TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 2B, pp. 789-804, April 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

OPTIMALITY CONDITIONS AND DUALITY FOR A CLASS OF NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING PROBLEMS

Do Sang Kim and Kwan Deok Bae

Abstract. In this paper, we formulate a general dual problem for a class of nondifferentiable multiobjective programs involving the support function of a compact convex set and linear functions. Fritz John and Kuhn-Tucker optimality conditions are presented. In addition, we establish weak and strong duality theorems for weakly efficient solutions under suitable generalized (F, α, ρ, d) convexity assumptions. Some special cases of our duality results are given.

1. Introduction and Preliminaries

There has been an increasing interest in developing optimality conditions and duality relations for nondifferentiable multiobjective programming problems. Mond and Schechter [12], firstly introduced nondifferentiable symmetric duality, in which the objective function contains a support function. Duality theorems for nondifferentiable static programming problem with a square root term are obtained by Lal *et al.* [7]. In nondifferentiable multiobjective programs involving a support function, further developments for duality relations are founded in Kim *et al.* [4] and Liang *et al.* [6].

In order to establish sufficient optimality conditions and duality relations we present the concept of generalized (F, α, ρ, d)-convexity which is related to various generalized convexity by several authors ([2, 3, 5, 7, 11, 13]).

Recently, Yang *et al.* [14] considered a class of nondifferentiable multiobjective programming problems, involving the support function of a compact convex set and constructed a more general dual model for a class of nondifferentiable multiobjective programs and established only weak duality theorems for efficient solutions

Received October 1, 2008.
2000 *Mathematics Subject Classification*: 90C46, 90C26, 90C29.
Key words and phrases: Nondifferentiable multiobjective programming problems, Generalized (F, α, ρ, d)-convex functions, Optimality conditions, Duality.

under the generalized (F, ρ)-convexity assumptions. Subsequently, Kim *et al.* [8] established generalized second order symmetric duality in nondifferentiable multiobjective programming problems.

In this paper, we introduce the concept of generalized (F, α, ρ, d)-convexity and consider a class of nondifferentiable multiobjective programs involving the support function of a compact convex set and linear functions. And we obtain the necessary and sufficient optimality theorems and generalized duality theorems for weakly efficient solutions under generalized (F, α, ρ, d)-convexity assumptions.

Not only weak duality theorems but also strong duality theorem are established by using necessary and sufficient optimality theorems under generalized (F, α, ρ, d)-convexity assumptions. Moreover we give some special cases of our duality results.

We consider the following multiobjective programming problem,

$$\begin{aligned} \text{(MPE)} \quad & \text{Minimize} \quad (f_1(x) + s(x|D_1), \cdots, f_p(x) + s(x|D_p)) \\ & \text{subject to} \quad g(x) \geqq 0, \quad l(x) = 0, \end{aligned}$$

where f and g are differentiable functions from $\mathbb{R}^n \to \mathbb{R}^p$ and $\mathbb{R}^n \to \mathbb{R}^m$, respectively; l is a linear vector function from $\mathbb{R}^n \to \mathbb{R}^q$ and D_i, for each $i \in P = \{1, 2, \cdots, p\}$, is a compact convex set of $\mathbb{R}^n$. The support function $s(x|D_i)$ of D_i defined by $s(x|D_i) = \max\{ \langle x, y \rangle \mid y \in D_i\}$ [1]. Further let, $S := \{x \in \mathbb{R}^n \mid g_i(x) \geqq 0,\ l_k(x) = 0,\ i = 1, \cdots, m,\ k = 1, \cdots, q\}$ and $I(x) := \{i \mid g_i(x) = 0\}$ for any $x \in \mathbb{R}^n$. Let $h_i(x) = s(x|D_i),\ i = 1, \cdots, p$. Then h_i is a convex function and $\partial h_i(x) = \{w \in D_i | \langle w, x \rangle = s(x|D_i)\}$ [12], where ∂h_i is the subdifferential of h_i.

We recall the definitions of (F, α, ρ, d)-convexity due to Liang *et al.* [6].

Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a sublinear functional; let the function $\phi : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $u \in \mathbb{R}^n, \rho \in \mathbb{R}$, and $d(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Definition 1.1. The function ϕ is said to be (F, α, ρ, d)-convex at u, if

$$\phi(x) - \phi(u) \geqq F(x, u; \alpha(x, u)\nabla\phi(u)) + \rho d^2(x, u), \quad \forall x \in \mathbb{R}^n.$$

Definition 1.2. The function ϕ is (F, α, ρ, d)-quasiconvex at u, if

$$\phi(x) \leqq \phi(u) \Rightarrow F(x, u; \alpha(x, u)\nabla\phi(u)) \leqq -\rho d^2(x, u), \quad \forall x \in \mathbb{R}^n.$$

Definition 1.3. The function ϕ is (F, α, ρ, d)-pseudoconvex at u, if

$$F(x, u; \alpha(x, u)\nabla\phi(u)) \geqq -\rho d^2(x, u) \Rightarrow \phi(x) \geqq \phi(u), \quad \forall x \in \mathbb{R}^n.$$

Definition 1.4. The function ϕ is strictly (F, α, ρ, d)-pseudoconvex at u, if for all $x \in \mathbb{R}^n,\ x \neq u$ such that

$$F(x, u; \alpha(x, u)\nabla\phi(u)) \geqq -\rho d^2(x, u) \Rightarrow \phi(x) > \phi(u), \quad \forall x \in \mathbb{R}^n.$$

Remark 1.1.

