

## RANDOM COINCIDENCE POINTS AND RANDOM FIXED POINTS OF MULTIFUNCTIONS IN METRIC SPACES

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**Abstract.** In this paper, we present some new random coincidence point and random fixed point theorems for multifunctions in separable complete metric spaces, which improve some existing results in the literature (even some results in the non-random case).

### 1. INTRODUCTION AND PRELIMINARIES

Random operator theory has received much attention in recent years because of its applications to random differential equations, random integral equations, random approximations, etc. (see, e.g., [2-5, 8] and references therein). In this article, we will give some new random coincidence point and random fixed point theorems for multifunctions in separable complete metric spaces. Even in the non-random case, our results also improve some known results. Moreover, we correct some errors in the proof of a related result in [5].

Throughout this paper,  $(X, d)$  is a separable complete metric space,  $2^X$  stands for the family of all subsets of  $X$ ,  $\mathbb{R}^+ = [0, +\infty)$ .  $(\Omega, \Sigma)$  denotes a measurable space with  $\Sigma$  a sigma-algebra of subsets of  $\Omega$ , Let  $CB(X)$  and  $CC(X)$  be the families of all nonempty bounded closed subsets and all nonempty compact subsets of  $X$ , respectively. For any non-empty subsets  $A$  and  $B$  of  $X$ , we denote

$$\begin{aligned}d(x, A) &:= \inf\{d(x, a) : a \in A\} \quad (x \in X), \\d(A, B) &:= \inf\{d(a, b) : a \in A, b \in B\}, \\H(A, B) &:= \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\},\end{aligned}$$

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where  $H(., .)$  is called the Hausdorff metric on  $CB(X)$ .

A mapping  $\mu : \Omega \rightarrow 2^X$  is called to be measurable if for any open subset  $C$  of  $X$ ,

$$\mu^{-1}(C) := \{\omega \in \Omega : \mu(\omega) \cap C \neq \emptyset\} \in \Sigma.$$

A mapping  $\xi : \Omega \rightarrow X$  is said to be measurable selector of a measurable mapping  $\mu : \Omega \rightarrow 2^X$  if  $\xi$  is measurable and for any  $\omega \in \Omega$ ,  $\xi(\omega) \in \mu(\omega)$ . A function  $f : \Omega \times X \rightarrow R$  is a Carathéodory function if  $f$  is measurable in  $\omega \in \Omega$  and is continuous in  $x \in X$ .

A mapping  $T : \Omega \times X \rightarrow CB(X)$  is called a multifunction if for every  $x \in X$ ,  $T(., x)$  is measurable. A mapping  $G : X \rightarrow CB(X)$  is said to be continuous on  $X$  (with respect to the Hausdorff metric  $H$ ) if  $H(Gx_n, Gx) \rightarrow 0$  whenever  $x_n \rightarrow x$ . A measurable mapping  $\xi : \Omega \rightarrow X$  is called a random fixed point of a multifunction  $T : \Omega \times X \rightarrow CB(X)$  if for every  $\omega \in \Omega$ ,  $\xi(\omega) \in T(\omega, \xi(\omega))$ . A measurable mapping  $\xi : \Omega \rightarrow X$  is called a random coincidence point of  $S, T : \Omega \times X \rightarrow CB(X)$  if for every  $\omega \in \Omega$ ,  $S(\omega, \xi(\omega)) \cap T(\omega, \xi(\omega)) \neq \emptyset$ .

**Lemma 1.1.** ([1]). *If  $T : \Omega \rightarrow 2^X$  is a measurable closed-valued operator, then  $T$  has a measurable selector.*

**Lemma 1.2.** ([7]). *Let  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be an increasing function such that*

$$\Phi(t+) < t, \text{ for all } t > 0 \tag{1.1}$$

and

$$\sum \Phi^n(t) \text{ is finite, for all } t > 0. \tag{1.2}$$

Then there exists a strictly increasing function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that

$$\Phi(t) < \phi(t), \text{ for all } t > 0 \tag{1.3}$$

and

$$\sum \phi^n(t) \text{ is finite, for all } t > 0. \tag{1.4}$$

**Lemma 1.3.** ([7]).

- (i) *If  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is strictly increasing and satisfies (1.2), then  $\Phi$  satisfies (1.1).*
- (ii) *Let  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be increasing and satisfy (1.1). If  $\sum \Phi^n(t_1)$  is convergent for some  $t_1 > 0$ , then (1.2) holds.*
- (iii) *Let  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be increasing and satisfy (1.1). If  $t \leq \Phi(t)$ , then  $t = 0$ .*

**Lemma 1.4.** (Castaing's Characteristic Theorem; cf. [1]). *If  $f : \Omega \rightarrow 2^X \setminus \{\emptyset\}$  is a closed-valued mapping, then the following conditions are equivalent:*

- (i)  $f$  is measurable.
- (ii) For each  $x \in X$ , the function  $\omega \rightarrow d(x, f(\omega))$  is measurable.
- (iii) There exists a sequence  $\{f_n(\omega)\}$  of measurable selectors of  $f$  such that

$$\text{cl}\{f_n(\omega) : n = 1, 2, \dots\} = f(\omega), \quad \text{for all } \omega \in \Omega,$$

where  $\text{cl}\{f_n(\omega) : n = 1, 2, \dots\}$  denotes the closure of  $\{f_n(\omega) : n = 1, 2, \dots\}$  in  $X$ .

