a \geq p$,

(i) $pA_{p,p}(u,n) < B_{p,p}(u,n)$ *for* $u > e_{n-1}$.

(ii) $(p+1)A_{p,a}(u,n) > B_{p,a}(u,n)$ *for* $u > \max\{\exp(a + p(n-1)), e_n\}$.

Lemma 3.3. *Let* $p > 1$. *Let* f *satisfy (2.4) and* $\rho \in [\rho_1, \rho_2] \subset I_p$. *Then for any* $n \in \mathbb{N}$, $a > p$, *there exists a constant* $M > \max\{\exp(a + p(n-1)), e_n\}$ *such that* $L_n(M) > 1$ *and*

$$F(u) - F(\rho) \geq \gamma u A_{p,a}(u,n) > 0 \ \ \textit{for } u > M,$$

where

$$\gamma = \begin{cases} \dfrac{\beta}{4(p+1)} & \textit{if } 0 < \beta < \infty, \\ \dfrac{1}{2(p+1)} & \textit{if } \beta = \infty. \end{cases} \tag{3.5}$$

We now prove Lemmas 3.1-3.3 as follows.

Proof of Lemma 3.1. We first prove Lemma 3.1(i) by mathematical induction on n. First, it is trivial that Lemma 3.1(i) holds when $n = 1$. Assume that, when $n = k$, Lemma 3.1(i) holds. Then when $n = k+1$, by (3.3) and (3.4), we obtain

$$\begin{aligned} & D_p(u,k+1) \\ = & \frac{d}{du}C_p(u,k+1) \\ = & \frac{d}{du}\left(C_p(u,k)L_{k+1}^p\right) \\ = & \left(\frac{d}{du}C_p(u,k)\right)L_{k+1}^p + C_p(u,k)\left(\frac{d}{du}L_{k+1}^p\right) \end{aligned}$$

$$
\begin{aligned}
&= D_p(u,k)L_{k+1}^p + C_p(u,k)\cdot p(uL_1L_2\cdots L_k)^{-1}L_{k+1}^{p-1} \\
&= pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_k^{p-1}(1+L_k+L_{k-1}L_k+\cdots+L_1L_2\cdots L_{k-1}L_k)L_{k+1}^p \\
&\quad +pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_k^{p-1}L_{k+1}^{p-1} \\
&\qquad \text{(by the assumption that Lemma 3.1(i) holds for } n=k\text{)} \\
&= pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_k^{p-1}L_{k+1}^{p-1}(1+L_{k+1}+L_kL_{k+1}+\cdots+L_1L_2\cdots L_kL_{k+1}).
\end{aligned}
$$

So Lemma 3.1(i) holds for $n = k+1$. Thus by mathematical induction, Lemma 3.1(i) holds for all $n \in \mathbb{N}$.

We then prove Lemma 3.1(ii). First when $n = 1$, it is trivial that the Lemma 3.1(ii) holds. Secondly, when $n \geq 2$, by (3.2)-(3.4) and Lemma 3.1(i), we obtain

$$
\begin{aligned}
&B_{p,a}(u,n) \\
&= \frac{d}{du}\left(C_p(u,n-1)L_n^a\right) \\
&= \left(\frac{d}{du}C_p(u,n-1)\right)L_n^a + C_p(u,n-1)\left(\frac{d}{du}L_n^a\right) \\
&= D_p(u,n-1)L_n^a + C_p(u,n-1)\cdot a(uL_1L_2\cdots L_{n-1})^{-1}L_n^{a-1} \\
&= pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^a(1+L_{n-1}+L_{n-2}L_{n-1}+\cdots \\
&\quad +L_1L_2\cdots L_{n-2}L_{n-1}) + au^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^{a-1} \\
&= pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^{a-1}\left(\frac{a}{p}+L_n+L_{n-1}L_n+\cdots+L_1L_2\cdots L_{n-1}L_n\right).
\end{aligned}
$$

So Lemma 3.1(ii) holds for $n \geq 2$. We conclude that Lemma 3.1(ii) holds for all $n \in \mathbb{N}$. ■

Proof of Lemma 3.2. We first prove Lemma 3.2(i). By Lemma 3.1(ii) and (3.1), when $n = 1$,

$$
\begin{aligned}
B_{p,p}(u,1) - pA_{p,p}(u,1) &= pu^{p-1}L_1^{p-1}(1+L_1) - pu^{p-1}L_1^p \\
&= pu^{p-1}L_1^{p-1} \\
&= pu^{p-1}(\ln u)^{p-1} \\
&> 0 \quad \text{for } u > e_0 = 1,
\end{aligned}
$$

and when $n \geq 2$, by Lemma 3.1(ii) and (3.1), it can be computed that

$$
\begin{aligned}
&B_{p,p}(u,n) - pA_{p,p}(u,n) \\
&= pu^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^{p-1}(1+L_n+L_{n-1}L_n+L_{n-2}L_{n-1}L_n \\
&\quad +\cdots+L_2L_3\cdots L_n) > 0 \quad \text{for } u > e_{n-1}.
\end{aligned}
$$

So Lemma 3.2(i) follows.