(i) When $\alpha(x, u) = 1$, the concept of (F, α, ρ, d)-convexity is the same as that of (F, ρ)-convexity in [13].

(ii) When $F(x, u; \alpha(x, u)\nabla\phi(u)) = \alpha(x, u)\nabla\phi(u)\eta(x, u)$, for a certain function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, the concept of (F, α, ρ, d)-convexity is the same as (V, ρ)-invexity in [6].

We give a generalization of Gordan's theorem for the convex and linear functions due to Mangasarian [9] and Mangasarian and Fromovitz [10].

Theorem 1.1. [9]. *Let Γ be a nonempty convex set in $\mathbb{R}^n$, let F be an m-dimensional convex vector function on Γ and let l be a q-dimensional linear vector function on $\mathbb{R}^n$. If*

$\langle F(x) < 0,\ l(x) = 0\rangle$ *has no solution* $x \in \Gamma$

then there exist $p \in \mathbb{R}^m$ *and* $q \in \mathbb{R}^q$ *such that*

$\langle pF(x)\ +\ ql(x) \geqq 0\rangle$ *for all* $x \in \Gamma,\ \ p \geqq 0\ ,\ \ (p, q) \neq 0.$

2. Optimality Conditions

In this section, we establish both Fritz John necessary and sufficient optimality conditions and Kuhn-Tucker necessary and sufficient optimality conditions for weakly efficient solutions of (MPE).

Theorem 2.1. (Fritz John Necessary Optimality Conditions). *Suppose that $f_i, g_j : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, p,\ j = 1, \cdots, m$, are differentiable and $l_k : \mathbb{R}^n \to \mathbb{R},\ k = 1, \cdots, q$, is a linear vector function. If $\bar{x} \in S$ is a weakly efficient solution of (MPE), then there exist $\lambda_i,\ i = 1, \cdots, p,\ \mu_j,\ j = 1, \cdots, m,\ \nu_k,\ k = 1, \cdots, q,\ w_i \in D_i,\ i = 1, \cdots, p$ such that*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k \nabla l_k(\bar{x}) = 0,$$

$$\langle w_i, \bar{x}\rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m, \nu_1, \cdots, \nu_q) \neq 0.$$

Proof. Let $h_i(x) = s(x|D_i),\ i = 1, \cdots, p$. Since D_i is convex and compact, $h_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function and hence $\forall d \in \mathbb{R}^n$,

$$h_i'(\bar{x}; d) = \lim_{\lambda \to 0+} \frac{h_i(\bar{x} + \lambda d) - h_i(\bar{x})}{\lambda}$$

is finite. Also, $\forall d \in \mathbb{R}^n$,

$$\begin{aligned}
(f_i + h_i)'(\bar{x}; d) &= \lim_{\lambda \to 0+} \frac{f_i(\bar{x} + \lambda d) + h_i(\bar{x} + \lambda d) - f_i(\bar{x}) - h_i(\bar{x})}{\lambda} \\
&= \lim_{\lambda \to 0+} \frac{f_i(\bar{x} + \lambda d) - f_i(\bar{x})}{\lambda} + \lim_{\lambda \to 0+} \frac{h_i(\bar{x} + \lambda d) - h_i(\bar{x})}{\lambda} \\
&= f_i'(\bar{x}; d) + h_i'(\bar{x}; d) \\
&= \langle \nabla f_i(\bar{x}), d \rangle + h_i'(\bar{x}; d).
\end{aligned}$$

Since $\bar{x}$ is a weakly efficient solution of (MPE),

$$\left\langle \begin{array}{c} \langle \nabla f_i(\bar{x}), d \rangle + h_i'(\bar{x}; d) < 0,\ i = 1, \cdots, p \\ -\langle \nabla g_j(\bar{x}), d \rangle < 0,\ j \in I(\bar{x}) \\ \langle \nabla l_k(\bar{x}), d \rangle = 0,\ k = 1, \cdots, q \end{array} \right\rangle$$

has no solution $d \in \mathbb{R}^n$. By Gordan theorem for convex functions, there exist $\lambda_i \geqq 0,\ i = 1, \cdots, p,\ \mu_j \geqq 0,\ j \in I(\bar{x})$ and $\nu_k,\ k = 1, \cdots, q$ are not all zero such that for any $d \in \mathbb{R}^n$,

$$\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d \rangle + \sum_{i=1}^{p} \lambda_i h_i'(\bar{x}; d) - \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d \rangle + \sum_{k=1}^{q} \nu_k \langle \nabla l_k(\bar{x}), d \rangle \geqq 0. \quad (1)$$