## 2. MAIN RESULTS

**Lemma 2.1.** Let  $\xi : \Omega \rightarrow X$  be a measurable mapping. Suppose  $T : \Omega \times X \rightarrow CB(X)$  be a multifunction such that  $T(\omega, \cdot)$  is continuous for each  $\omega \in \Omega$ . Then

- (i) the mapping  $\omega \rightarrow T(\omega, \xi(\omega))$  is measurable;
- (ii) the mapping  $\omega \rightarrow d(\xi(\omega), T(\omega, y))$  is real-valued measurable for each  $y \in X$ .

*Proof.*

- (i) By the definition of a multifunction (see section 1), the mapping  $\omega \rightarrow T(\omega, x)$  is measurable for each  $x \in X$ . It follows from Lemma 1.4 that the mapping  $\omega \rightarrow d(v, T(\omega, x))$  is measurable for each  $v, x \in X$ . By assumption, it follows that the mapping  $x \rightarrow d(v, T(\omega, x))$  is continuous for each  $v \in X, \omega \in \Omega$ . Thus the mapping  $(\omega, x) \rightarrow d(v, T(\omega, x))$  is a Carathéodory function for each  $v \in X$ . As a result, it is jointly measurable [1, pp.313]. In view of the measurability of  $\xi$ , the mapping  $\omega \rightarrow d(v, T(\omega, \xi(\omega)))$  is measurable. By Lemma 1.4, we have  $\omega \rightarrow T(\omega, \xi(\omega))$  is measurable.
- (ii) By a similar argument as in (i), we deduce that the mapping  $(\omega, x) \rightarrow d(x, T(\omega, y))$  is also jointly measurable for each  $y \in X$ . Thus the mapping  $\omega \rightarrow d(\xi(\omega), T(\omega, y))$  is real-valued measurable for each  $y \in X$ .

**Lemma 2.2.** Let  $s : \Omega \rightarrow X$  be a measurable mapping,  $F : \Omega \times X \rightarrow CB(X)$  be a multifunction such that

- (1)  $F(\omega, \cdot)$  is continuous for all  $\omega \in \Omega$ ;
- (2) for any  $\omega \in \Omega, s(\omega) \in F(\omega, X)$  and  $F(\omega, M)$  is closed for any bounded closed subset  $M$  in  $X$ .

Then there exists a measurable mapping  $\xi : \Omega \rightarrow X$  such that  $s(\omega) \in F(\omega, \xi(\omega))$  for any  $\omega \in \Omega$ .

*Proof.* We denote  $G(\omega) := \{x \in X : s(\omega) \in F(\omega, x)\}$  for any  $\omega \in \Omega$ . Then for any  $\omega \in \Omega, s(\omega) \in F(\omega, G(\omega))$  and  $G(\omega)$  is nonempty by assumption. We first prove the mapping  $G : \Omega \rightarrow 2^X$  is measurable.

(a) For any nonempty bounded closed subset  $M$  of  $X$ , let  $G^{-1}(M) := \{\omega : G(\omega) \cap M \neq \emptyset\}$ . Then  $G^{-1}(M) = \{\omega : \exists x \in M, x \in G(\omega)\} = \{\omega : \exists x \in M, s(\omega) \in F(\omega, x)\}$ . Let

$$L(M) := \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \{\omega \in \Omega : d(s(\omega), F(\omega, x_i)) < \frac{1}{n}\},$$

where  $\{x_i\}$  is a countable dense subset of  $M$ .

If  $\omega \in G^{-1}(M)$ , then there exists a point  $x_0 \in M$  such that  $s(\omega) \in F(\omega, x_0)$ . Since  $F(\omega, \cdot) : M \rightarrow CB(X)$  is continuous and  $\{x_i\}$  is a countable dense subset of  $M$ , for any positive integer  $n$ , there exists  $x_{i(n)} \in X$  such that  $H(F(\omega, x_0), F(\omega, x_{i(n)})) < \frac{1}{n}$ . We have

$$\begin{aligned} d(s(\omega), F(\omega, x_{i(n)})) &\leq d(s(\omega), F(\omega, x_0)) + H(F(\omega, x_0), F(\omega, x_{i(n)})) \\ &\leq H(F(\omega, x_0), F(\omega, x_{i(n)})) \\ &< \frac{1}{n}, \end{aligned}$$

so  $w \in L(M)$ .

If  $\omega \in L(M)$ , then for any positive integer  $n$ , there exists  $x_{i(n)} \in M$  such that  $d(s(\omega), F(\omega, x_{i(n)})) < \frac{1}{n}$ . It follows that  $d(s(\omega), F(\omega, M)) = 0$ . Since  $F(\omega, M)$  is closed, we see  $s(\omega) \in F(\omega, M)$  and  $\omega \in G^{-1}(M)$ .