We then prove Lemma 3.2(ii). By (3.1), (3.2) and Lemma 3.1(ii), when $n = 1$,

$$\begin{aligned}(p+1)A_{p,a}(u,1) - B_{p,a}(u,1) &= (p+1)u^{p-1}L_1^a - pu^{p-1}L_1^{a-1}\left(\frac{a}{p}+L_1\right) \\ &= u^{p-1}L_1^{a-1}(L_1 - a) \\ &= u^{p-1}(\ln u)^{a-1}((\ln u) - a) \\ &> 0 \quad \text{for } u > \exp(a),\end{aligned} \tag{3.6}$$

and when $n \geq 2$, for $u > e_n$, it can be computed that

$$\begin{aligned}&(p+1)A_{p,a}(u,n) - B_{p,a}(u,n) \\ = \ & u^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^{a-1}(L_1L_2\cdots L_{n-1}L_n - a \\ &-pL_n - pL_{n-1}L_n - pL_{n-2}L_{n-1}L_n - \cdots - pL_2\cdots L_{n-1}L_n) \\ > \ & u^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^{a-1}(L_1L_2\cdots L_{n-1}L_n - aL_2\cdots L_{n-1}L_n \\ &\underbrace{-pL_2\cdots L_{n-1}L_n - pL_2\cdots L_{n-1}L_n - \cdots - pL_2\cdots L_{n-1}L_n}_{(n-1)-\text{times}}) \text{ (by (2.2))} \\ = \ & u^{p-1}L_1^{p-1}L_2^{p-1}\cdots L_{n-1}^{p-1}L_n^{a-1}(L_1L_2\cdots L_{n-1}L_n - (a+p(n-1))L_2\cdots L_{n-1}L_n) \\ = \ & u^{p-1}L_1^{p-1}L_2^{p}\cdots L_{n-1}^{p}L_n^{a}(L_1 - (a+p(n-1))).\end{aligned}$$

Since

$$L_1 - (a+p(n-1)) = \ln u - (a+p(n-1)) > 0 \quad \text{for } u > \exp(a+p(n-1)),$$

we conclude that, for $n \geq 2$,

$$(p+1)A_a(u,n) - B_a(u,n) > 0 \quad \text{for } u > \max\{\exp(a+p(n-1)), e_n\}. \tag{3.7}$$

So Lemma 3.2(ii) follows immediately from (3.6) and (3.7). ∎

Proof of Lemma 3.3. Suppose that f satisfies (2.4), by (2.2), (3.1) and Lemma 3.2(ii), it is easy to see that there exists a constant $M_1 > \max\{\exp(a+p(n-1)), e_n\}$ such that $L_n(M_1) > 1$, $A_{p,a}(M_1, n) > 1$, and

$$f(u) > 2(p+1)\gamma A_{p,a}(u,n) > 2\gamma B_{p,a}(u,n) \quad \text{for } u > M_1, \tag{3.8}$$

where γ is defined in (3.5). Then for $u > M_1$,

$$\begin{aligned} F(u) &= F(M_1) + \int_{M_1}^{u} f(t)dt \\ &\geq F(M_1) + \int_{M_1}^{u} 2\gamma B_{p,a}(t,n)dt \\ &= F(M_1) + 2\gamma \left[uA_{p,a}(u,n) - M_1 A_{p,a}(M_1,n) \right] \end{aligned}$$

by (3.2). Let $\rho \in [\rho_1, \rho_2] \subset I_p$, then

$$F(u) - F(\rho) \geq F(M_1) - F(\rho) + 2\gamma \left[uA_{p,a}(u,n) - M_1 A_{p,a}(M_1,n) \right].$$

Let $K = F(M_1) - 2\gamma M_1 A_{p,a}(M_1,n) - \sup_{\rho\in[\rho_1,\rho_2]} F(\rho)$. We obtain

$$F(u) - F(\rho) \geq K + 2\gamma u A_{p,a}(u,n) \quad \text{for } u > M_1. \tag{3.9}$$