Let $A = \{\sum_{i=1}^{p} \lambda_i [\nabla f_i(\bar{x}) + \xi_i] - \sum_{j \in I(\bar{x})} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k \nabla l_k(\bar{x}) \mid \xi_i \in \partial h_i(\bar{x}),\ i = 1, \cdots, p\}$. Then $0 \in A$. Ab absurdo, suppose that $0 \notin A$. By separation theorem, there exists $d^* \in \mathbb{R}^n, d^* \neq (0, \cdots, 0)$, such that $\forall a \in A,\ \langle a, d^* \rangle < 0$, that is, $\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d^* \rangle + \sum_{i=1}^{p} \lambda_i \langle \xi_i, d^* \rangle - \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d^* \rangle + \sum_{k=1}^{q} \nu_k \langle \nabla l_k(\bar{x}), d^* \rangle < 0,\ \forall \xi_i \in \partial h_i(\bar{x})$. Hence $\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d^* \rangle + \sum_{i=1}^{p} \lambda_i h_i'(\bar{x}; d) - \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d^* \rangle + \sum_{k=1}^{q} \nu_k \langle \nabla l_k(\bar{x}), d^* \rangle < 0$, which contradicts (1). Letting $\mu_j = 0,\ \forall j \notin I(\bar{x})$, we have

$0 \in \sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i \partial h_i(\bar{x}) - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k \nabla l_k(\bar{x})$ and $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0$, $(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots,\ \mu_m, \nu_1, \cdots, \nu_q,) \neq 0$. Since $\partial h_i(\bar{x}) = \{w_i \mid \langle w_i, \bar{x} \rangle = s(\bar{x}|D_i)\}$, we obtain the desired result. ∎

Theorem 2.2. (Kuhn-Tucker Necessary Optimality Conditions). *Suppose that* $f_i, g_j : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, p,\ j = 1, \cdots, m$ *are differentiable and the vectors* $\nabla l_k(\bar{x}),\ k = 1, \cdots, q,$ *are linearly independent. Assume that* $\exists z^* \in \mathbb{R}^n$ *such that* $\langle \nabla g_j(\bar{x}), z^* \rangle > 0,\ j \in I(\bar{x}),\ \langle \nabla l_k(\bar{x}), z^* \rangle = 0,\ k = 1, \cdots, q.$ *If* $\bar{x} \in S$ *is a weakly efficient solution of (MPE), then there exist* $\lambda_i,\ i = 1, \cdots, p,\ \mu_j,\ j = 1, \cdots, m,\ \nu_k,\ k = 1, \cdots, q,\ w_i \in D_i,\ i = 1, \cdots, p,$ *such that*

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k \nabla l_k(\bar{x}) = 0,\\
&\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,\\
&\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,\\
&(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geq 0,\\
&(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0).
\end{aligned}$$

Proof. Since $\bar{x}$ is a weakly efficient solution of (MPE), by Theorem 2.1, there exists $\lambda_i,\ i = 1, \cdots, p,\ \mu_j,\ j = 1, \cdots, m,\ \nu_k,\ k = 1, \cdots, q\ and\ w_i \in D_i,\ i = 1, \cdots, p$ such that

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k \nabla l_k(\bar{x}) = 0,\\
&\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,\\
&\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,\\
&(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,\\
&(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m, \nu_1, \cdots, \nu_q) \neq 0.
\end{aligned}$$

Assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle > 0,\ \forall j \in I(\bar{x}),$ $\langle \nabla l_k(\bar{x}), z^* \rangle = 0,\ k = 1, \cdots, q.$ Then $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$. Ab absurdo, suppose that $(\lambda_1, \cdots, \lambda_p) = (0, \cdots, 0)$. Then $(\mu_1, \cdots, \mu_m, \nu_1, \cdots, \nu_q) \neq (0, \cdots, 0)$. If $\mu = 0$, then $\nu \neq 0$. Since $\nabla l_k(\bar{x}),\ k = 1, \cdots, q,$ are linearly independent, $\nu_1 \nabla l_1(\bar{x}) + \cdots + \nu_q \nabla l_q(\bar{x}) = 0$ has trivial solution $\nu = 0$, this contradicts $\nu \neq 0$. So $\mu \geq 0$. Since $\langle \nabla g_j(\bar{x}), z^* \rangle > 0,\ j \in I(\bar{x})$. Defining $\mu_j > 0$ for some $j \in \{1, \cdots, m\}$ then $\sum_{j=1}^{m} \mu_j \langle \nabla g_j(\bar{x}), z^* \rangle > 0$ and so $\sum_{j=1}^{m} \mu_j \langle \nabla g_j(\bar{x}), z^* \rangle + \sum_{k=1}^{q} \nu_k \langle \nabla l_k(\bar{x}), z^* \rangle > 0$. This is contradiction. Hence $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$. ∎