Thus,  $G^{-1}(M) = L(M)$ . By Lemma 2.1, the mapping  $\omega \rightarrow d(s(\omega), F(\omega, x))$  is measurable for each  $x \in X$ . Thus  $L(M)$  is measurable, and so is  $G^{-1}(M)$ .

(b) For any nonempty unbounded closed subset  $M$  of  $X$ , there exists a sequence  $\{M_j\}$  of nonempty bounded closed sets in  $X$  such that  $\bigcup_{j=1}^{\infty} M_j = M$ . Because  $G^{-1}(M) = \bigcup_{j=1}^{\infty} G^{-1}(M_j)$ ,  $G^{-1}(M)$  is a measurable subset in  $\Omega$ .

It follows from (a) and (b) that  $G$  is measurable. Now we prove  $G(\omega)$  is closed in  $X$  for each  $\omega \in \Omega$ . In fact, if  $\{x_j\} \subset G(\omega)$  converges to some point  $y$  in  $X$ , then we have  $s(\omega) \in F(\omega, x_j)$  and

$$\begin{aligned} d(s(\omega), F(\omega, y)) &\leq d(s(\omega), F(\omega, x_j)) + H(F(\omega, y), F(\omega, x_j)) \\ &\leq 0 + H(F(\omega, y), F(\omega, x_j)). \end{aligned}$$

Since  $x \rightarrow F(\omega, x)$  is continuous for each  $\omega \in \Omega$ , we have  $d(s(\omega), F(\omega, y)) = 0$ . Thus  $s(\omega) \in F(\omega, y)$  and  $G(\omega)$  is closed in  $X$ .

By Lemma 1.1, there exists a measurable selector  $\xi : \Omega \rightarrow X$  of  $G$  such that  $s(\omega) \in F(\omega, \xi(\omega))$  for any  $\omega \in \Omega$ . This completes the proof.

Let  $S, T$  and  $F : \Omega \times X \rightarrow CB(X)$  be multifunctions such that

$$\begin{aligned} & H(S(\omega, x), T(\omega, y)) \\ & \leq \Phi(\max\{d(F(\omega, x), F(\omega, y)), d(F(\omega, x), S(\omega, x)), \\ & \quad d(F(\omega, y), T(\omega, y)), [d(F(\omega, x), T(\omega, y)) + d(F(\omega, y), S(\omega, x))]/2\}), \end{aligned} \tag{2.1}$$

for all  $x, y \in X$  and for all  $\omega \in \Omega$ , where  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing function satisfying conditions (1.1) and (1.2).

**Theorem 2.3.** *Let  $S, T$  and  $F : \Omega \times X \rightarrow CB(X)$  be multifunctions such that*

- (1)  $S(\omega, \cdot), T(\omega, \cdot), F(\omega, \cdot)$  are all continuous for all  $\omega \in \Omega$ ;
- (2)  $S(\omega, X) \cup T(\omega, X) \subset F(\omega, X)$ ,  $F(\omega, X)$  is closed and  $F(\omega, M)$  is closed for each bounded closed subset  $M$  in  $X$ ;
- (3)  $S, T$  and  $F$  satisfy (2.1) for all  $\omega \in \Omega$  and all  $x, y \in X$ .

Then there exists a measurable mapping  $\xi : \Omega \rightarrow X$  such that

$$F(\omega, \xi(\omega)) \cap S(\omega, \xi(\omega)) \cap T(\omega, \xi(\omega)) \neq \emptyset.$$

*Proof.* For any  $x, y \in X$  and  $\omega \in \Omega$ , we denote

$$\begin{aligned} & D(x, y, \omega) : \\ & = \max\{d(F(\omega, x), F(\omega, y)), d(F(\omega, x), S(\omega, x)), d(F(\omega, y), T(\omega, y)), \\ & \quad [d(F(\omega, x), T(\omega, y)) + d(F(\omega, y), S(\omega, x))]/2\}. \end{aligned}$$

By Lemma 1.2, there exists a strictly increasing function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying conditions (1.3) and (1.4). Hence (2.1) reads:

$$H(S(\omega, x), T(\omega, y)) \leq \Phi(D(x, y, \omega)) \leq \phi(D(x, y, \omega)). \tag{2.2}$$

Let  $\xi_0 : \Omega \rightarrow X$  be an arbitrary measurable mapping. By Lemma 2.1,  $\omega \rightarrow S(\omega, \xi_0(\omega))$  is measurable. We deduce from the Kuratowski-Ryll Nardzewski Selection Theorem [4] that there is a measurable selector  $s_1(\omega) \in S(\omega, \xi_0(\omega))$ . Since  $S(\omega, X) \cup T(\omega, X) \subset F(\omega, X)$ , by Lemma 2.2 there exists a measurable mapping  $\xi_1 : \Omega \rightarrow X$  such that  $s_1(\omega) \in S(\omega, \xi_0(\omega)) \cap F(\omega, \xi_1(\omega))$ . Then by (2.2) we get

$$\begin{aligned} & d(s_1(\omega), T(\omega, \xi_1(\omega))) \\ & \leq d(s_1(\omega), S(\omega, \xi_0(\omega))) + H(S(\omega, \xi_0(\omega)), T(\omega, \xi_1(\omega))) \\ & \leq \Phi(D(\xi_0(\omega), \xi_1(\omega)), \omega) \\ & \leq \phi(D(\xi_0(\omega), \xi_1(\omega)), \omega). \end{aligned} \tag{2.3}$$