Now there exists $M_2 > 0$ such that

$$\gamma u A_{p,a}(u,n) \geq -K \quad \text{for } u > M_2,$$

which implies

$$K + 2\gamma u A_{p,a}(u,n) \geq \gamma u A_{p,a}(u,n) > 0 \quad \text{for } u > M_2. \tag{3.10}$$

Letting $M = \max\{M_1, M_2\}$, by (3.9) and (3.10), we obtain

$$F(u) - F(\rho) \geq \gamma u A_{p,a}(u,n) > 0 \quad \text{for } u > M.$$

This completes the proof of Lemma 3.3. ∎

4. Proofs of Main Results

The proofs of Theorems 2.1-2.3 are based upon modification of methods of [2, Theorems 3.1-3.3] and of [30, Theorems 2.1-2.3].

Proof of Theorem 2.1. We prove Theorem 2.1 by method of contradiction. Assume that $\limsup_{u\to\infty} f(u)/A_{p,p}(u,n) \neq \infty$ for some $n \in \mathbb{N}$. Then there exist constants $K > 0$, $M_1 > e_{n-1}$ such that, for $u \geq M_1$,

$$\begin{aligned} f(u) &\leq K A_{p,p}(u,n) \\ &< \frac{K}{p} B_{p,p}(u,n) \end{aligned}$$

by Lemma 3.2(i). This and (3.2) imply that

$$\begin{aligned}
F(u) &= \int_0^u f(t)dt = \int_0^{M_1} f(t)dt + \int_{M_1}^u f(t)dt \\
&< F(M_1) + \int_{M_1}^u \frac{K}{p} B_{p,p}(t,n)dt \\
&= F(M_1) + \frac{K}{p} uA_{p,p}(u,n) - \frac{K}{p} M_1 A_{p,p}(M_1,n) \quad \text{for } u \geq M_1.
\end{aligned}$$

Let $\rho \in I_p$ and $K_1 = F(M_1) - F(\rho) - \frac{K}{p} M_1 A_{p,p}(M_1,n)$, then

$$\begin{aligned}
F(u) - F(\rho) &< F(M_1) - F(\rho) + \frac{K}{p} uA_{p,p}(u,n) - \frac{K}{p} M_1 A_{p,p}(M_1,n) \\
&= K_1 + \frac{K}{p} uA_{p,p}(u,n) \quad \text{for } u \geq M_1.
\end{aligned} \tag{4.1}$$

It is easy to see that there exists $M_2 > 0$ such that

$$\frac{(p-1)K}{p} uA_{p,p}(u,n) = \frac{(p-1)K}{p} u^p L_1^p L_2^p \cdots L_{n-1}^p L_n^p > K_1 \quad \text{for } u \geq M_1. \tag{4.2}$$

Let $M = \max\{M_1, M_2\}$, then (4.1) and (4.2) imply

$$F(u) - F(\rho) < KuA_{p,p}(u,n) \quad \text{for } u \geq M. \tag{4.3}$$

Without loss of generality, we may assume $M > \rho$, and we obtain from (1.6), (4.3) and (3.1) that

$$\begin{aligned}
G(\rho) &= 2\left(p'\right)^{-1/p} \int_\rho^\infty \frac{du}{(F(u) - F(\rho))^{1/p}} \\
&\geq 2\left(p'\right)^{-1/p} \int_M^\infty \frac{du}{(F(u) - F(\rho))^{1/p}} \\
&\geq 2\left(p'\right)^{-1/p} \int_M^\infty \frac{du}{(KuA_{p,p}(u,n))^{1/p}} \\
&= 2\left(p'K\right)^{-1/p} \int_M^\infty \frac{du}{uL_1 L_2 \cdots L_{n-1} L_n} \\
&= 2\left(p'K\right)^{-1/p} L_{n+1}(u)\big|_{u=M}^{u=\infty} \\
&= \infty.
\end{aligned}$$

Thus $G(\rho)$ does not exist if $\limsup_{u\to\infty} f(u)/A_{p,p}(u,n) \neq \infty$ for some $n \in \mathbb{N}$, and Theorem 2.1 follows from Lemma 1.1. ■