Theorem 2.3. (Fritz John Sufficient Optimality Conditions). *Let* $(\bar{x}, \lambda, w, \mu, \nu)$ *satisfy the Fritz John optimality conditions as follows:*

$$\sum_{i=1}^{p}\lambda_i\nabla f_i(\bar{x})+\sum_{i=1}^{p}\lambda_i w_i-\sum_{j=1}^{m}\mu_j\nabla g_j(\bar{x})+\sum_{k=1}^{q}\nu_k\nabla l_k(\bar{x})=0,$$
$$\langle w_i,\bar{x}\rangle=s(\bar{x}|D_i),\ i=1,\cdots,p,$$
$$\sum_{j=1}^{m}\mu_j g_j(\bar{x})=0,$$
$$(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_m)\geqq 0,$$
$$(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_m,\nu_1,\cdots,\nu_q)\neq 0.$$

(a) *$f_i(\cdot)+(\cdot)^T w_i$ is (F,α,ρ_i,d)-pseudoconvex at $\bar{x}$, and $-\sum_{j=1}^{m}\mu_j g_j(\cdot)+\sum_{k=1}^{q}\nu_k l_k(\cdot)$ is strictly (F,α,β,d)-pseudoconvex at $\bar{x}$, with $\beta+\sum_{i=1}^{p}\lambda_i\rho_i\geqq 0$; or*

(b) *$\sum_{i=1}^{p}\lambda_i(f_i(\cdot)+(\cdot)^T w_i)$ is (F,α,ρ,d)-quasiconvex at $\bar{x}$, and $-\sum_{j=1}^{m}\mu_j g_j(\cdot)+\sum_{k=1}^{q}\nu_k l_k(\cdot)$ is strictly (F,α,β,d)-pseudoconvex at $\bar{x}$, with $\beta+\rho\geqq 0$.*

Then $\bar{x}$ is a weakly efficient solution of (MPE).

Proof. (a) Suppose that $\bar{x}$ is not a weakly efficient solution of (MPE). Then there exists $x^*\in S$ such that $f_i(x^*)+s(x^*|D_i)<f_i(\bar{x})+s(\bar{x}|D_i)$. Since $\langle w_i,\bar{x}\rangle=s(\bar{x}|D_i),\ i=1,\cdots,p$,

$$\begin{aligned} f_i(x^*)+x^{*T}w_i &= f_i(x^*)+s(x^*|D_i)\\ &< f_i(\bar{x})+s(\bar{x}|D_i)\\ &= f_i(\bar{x})+\bar{x}^T w_i. \end{aligned}$$

By the (F,α,ρ_i,d)-pseudoconvexity of $f_i(\bar{x})+\bar{x}^T w_i$, we have

$$F(x^*,\bar{x};\alpha(x^*,\bar{x})(\nabla f_i(\bar{x})+w_i))<-\rho_i d^2(x^*,\bar{x}).$$

By sublinearity, there exists $\lambda_i\geqq 0$,

$$F(x^*,\bar{x};\alpha(x^*,\bar{x})\sum_{i=1}^{p}\lambda_i(\nabla f_i(\bar{x})+w_i))\leqq-\sum_{i=1}^{p}\lambda_i\rho_i d^2(x^*,\bar{x}).$$

Since $\beta+\sum_{i=1}^{p}\lambda_i\rho_i\geqq 0$,

$$F(x^*,\bar{x};-\alpha(x^*,\bar{x})(\sum_{j=1}^{m}\mu_i\nabla g_j(\bar{x})-\sum_{k=1}^{q}\nu_k\nabla l_k(\bar{x})))\geqq-\beta d^2(x^*,\bar{x}).$$

Since $-\sum_{j=1}^{m}\mu_j g_j(\bar{x})+\sum_{k=1}^{q}\nu_k l_k(\bar{x})$ is strictly (F,α,β,d)-pseudoconvex,

$$-\sum_{j=1}^{m}\mu_j g_j(x^*)+\sum_{k=1}^{q}\nu_k l_k(x^*)>-\sum_{j=1}^{m}\mu_j g_j(\bar{x})+\sum_{k=1}^{q}\nu_k l_k(\bar{x}).$$

Since $\mu_j g_j(\bar{x}) = 0,\ j = 1, \cdots, m,\ \ \nu_k l_k(\bar{x}) = 0,\ \ \nu_k l_k(x^*) = 0,\ k = 1, \cdots, q$, we obtain

$$\sum_{j=1}^{m} \mu_j g_j(x^*) < 0,$$

which contradicts the condition $\mu_j \geqq 0$ and $g_j(x^*) \geqq 0$.