Let

$$\begin{aligned} \Omega_1 &:= \{\omega \in \Omega : D(\xi_0(\omega), \xi_1(\omega), \omega) = 0\}, \\ \Omega_2 &:= \{\omega \in \Omega : D(\xi_0(\omega), \xi_1(\omega), \omega) > 0\}. \end{aligned}$$

By Lemma 2.1, we know that  $\Omega_1$  and  $\Omega_2$  are both measurable and  $\Omega_1 \cup \Omega_2 = \Omega$ .

- (a) If  $\omega \in \Omega_1$ , then by (2.3),  $d(s_1(\omega), T(\omega, \xi_1(\omega))) = 0$ . Since  $T(\omega, \xi_1(\omega))$  is closed, we have  $s_1(\omega) \in T(\omega, \xi_1(\omega))$ . Thus  $s_1(\omega) \in F(\omega, \xi_1(\omega)) \cap T(\omega, \xi_1(\omega))$ .
- (b) If  $\omega \in \Omega_2$ , then  $D(\xi_0(\omega), \xi_1(\omega), \omega) > 0$ . Denoting  $Q(\omega) = T(\omega, \xi_1(\omega))$ , by Lemma 2.1 and (2.3) we have  $Q : \Omega_2 \rightarrow X$  is measurable and

$$d(s_1(\omega), Q(\omega)) \leq \Phi(D(\xi_0(\omega), \xi_1(\omega), \omega)) < \phi(D(\xi_0(\omega), \xi_1(\omega), \omega)).$$

By Lemma 1.4 there exists a sequence  $\{f_n(\omega) : n = 1, 2, \dots\}$  of measurable selectors of  $Q$  such that  $cl\{f_n(\omega) : n = 1, 2, \dots\} = Q(\omega)$  for all  $\omega \in \Omega_2$ . Let

$$E_1 := \{\omega \in \Omega_2 : d(s_1(\omega), f_1(\omega)) < \phi(D(\xi_0(\omega), \xi_1(\omega), \omega))\},$$

$$E_n := \{\omega \in \Omega_2 : d(s_1(\omega), f_n(\omega)) < \phi(D(\xi_0(\omega), \xi_1(\omega), \omega))\} \setminus \bigcup_{i=1}^{n-1} E_i, n = 2, 3, \dots$$

Then every  $E_i$  is measurable (perhaps, some  $E_i$  are empty-sets ) and

$$\bigcup_{i=1}^{\infty} E_i = \Omega_2.$$

The mapping  $s_2^* : \Omega_2 \rightarrow X$ , defined by  $s_2^*(\omega) = f_n(\omega), \omega \in E_n, n = 1, 2, \dots$ , is measurable in  $\Omega_2$  and for any  $\omega \in \Omega_2$ ,  $s_2^*(\omega) \in Q(\omega)$  and

$$d(s_1(\omega), s_2^*(\omega)) \leq \phi(D(\xi_0(\omega), \xi_1(\omega), \omega)).$$

Since  $s_2^*(\omega) \in Q(\omega) = T(\omega, \xi_1(\omega)) \subset F(\omega, X)$ , by Lemma 2.2, there exists a measurable mapping, say  $\xi_2^* : \Omega_2 \rightarrow X$  such that  $s_2^*(\omega) \in F(\omega, \xi_2^*(\omega))$ . Thus for each  $\omega \in \Omega_2$ ,  $s_2^*(\omega) \in T(\omega, \xi_1(\omega)) \cap F(\omega, \xi_2^*(\omega))$ .

Let

$$s_2(\omega) := \begin{cases} s_1(\omega), & \omega \in \Omega_1, \\ s_2^*(\omega), & \omega \in \Omega_2, \end{cases} \quad \xi_2(\omega) := \begin{cases} \xi_1(\omega), & \omega \in \Omega_1, \\ \xi_2^*(\omega), & \omega \in \Omega_2. \end{cases}$$

Then  $s_2 : \Omega \rightarrow X$  is measurable in  $\Omega$  and for any  $\omega \in \Omega$ ,  $s_2(\omega) \in T(\omega, \xi_1(\omega)) \cap F(\omega, \xi_2(\omega))$  and

$$d(s_1(\omega), s_2(\omega)) \leq \phi(D(\xi_0(\omega), \xi_1(\omega), \omega)).$$