Proof of Theorem 2.2. First, we suppose $\rho \in I_p$. Since I_p is open, there exist $\rho_1, \rho_2 \in I_p$ such that $\rho \in (\rho_1, \rho_2) \subset I_p$. Suppose that f satisfies (2.4), by Lemma 3.3, then there exists a constant $M > \max\{\exp(a + p(n-1)), e_n, \rho_2\}$ (we assume without loss of generality that $M > \rho_2$) such that $L_n(M) > 1$ and

$$F(u) - F(\rho) \geq \gamma u A_{p,a}(u,n) > 0 \quad \text{for } u > M,\ \rho \in [\rho_1, \rho_2], \tag{4.4}$$

where γ is defined in (3.5). Note that

$$G(\rho) = 2\left(p'\right)^{-1/p} \int_\rho^\infty \frac{du}{(F(u) - F(\rho))^{1/p}} < \infty$$

if and only if there exists $\tilde{\delta} \in (0, \rho_2 - \rho)$ such that

$$\int_\rho^{\rho+\tilde{\delta}} \frac{du}{(F(u) - F(\rho))^{1/p}} < \infty \text{ and } \int_M^\infty \frac{du}{(F(u) - F(\rho))^{1/p}} < \infty.$$

Let $\alpha := \min_{z \in [\rho, \rho+\tilde{\delta}]} f(z) > 0$. For $u \in \left[\rho, \rho + \tilde{\delta}\right] \subset I_p$, by the mean value theorem, there exists $z \in [\rho, u] \subset \left[\rho, \rho + \tilde{\delta}\right]$ such that

$$F(u) - F(\rho) = f(z)(u - \rho) \geq \left[\min_{z \in [\rho, \rho+\tilde{\delta}]} f(z)\right](u - \rho) = \alpha(u - \rho).$$

Thus

$$\int_\rho^{\rho+\tilde{\delta}} \frac{du}{(F(u) - F(\rho))^{1/p}} \leq \alpha^{-1/p} \int_\rho^{\rho+\tilde{\delta}} \frac{du}{(u-\rho)^{1/p}} = \frac{p}{p-1} \alpha^{-1/p} \tilde{\delta}^{(p-1)/p} < \infty. \tag{4.5}$$

In addition, by (4.4), (3.1) and (2.1), if f satisfies (2.4) with $n = 1$ and $a > p$, then

$$\begin{aligned} \int_M^\infty \frac{du}{(F(u) - F(\rho))^{1/p}} &\leq \gamma^{-1/p} \int_M^\infty \frac{du}{(u A_{p,a}(u,1))^{1/p}} \\ &= \gamma^{-1/p} \int_M^\infty \frac{du}{u L_1^{a/p}} \\ &= \gamma^{-1/p} \int_M^\infty \frac{du}{u (\ln u)^{a/p}} \\ &= \gamma^{-1/p} \frac{p}{p-a} \left. (\ln u)^{(p-a)/p} \right|_{u=M}^{u=\infty} \\ &= \gamma^{-1/p} \frac{p}{a-p} (\ln M)^{(p-a)/p} \\ &< \infty; \end{aligned} \tag{4.6}$$

if f satisfies (2.4) with $n \geq 2$ and $a > p$, then

$$\begin{aligned}\int_M^\infty \frac{du}{(F(u)-F(\rho))^{1/p}} &\leq \gamma^{-1/p}\int_M^\infty \frac{du}{(uA_{p,a}(u,n))^{1/p}} \\ &= \gamma^{-1/p}\int_M^\infty \frac{du}{uL_1L_2\cdots L_{n-1}L_n^{a/p}} \\ &= \gamma^{-1/p}\int_{u=M}^{u=\infty} \frac{dL_n}{L_n^{a/p}} \\ &= \gamma^{-1/p}\frac{p}{p-a}\left(L_n(u)\right)^{(p-a)/p}\Big|_{u=M}^{u=\infty} \\ &= \gamma^{-1/p}\frac{p}{a-p}\left(L_n(M)\right)^{(p-a)/p} \\ &< \infty.\end{aligned} \tag{4.7}$$

By (4.5)-(4.7), it follows immediately that $G(\rho) < \infty$ for $\rho \in I_p$. Hence $G(\rho)$ is well defined for all $\rho \in I_p$, and by Lemma 1.1, there exists a nonnegative solution to (1.1) for some $\lambda = (G(\rho))^p$ given by any $\rho \in I_p$. By using (4.4), the arguments in the proof of [2, Theorem 3.2] can be modified to prove that $G(\rho)$ is continuous for all $\rho \in I_p$.

The proof of Theorem 2.2 is now complete. ■

The proof of Theorem 2.3 follows by slight modification of the proof of [30, Theorem 2.3] and by Lemma 3.3 and (3.8). We omit the proof.

The proof of Theorem 2.4 follows by slight modification of the proof of [2, Lemma 4.2] and by (2.5). We omit the proof.

The proof of Theorem 2.5 follows by slight modification of the proof of [30, Theorem 2.5] and by Lemma 3.3. We omit the proof.

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