By a method similar to that used in the proof of (a), we can prove for (b). ■

Theorem 2.4. (Kuhn-Tucker Sufficient Optimality Conditions). *Let* $(\bar{x}, \lambda, w, \mu, \nu)$ *satisfy the Kuhn-Tucker optimality conditions as follows:*

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k \nabla l_k(\bar{x}) = 0,\\
&\langle w_i, \bar{x}\rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,\\
&\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,\\
&(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq (0, \cdots, 0), \quad (\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0).
\end{aligned}$$

(a) $f_i(\cdot) + (\cdot)^T w_i$ *is* (F, α, ρ_i, d)*-pseudoconvex at* $\bar{x}$*, and* $-\sum_{j=1}^{m} \mu_j g_j(\cdot) + \sum_{k=1}^{q} \nu_k l_k(\cdot)$ *is* (F, α, β, d)*-quasiconvex at* $\bar{x}$*, with* $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$*; or*

(b) $\sum_{i=1}^{p} \lambda_i (f_i(\cdot) + (\cdot)^T w_i)$ *is* (F, α, ρ, d)*-pseudoconvex at* $\bar{x}$*, and* $-\sum_{j=1}^{m} \mu_j g_j(\cdot) + \sum_{k=1}^{q} \nu_k l_k(\cdot)$ *is* (F, α, β, d)*-quasiconvex at* $\bar{x}$*, with* $\beta + \rho \geqq 0$.

Then $\bar{x}$ *is a weakly efficient solution of (MPE).*

Proof. (a) Suppose that $\bar{x}$ is not a weakly efficient solution of (MPE). Then there exists $x^* \in S$ such that $f_i(x^*) + s(x^*|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i)$. Since $\langle w_i, \bar{x}\rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p$,

$$\begin{aligned}
f_i(x^*) + x^{*T} w_i &= f_i(x^*) + s(x^*|D_i)\\
&< f_i(\bar{x}) + s(\bar{x}|D_i)\\
&= f_i(\bar{x}) + \bar{x}^T w_i.
\end{aligned}$$

By the (F, α, ρ_i, d)-pseudoconvexity of $f_i(\bar{x}) + \bar{x}^T w_i$, we have

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\nabla f_i(\bar{x}) + w_i)) < -\rho_i d^2(x^*, \bar{x}).$$

By sublinearity, there exists $\lambda_i \geq 0$,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i (\nabla f_i(\bar{x}) + w_i)) < -\sum_{i=1}^{p} \lambda_i \rho_i d^2(x^*, \bar{x}).$$

Since $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$,

$$F(x^*, \bar{x}; -\alpha(x^*, \bar{x})(\sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x}) - \sum_{k=1}^{q} \nu_k \nabla l_k(\bar{x}))) > -\beta d^2(x^*, \bar{x}).$$

Since $-\sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k l_k(\bar{x})$ is (F, α, β, d)-quasiconvex,

$$-\sum_{j=1}^{m} \mu_j g_j(x^*) + \sum_{k=1}^{q} \nu_k l_k(x^*) > -\sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k l_k(\bar{x}).$$

Since $\mu_j g_j(\bar{x}) = 0,\ j = 1, \cdots, m,\ \ \nu_k l_k(\bar{x}) = 0,\ \ \nu_k l_k(x^*) = 0,\ k = 1, \cdots, q$, we obtain

$$\sum_{j=1}^{m} \mu_j g_j(x^*) < 0,$$

which contradicts the condition $\mu_j \geqq 0$ and $g_j(x^*) \geqq 0$.

By a method similar to that used in the proof of (a), we can prove for (b). ∎

3. Duality Theorems

In this section, we formulate the generalized dual programming problem and establish weak and strong duality theorems under generalized (F, α, ρ, d)-convexity assumptions. Now we propose the following general dual (MDE) to (MPE):

(MDE) Maximize

$$(f_1(u) + u^T w_1 - \sum_{i \in I_0} y_i g_i(u) + z^T l(u),$$

$$\cdots, f_p(u) + u^T w_p - \sum_{i \in I_0} y_i g_i(u) + z^T l(u))$$

subject to

$$\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) - y^T \nabla g(u) + \sum_{k=1}^{q} z_k \nabla l_k(u) = 0, \quad (2)$$

$$\sum_{i \in I_\alpha} y_i g_i(u) \leqq 0,\ \alpha = 1, \cdots, r, \quad (3)$$

$$y \geqq 0, \quad w_i \in D_i,\ i = 1, \cdots, p,$$

$$\lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+,$$

where $I_\alpha \subset M = \{1, \cdots, m\},\ \alpha = 0, 1, \cdots, r$ with $\cup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$. Let $\Lambda^+ = \{\lambda \in \mathbb{R}^p :\ \lambda \geqq 0,\ \lambda^T e = 1,\ e = (1, \cdots, 1)^T \in \mathbb{R}^p\}$.