By (2.2), we have

$$\begin{aligned} d(s_2(\omega), S(\omega, \xi_2(\omega))) &\leq d(s_2(\omega), T(\omega, \xi_1(\omega))) + H(S(\omega, \xi_2(\omega)), T(\omega, \xi_1(\omega))) \\ &\leq \Phi(D(\xi_2(\omega), \xi_1(\omega)), \omega) \\ &\leq \phi(D(\xi_2(\omega), \xi_1(\omega)), \omega). \end{aligned}$$

Similarly, we can find two measurable mappings  $\xi_3, s_3 : \Omega \rightarrow X$ , for any  $\omega \in \Omega$  such that  $s_3(\omega) \in S(\omega, \xi_2(\omega)) \cap F(\omega, \xi_3(\omega))$  and

$$d(s_2(\omega), s_3(\omega)) \leq \phi(D(\xi_2(\omega), \xi_1(\omega)), \omega).$$

Proceeding inductively, we obtain two sequences  $\{s_n\}, \{\xi_n\} \subset X$  such that

$$s_{2n+1}(\omega) \in F(\omega, \xi_{2n+1}(\omega)) \cap S(\omega, \xi_{2n}(\omega)), \tag{2.4}$$

$$s_{2n+2}(\omega) \in F(\omega, \xi_{2n+2}(\omega)) \cap T(\omega, \xi_{2n+1}(\omega)), \tag{2.5}$$

and

$$d(s_{2n+1}(\omega), s_{2n+2}(\omega)) \leq \phi(D(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), \omega), \tag{2.6}$$

$$d(s_{2n+2}(\omega), s_{2n+3}(\omega)) \leq \phi(D(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)), \omega). \tag{2.7}$$

Next we claim that  $\{s_n(\omega)\}$  is convergent in  $X$ . In fact, by (2.4), (2.5) and (2.7),

$$\begin{aligned} &d(s_{2n+2}(\omega), s_{2n+3}(\omega)) \\ &\leq \phi(D(\xi_{2n+2}(\omega), \xi_{2n+1}(\omega)), \omega) \\ &\leq \phi(\max\{d(s_{2n+2}(\omega), s_{2n+1}(\omega)), d(s_{2n+2}(\omega), s_{2n+3}(\omega)), \\ &\quad d(s_{2n+1}(\omega), s_{2n+2}(\omega)), \\ &\quad [d(s_{2n+2}(\omega), s_{2n+2}(\omega)) + d(s_{2n+1}(\omega), s_{2n+3}(\omega))]/2\}) \\ &\leq \phi(\max\{d(s_{2n+1}(\omega), s_{2n+2}(\omega)), d(s_{2n+2}(\omega), s_{2n+3}(\omega))\}), \quad \omega \in \Omega. \end{aligned} \tag{2.8}$$

If  $d(s_{2n+2}(\omega), s_{2n+3}(\omega)) > d(s_{2n+1}(\omega), s_{2n+2}(\omega))$ , then by (2.8) and Lemma 1.3 we have

$$d(s_{2n+2}(\omega), s_{2n+3}(\omega)) \leq \phi(d(s_{2n+2}(\omega), s_{2n+3}(\omega))) < d(s_{2n+2}(\omega), s_{2n+3}(\omega)).$$

This is a contradiction, and so we obtain

$$d(s_{2n+2}(\omega), s_{2n+3}(\omega)) \leq \phi(d(s_{2n+1}(\omega), s_{2n+2}(\omega))).$$

By a similar argument, it can be proved, in view of (2.4), (2.5) and (2.6), that

$$d(s_{2n+1}(\omega), s_{2n+2}(\omega)) \leq \phi(d(s_{2n}(\omega), s_{2n+1}(\omega))).$$

Hence

$$d(s_{n+1}(\omega), s_{n+2}(\omega)) \leq \phi(d(s_n(\omega), s_{n+1}(\omega))) \leq \phi^n(d(s_1(\omega), s_2(\omega))). \quad (2.9)$$

If  $\omega \in \Omega$  with  $d(s_1(\omega), s_2(\omega)) = 0$ , then, by (2.9) we have  $s_n(\omega) = s_1(\omega)$ ,  $n = 1, 2, \dots$

If  $\omega \in \Omega$  with  $d(s_1(\omega), s_2(\omega)) > 0$ , then  $\sum \phi^n(d(s_1(\omega), s_2(\omega)))$  is convergent by (1.4). Now (2.9) implies that  $\sum d(s_n(\omega), s_{n+1}(\omega))$  is convergent too. Thus, for each  $\omega \in \Omega$ ,  $\{s_n(\omega)\}$  is a Cauchy sequence in  $X$ . From the completeness of  $X$ , there exists a measurable mapping  $s^* : \Omega \rightarrow X$  such that  $s_n(\omega) \rightarrow s^*(\omega)$ . Since  $s_n(\omega) \in F(\omega, \xi_n(\omega)) \subset F(\omega, X)$  and  $F(\omega, X)$  is closed, we see that  $s^*(\omega) \in F(\omega, X)$ . Thus, by Lemma 2.2 there exists a measurable mapping  $\xi : \Omega \rightarrow X$  such that  $s^*(\omega) \in F(\omega, \xi(\omega))$  for each  $\omega \in \Omega$ . Combining (2.1), (2.4) with (2.5), we get, for each  $\omega \in \Omega$ ,