Theorem 3.1. (Weak Duality). *Assume that for all feasible x of (MPE) and all feasible (u,λ,w,y,z) of (MDE), if $-\sum_{i\in I_\alpha} y_i g_i(\cdot)$ $(\alpha=1,\cdots,r)$ is $(F,\alpha,\beta_\alpha,\rho)$-quasiconvex at u and assuming that one of the following conditions hold:*

- (a) *$f_i(\cdot)+(\cdot)^T w_i-\sum_{i\in I_0} y_i g_i(\cdot)+z^T l(\cdot)$ is (F,α,ρ_i,d)-pseudoconvex at u, with $\sum_{\alpha=1}^r \beta_\alpha+\sum_{i=1}^p \lambda_i\rho_i \geqq 0$; or*
- (b) *$\sum_{i=1}^p \lambda_i(f_i(\cdot)+(\cdot)^T w_i)-\sum_{i\in I_0} y_i g_i(\cdot)+\sum_{k=1}^q z_k l_k(\cdot)$ is (F,α,ρ,d)-pseudoconvex at u, with $\sum_{\alpha=1}^r \beta_\alpha+\rho\geqq 0$.*

Then the following cannot hold:

$$f(x)+s(x|D) < f(u)+u^T w-\sum_{i\in I_0} y_i g_i(u)e+z^T l(u)e. \tag{4}$$

Proof. As x is feasible for (MPE) and (u,λ,w,y,z) is feasible for (MDE), we have

$$\sum_{i\in I_\alpha} y_i g_i(x) \geqq 0 \geqq \sum_{i\in I_\alpha} y_i g_i(u),\ \ \alpha=1,\cdots,r.$$

By the $(F,\alpha,\beta_\alpha,d)$-quasiconvexity of $-\sum_{i\in I_\alpha} y_i g_i(u),\ \alpha=1,\cdots,r$, it follows that

$$F(x,u;-\alpha(x,u)(\textstyle\sum_{i\in I_\alpha} y_i\nabla g_i(u)) \leqq -\beta_\alpha d^2(x,u),\ \alpha=1,\cdots,r. \tag{5}$$

On the other hand, by (2) and the sublinearity of F, we have

$$\begin{aligned} &F(x,u;\alpha(x,u)\left(\sum_{i=1}^p\lambda_i\left(\nabla f_i(u)+w_i)-\sum_{i\in I_0} y_i\nabla g_i(u)+\sum_{k=1}^q z_k\nabla l_k(u)\right)\right)\\ &+\sum_{\alpha=1}^r F(x,u;-\alpha(x,u)\left(\sum_{i\in I_\alpha} y_i\nabla g_i(u)\right) \geqq F(x,u;\alpha(x,u)\\ &\left(\sum_{i=1}^p \lambda_i(\nabla f_i(u)+w_i)-y^T\nabla g(u)+\sum_{k=1}^q z_k\nabla l_k(u))\right)=0. \end{aligned} \tag{6}$$

Combination (5) and (6) gives

$$\begin{aligned} &F(x,u;\alpha(x,u)(\sum_{i=1}^p\lambda_i(\nabla f_i(u)+w_i)-\sum_{i\in I_0} y_i\nabla g_i(u)\\ &+\sum_{k=1}^q z_k\nabla l_k(u))) \geqq (\sum_{\alpha=1}^r\beta_\alpha)d^2(x,u). \end{aligned} \tag{7}$$

Now suppose, contrary to the result, that (4) holds. Since $x^T w_i \leqq s(x|D_i)$, we have for all $i\in\{1,\cdots,p\}$

$$\begin{aligned} &f_i(x)+x^T w_i-\sum_{i\in I_0} y_i g_i(x)+z^T l(x) \leqq f_i(x)+x^T w_i\\ &\leqq f_i(x)+s(x|D_i) < f_i(u)+u^T w_i-\sum_{i\in I_0} y_i g_i(u)+z^T l(u). \end{aligned} \tag{8}$$

By (a), we get

$$F(x,u;\alpha(x,u)\left(\nabla f_i(u)+w_i-\sum_{i\in I_0}y_i\nabla g_i(u)+z^T\nabla l(u))\right) \\ < -\rho_i d^2(x,u), \quad \forall i\in\{1,\cdots p\}. \tag{9}$$

From $\lambda\in\Lambda^+$, (9) and the sublinearity of F, we obtain

$$F(x,u;\alpha(x,u)\left(\sum_{i=1}^p\lambda_i\left(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+\sum_{k=1}^q z_k\nabla l_k(u)\right)\right) \\ < \left(-\sum_{i=1}^p\lambda_i\rho_i\right)d^2(x,u). \tag{10}$$

Since $\sum_{\alpha=1}^r\beta_\alpha+\sum_{i=1}^p\lambda_i\rho_i\geqq 0$, it follows from (10) that

$$F(x,u;\alpha(x,u)\left(\sum_{i=1}^p\lambda_i\left(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+\sum_{k=1}^q z_k\nabla l_k(u)\right)\right) \\ < \left(\sum_{\alpha=1}^r\beta_\alpha\right)d^2(x,u),$$

which contradicts (7). Hence (4) cannot hold.

Suppose now that (b) is satisfied. From $\lambda\in\Lambda^+$ and (8), it follows that

$$\sum_{i=1}^p\lambda_i(f_i(x)+x^Tw_i)-\sum_{i\in I_0}y_ig_i(x)+z^Tl(x) \\ < \sum_{i=1}^p\lambda_i(f_i(u)+u^Tw_i)-\sum_{i\in I_0}y_ig_i(u)+z^Tl(u).$$

Then, by the (F,α,ρ,d)-pseudoconvexity of $\sum_{i=1}^p\lambda_i(f_i(\cdot)+(\cdot)^Tw_i)-\sum_{i\in I_0}y_ig_i(\cdot)$ $+\sum_{k=1}^q z_kl_k(\cdot)$ at u,

$$F(x,u;\alpha(x,u)\left(\sum_{i=1}^p\lambda_i\left(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+z^T\nabla l(u)\right)\right) \\ < -\rho d^2(x,u). \tag{11}$$

Since $\sum_{\alpha=1}^r\beta_\alpha+\rho\geqq 0$, it follows from (11) that

$$F(x,u;\alpha(x,u)\left(\sum_{i=1}^{p}\lambda_i\left(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+z^T\nabla l(u)\right)\right)$$
$$<\left(\sum_{\alpha=1}^{r}\beta_\alpha\right)d^2(x,u),$$

which contradicts (7). Hence (4) cannot hold. ∎

Theorem 3.2. (Weak Duality). *Assume that for all feasible x of (MPE) and all feasible (u,λ,w,y,z) of (MDE), if $-\sum_{i\in I_\alpha}y_ig_i(\cdot)$ $(\alpha=1,\cdots,r)$ is $(F,\alpha,\beta_\alpha,\rho)$-quasiconvex at u and assuming that one of the following three conditions hold:*

- (a) *$f_i(\cdot)+(\cdot)^Tw_i-\sum_{i\in I_0}y_ig_i(\cdot)+z^Tl(\cdot)$ is both (F,α,ρ_i,d)-quasiconvex and (F,α,ρ_i,d)-pseudoconvex at u, $i\in P$ with $\sum_{\alpha=1}^{r}\beta_\alpha+\sum_{i=1}^{p}\lambda_i\rho_i\geqq 0$; or*
- (b) *$f_i(\cdot)+(\cdot)^Tw_i-\sum_{i\in I_0}y_ig_i(\cdot)+z^Tl(\cdot)$ is (F,α,ρ_i,d)-quasiconvex at u, $\forall i\in P$ and there exist $k\in P$ such that $f_k(\cdot)+(\cdot)^Tw_k-\sum_{i\in I_0}y_ig_i(\cdot)+z^Tl(\cdot)$ is strictly (F,α,ρ_i,d)-pseudoconvex at u, with $\sum_{\alpha=1}^{r}\beta_\alpha+\sum_{i=1}^{p}\lambda_i\rho_i\geqq 0$; or*
- (c) *$\sum_{i=1}^{p}\lambda_i(f_i(\cdot)+(\cdot)^Tw_i)-\sum_{i\in I_0}y_ig_i(\cdot)+\sum_{k=1}^{q}z_kl_k(\cdot)$ is (F,α,ρ,d)-pseudoconvex at u, with $\sum_{\alpha=1}^{r}\beta_\alpha+\rho\geqq 0$, then the following cannot hold:*

$$f(x)+s(x|D)\leq f(u)+u^Tw-\sum_{i\in I_0}y_ig_i(u)e+z^Tl(u)e,\ for\ all\ i\in P, \tag{12}$$

and

$$f(x)+s(x|D)<f(u)+u^Tw-\sum_{i\in I_0}y_ig_i(u)e+z^Tl(u)e,\ some\ i\in P. \tag{13}$$

Proof. As x is feasible for (MPE) and (u,λ,w,y,z) is feasible for (MDE), we have

$$\sum_{i\in I_\alpha}y_ig_i(x)\geqq 0\geqq\sum_{i\in I_\alpha}y_ig_i(u),\ \alpha=1,\cdots,r.$$

By the $(F,\alpha,\beta_\alpha,d)$-quasiconvexity of $-\sum_{i\in I_\alpha}y_ig_i(u),\ \alpha=1,\cdots,r$, it follows that

$$F(x,u;-\alpha(x,u)\left(\sum_{i\in I_\alpha}y_i\nabla g_i(u)\right)\leqq-\beta_\alpha d^2(x,u),\ \alpha=1,\cdots,r. \tag{14}$$