$$\begin{aligned} & d(s^*(\omega), T(\omega, \xi(\omega))) \\ & \leq d(s^*(\omega), s_{2n+1}(\omega)) + d(s_{2n+1}(\omega), T(\omega, \xi(\omega))) \\ & \leq d(s^*(\omega), s_{2n+1}(\omega)) + d(s_{2n+1}(\omega), S(\omega, \xi_{2n}(\omega))) \\ & \quad + H(S(\omega, \xi_{2n}(\omega)), T(\omega, \xi(\omega))) \\ & \leq d(s^*(\omega), s_{2n+1}(\omega)) + 0 + \Phi(\max\{d(F(\omega, \xi_{2n}(\omega)), \\ & \quad F(\omega, \xi(\omega))), d(F(\omega, \xi_{2n}(\omega)), S(\omega, \xi_{2n}(\omega))), \\ & \quad d(F(\omega, \xi(\omega)), T(\omega, \xi(\omega))), [d(F(\omega, \xi_{2n}(\omega)), T(\omega, \xi(\omega))) \\ & \quad + d(F(\omega, \xi(\omega), S(\omega, \xi_{2n}(\omega)))]/2\}) \\ & \leq d(s^*(\omega), s_{2n+1}(\omega)) + \Phi(\max\{d(s_{2n}(\omega), s^*(\omega)), d(s_{2n}(\omega), S(\omega, \xi_{2n}(\omega))), \\ & \quad d(s^*(\omega), T(\omega, \xi(\omega))), [d(s_{2n}(\omega), T(\omega, \xi(\omega))) + d(s^*(\omega), S(\omega, \xi_{2n}(\omega)))]/2\}). \end{aligned}$$

Letting  $n \rightarrow \infty$  gives

$$d(s^*(\omega), T(\omega, \xi(\omega))) \leq \Phi(d(s^*(\omega), T(\omega, \xi(\omega)))).$$

Thus, by Lemma 1.3,  $d(s^*(\omega), T(\omega, \xi(\omega))) = 0$ . From the closedness of  $T(\omega, \xi(\omega))$  it follows that  $s^*(\omega) \in T(\omega, \xi(\omega))$ . Similarly, we can prove that  $s^*(\omega) \in S(\omega, \xi(\omega))$ . Thus  $s^*(\omega) \in F(\omega, \xi(\omega)) \cap S(\omega, \xi(\omega)) \cap T(\omega, \xi(\omega))$  for each  $\omega \in \Omega$ . The proof is finished.

**Remark 1.** In Theorem 2.3, the condition  $F : \Omega \times X \rightarrow CC(X)$  in [5, Theorem 2.2] is weakened to  $F : \Omega \times X \rightarrow CB(X)$ . So even in the non-random condition, our results also improve some known ones.



**Remark 2.** In step (a) of the proof of [5, Theorem 2.2], the random coincidence point  $s : \Omega \rightarrow X$  is not well defined. In fact, the domain of  $s$  is only the set of  $\omega \in \Omega$  with  $A(\xi_0(\omega), \xi_1(\omega)) = 0$ . A similar error exists in step (b) of the proof of [5, Theorem 2.2], and also the conclusion in step (a) is used in step (b) there.

Moreover, even if one gets the existence of a coincidence point  $s : \Omega \rightarrow X$ , its measurability still needs to be proved.

These problems, together with other errors in the proof of [5, Theorem 2.2], have been clarified in our proof.

**Corollary 2.4.** Let  $T_i : \Omega \times X \rightarrow CB(X), i \in \mathbb{N}$  (the set of positive integers), be multifunctions such that

- (1)  $T_i(\omega, \cdot), i \in \mathbb{N}$  are all continuous for all  $\omega \in \Omega$ ;
- (2) for every  $i, j \in \mathbb{N}, i \neq j$ ,

$$H(T_i(\omega, x), T_j(\omega, y)) \leq \Phi(\max\{d(x, y), d(x, T_i(\omega, x)), d(y, T_j(\omega, y)), [d(x, T_j(\omega, y)) + d(y, T_i(\omega, x))]/2\}),$$

for all  $\omega \in \Omega$  and all  $x, y \in X$ , where  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is an increasing function satisfying conditions (1.1) and (1.2). Then

- (i) the random fixed point sets  $\{\xi : \Omega \rightarrow X : \xi(\omega) \in T_i(\omega, \xi(\omega))\}, i = 1, 2, \dots$  are nonempty and equal to each other;
- (ii) if  $\{\xi_n : n = 1, 2, \dots\} \subset \{\xi : \Omega \rightarrow X : \xi(\omega) \in T_i(\omega, \xi(\omega))\}$  and  $\xi_n(\omega) \rightarrow \xi_0(\omega)$  as  $n \rightarrow \infty$ , then  $\xi_0 \in \{\xi : \Omega \rightarrow X : \xi(\omega) \in T_i(\omega, \xi(\omega))\}$ .

*Proof.* Let  $F(\omega, x) = x$ . Following the reasoning in the proof of [5, Theorem 2.3], the conclusion of Corollary 2.4 can be obtained by using Theorem 2.3 of this paper.

**Definition 2.1.** A function  $\Psi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$  is said to satisfy the condition  $(\Psi)$ , if it is nondecreasing in each variable and there exists an increasing function  $\Phi(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying the conditions (1.1) and (1.2) such that

$$\Psi(t, t, t, at, bt) \leq \Phi(t), \quad \forall t \geq 0, a + b = 3, a, b = 1, 2.$$

Let  $S, T$  and  $F : \Omega \times X \rightarrow CB(X)$  be multifunctions such that

$$\begin{aligned} & H(S(\omega, x), T(\omega, y)) \\ & \leq \Psi(d(F(\omega, x), F(\omega, y)), d(F(\omega, x), S(\omega, x)), d(F(\omega, y), \\ & T(\omega, y)), d(F(\omega, x), T(\omega, y)), d(F(\omega, y), S(\omega, x))), \end{aligned} \tag{2.10}$$

for all  $x, y \in X$  and for all  $\omega \in \Omega$ , where  $\Psi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$  satisfies condition  $(\Psi)$ .

**Theorem 2.5.** *Let  $S, T$  and  $F : \Omega \times X \rightarrow CB(X)$  be multifunctions such that*

- (1)  $S(\omega, \cdot), T(\omega, \cdot), F(\omega, \cdot)$  are all continuous for all  $\omega \in \Omega$ ;
- (2)  $S(\omega, X) \cup T(\omega, X) \subset F(\omega, X)$ ,  $F(\omega, M)$  is closed for any bounded closed subset  $M$  in  $X$ ;
- (3)  $S, T$  and  $F$  satisfy (2.10) for all  $\omega \in \Omega$  and all  $x, y \in X$ .

*Then there exists a measurable mapping  $\xi : \Omega \rightarrow X$  such that*

$$F(\omega, \xi(\omega)) \cap S(\omega, \xi(\omega)) \cap T(\omega, \xi(\omega)) \neq \emptyset.$$

*Proof.* Following the reasoning in the proof of [5, Theorem 2.4], the conclusion of Theorem 2.5 can be obtained by using Theorem 2.3 of this paper.

From Theorem 2.5, we can obtain the following.

**Corollary 2.6.** *Let  $T_i : \Omega \times X \rightarrow CB(X)$ ,  $i \in \mathbb{N}$ , be multifunctions such that*

- (1)  $T_i(\omega, \cdot)$ ,  $i \in \mathbb{N}$  are all continuous for all  $\omega \in \Omega$ ;
- (2) for every  $i, j \in \mathbb{N}$ ,  $i \neq j$ ,

$$H(T_i(\omega, x), T_j(\omega, y)) \leq \Psi(d(x, y), d(x, T_i(\omega, x)), d(y, T_j(\omega, y)), d(x, T_j(\omega, y)) + d(y, T_i(\omega, x))),$$

for all  $\omega \in \Omega$  and all  $x, y \in X$ , where  $\Psi(t_1, t_2, t_3, t_4, t_5) : \mathbb{R}^{+5} \rightarrow \mathbb{R}^+$  satisfies conditions  $(\Psi)$ . Then

- (i) the random fixed point sets  $\{\xi : \Omega \rightarrow X : \xi(\omega) \in T_i(\omega, \xi(\omega))\}$ ,  $i \in \mathbb{N}$ , are nonempty and equal to each other;
- (ii) if  $\{\xi_n : n = 1, 2, \dots\} \subset \{\xi : \Omega \rightarrow X : \xi(\omega) \in T_i(\omega, \xi(\omega))\}$  and  $\xi_n(\omega) \rightarrow \xi_0(\omega)$  as  $n \rightarrow \infty$ , then  $\xi_0 \in \{\xi : \Omega \rightarrow X : \xi(\omega) \in T_i(\omega, \xi(\omega))\}$ .

**Example 2.7.** Let  $X = \mathbb{R}^+$ ,  $d(x, y) = \min\{1, |x - y|\}$ , and  $\Omega = [0, 1]$ . It is easy to verify that  $(X, d)$  is a bounded separable complete metric space. Assume that  $E$  is a Lebesgue measurable subset of  $\Omega$  with  $0 < m(E) < 1$ .

For each  $x \in X$  and  $\omega \in \Omega$ , define

$$S(\omega, x) := \begin{cases} A(\omega, x), & \omega \in E \cap [0, 1/2), \\ B(\omega, x), & \omega \in E \cap [1/2, 1], \\ 1 - \min\{1/4, 2x\}, & \omega \notin E, \end{cases}$$

$$T(\omega, x) := \begin{cases} A(\omega, x), & \omega \in E \cap [0, 1/2), \\ B(\omega, x), & \omega \in E \cap [1/2, 1], \\ 1 - \min\{1/4, x\}, & \omega \notin E, \end{cases}$$

$$F(\omega, x) := x,$$

where

$$A(\omega, x) = \left\{ z \in X : \frac{\omega x}{2(1+x)} \leq z \leq \frac{x}{2(1+x)} + \omega \right\},$$

$$B(\omega, x) = \left\{ z \in X : z \geq \frac{x}{2(1+x)} + 2\omega \right\}.$$

We now show for fixed  $\omega \in E \cap [0, 1/2)$ ,  $S(\omega, \cdot)$  and  $T(\omega, \cdot)$  are continuous on  $X$ , by recalling such multivalued maps (i.e. compact valued) are continuous iff they are continuous in the Hausdorff metric. Actually, in this case  $S(\omega, x) = T(\omega, x) = A(\omega, x)$  for all  $x \in X$ . Clearly,

$$\begin{aligned} & \sup_{a \in A(\omega, x_n)} d(a, A(\omega, x)) \\ & \leq \max\left\{ \frac{\omega|x-x_n|}{2(1+x)(1+x_n)}, \frac{|x-x_n|}{2(1+x)(1+x_n)} \right\} \leq \frac{|x-x_n|}{2(1+|x-x_n|)} \end{aligned}$$

and

$$\begin{aligned} & \sup_{b \in A(\omega, x)} d(b, A(\omega, x_n)) \\ & \leq \max\left\{ \frac{\omega|x_n-x|}{2(1+x_n)(1+x)}, \frac{|x_n-x|}{2(1+x_n)(1+x)} \right\} \leq \frac{|x_n-x|}{2(1+|x_n-x|)}. \end{aligned}$$

Hence

$$\begin{aligned} H(A(\omega, x_n), A(\omega, x)) &= \max\left\{ \sup_{a \in A(\omega, x_n)} d(a, A(\omega, x)), \sup_{b \in A(\omega, x)} d(b, A(\omega, x_n)) \right\} \\ &\leq \frac{|x_n-x|}{2(1+|x_n-x|)} \rightarrow 0, \quad \text{as } x_n \rightarrow x \text{ in } X. \end{aligned}$$

Likewise, we can see  $S(\omega, \cdot)$  and  $T(\omega, \cdot)$  are continuous on  $X$  for fixed  $\omega \in E \cap [1/2, 1]$  or fixed  $\omega \notin E$ .

Take

$$\Phi(t) := ht, \quad t \in \mathbb{R}^+,$$

where  $\frac{1}{2} \leq h < 1$ .

By an argument similar to the above we get

$$\begin{aligned} H(S(\omega, x), T(\omega, y)) &\leq \frac{|x - y|}{2(1 + |x - y|)} \\ &\leq \Phi(d(F(\omega, x), F(\omega, y))) \\ &\leq \Phi(D(x, y, \omega)), \end{aligned}$$

for every  $x, y \in X$ ,  $\omega \in E$ . Note also that

(1) if  $0 \leq x \leq \frac{1}{8}$ ,  $0 \leq y \leq \frac{1}{4}$ ,  $\omega \notin E$ , then

$$H(S(\omega, x), T(\omega, y)) \leq |2x - y| \leq \frac{1}{4} \leq \frac{1 - 2x - x}{2} \leq \Phi(d(F(\omega, x), S(\omega, x)));$$

(2) if  $x > \frac{1}{8}$ ,  $0 \leq y \leq \frac{1}{4}$ ,  $\omega \notin E$ , then

$$H(S(\omega, x), T(\omega, y)) \leq |\frac{1}{4} - y| \leq \frac{1}{4} \leq \frac{1 - y - y}{2} \leq \Phi(d(F(\omega, y), T(\omega, y)));$$

(3) if  $0 \leq x \leq \frac{1}{8}$ ,  $y > \frac{1}{4}$ ,  $\omega \notin E$ , then

$$H(S(\omega, x), T(\omega, y)) \leq |2x - \frac{1}{4}| \leq \frac{1}{4} \leq \frac{1 - 2x - x}{2} \leq \Phi(d(F(\omega, x), S(\omega, x)));$$

(4) if  $x > \frac{1}{8}$ ,  $y > \frac{1}{4}$ ,  $\omega \notin E$ ,

$$H(S(\omega, x), T(\omega, y)) \leq 0 \leq \Phi(d(F(\omega, x), S(\omega, x))).$$

Therefore,

$$\begin{aligned} &H(S(\omega, x), T(\omega, y)) \\ &\leq \Phi(\max\{d(F(\omega, x), S(\omega, x)), d(F(\omega, y), T(\omega, y))\}) \\ &\leq \Phi(D(x, y, \omega)), \end{aligned}$$

for every  $x, y \in X$ ,  $\omega \notin E$ . On the other hand, it is not hard to see other conditions in Theorem 2.3 are satisfied. Consequently, Theorem 2.3 of this paper guarantees the existence of a measurable mapping  $\xi : \Omega \rightarrow X$  such that

$$\xi(\omega) \in S(\omega, \xi(\omega)) \cap T(\omega, \xi(\omega)), \text{ for all } \omega \in \Omega.$$

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