On the other hand, by (2) and the sublinearity of F, we have

$$F(x,u;\alpha(x,u)\left(\sum_{i=1}^{p}\lambda_i\left(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+\sum_{k=1}^{q}z_k\nabla l_k(u)\right)\right)$$
$$+\sum_{\alpha=1}^{r}F(x,u;-\alpha(x,u)\sum_{i\in I_\alpha}y_i\nabla g_i(u)) \tag{15}$$
$$\geqq F(x,u;\alpha(x,u)\left(\sum_{i=1}^{p}\lambda_i\left(\nabla f_i(u)+w_i)-y^T\nabla g(u)+\sum_{k=1}^{q}z_k\nabla l_k(u)\right)\right)=0.$$

Combination (14) and (15) gives

$$F(x,u;\alpha(x,u)\left(\sum_{i=1}^{p}\lambda_i\left(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+\sum_{k=1}^{q}z_k\nabla l_k(u)\right)\right)$$
$$\geqq\left(\sum_{\alpha=1}^{r}\beta_\alpha\right)d^2(x,u). \tag{16}$$

Now suppose, contrary to the result, that (12)and (13) hold. Since $x^Tw_i \leqq s(x|D_i)$, we have

$$f_i(x)+x^Tw_i-\sum_{i\in I_0}y_ig_i(x)+z^Tl(x)\leqq f_i(x)+x^Tw_i\leqq f_i(x)+s(x|D_i)$$
$$\leqq f_i(u)+u^Tw_i-\sum_{i\in I_0}y_ig_i(u)+z^Tl(u),\ \forall i\in P, \tag{17}$$

$$f_i(x)+x^Tw_i-\sum_{i\in I_0}y_ig_i(x)+z^Tl(x)\leqq f_i(x)+x^Tw_i\leqq f_i(x)+s(x|D_i)$$
$$< f_i(u)+u^Tw_i-\sum_{i\in I_0}y_ig_i(u)+z^Tl(u),\ for\ some\ i\in P. \tag{18}$$

By (a), we get

$$F\left(x,u;\alpha(x,u)\left(\nabla f_i(u)+w_i-\sum_{i\in I_0}y_i\nabla g_i(u)+z^T\nabla l(u)\right)\right)$$
$$\leqq-\rho_i d^2(x,u),\quad \forall i\in P, \tag{19}$$

$$F\left(x,u;\alpha(x,u)\left(\nabla f_i(u)+w_i-\sum_{i\in I_0}y_i\nabla g_i(u)+z^T\nabla l(u)\right)\right)$$
$$<-\rho_i d^2(x,u),\quad for\ some\ i\in P. \tag{20}$$

From $\lambda\in\Lambda^+$, (19), (20) and the sublinearity of F, we have

$$F\left(x,u;\alpha(x,u)\left(\sum_{i=1}^{p}\lambda_i(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+\sum_{k=1}^{q}z_k\nabla l_k(u)\right)\right)$$
$$<(-\sum_{i=1}^{p}\lambda_i\rho_i)d^2(x,u). \tag{21}$$

Since $\sum_{\alpha=1}^{r}\beta_\alpha+\sum_{i=1}^{p}\lambda_i\rho_i \geqq 0$, it follows from (21) that

$$F\left(x,u;\alpha(x,u)\left(\sum_{i=1}^{p}\lambda_i(\nabla f_i(u)+w_i)-\sum_{i\in I_0}y_i\nabla g_i(u)+\sum_{k=1}^{q}z_k\nabla l_k(u)\right)\right)$$
$$<\left(\sum_{\alpha=1}^{r}\beta_\alpha\right)d^2(x,u),$$

which contradicts (16). Hence (12)and (13) cannot hold.

The proof for (b) or (c) is similar to the one used for the proof of (a). ■

Remark 3.1. *If* $l=0$ *and* $\alpha(x,u)=1$, *then Theorem 3.2 reduces to Theorem 2.1 of Yang et al. [14] in the sense of efficient solutions.*

Theorem 3.3. (Strong Duality). *If* $\bar{x}\in S$ *is a weakly efficient solution of (MPE), and assume that there exists* $z^*\in\mathbb{R}^n$ *such that* $\langle\nabla g_j(\bar{x}),z^*\rangle>0,\ \forall j\in I(\bar{x})$, $\langle\nabla l_k(\bar{x}),z^*\rangle=0,\ k=1,\cdots,q$, *the vector* $\nabla l_k(\bar{x}),\ k=1,\cdots,q$ *are linearly independent. Then there exist* $\bar{\lambda}\in\mathbb{R}^p,\ \bar{w}_i\in D_i,\ i=1,\cdots,p,\ \bar{y}\in\mathbb{R}^m,\ \bar{z}\in\mathbb{R}^q$ *such that* $(\bar{x},\bar{\lambda},\bar{w},\bar{y},\bar{z})$ *is feasible for (MDE) and* $\bar{x}^T\bar{w}_i=s(\bar{x}|D_i),\ i=1,\cdots,p$. *Moreover, if the assumptions of Theorem 3.1 are satisfied, then* $(\bar{x},\bar{\lambda},\bar{w},\bar{y},\bar{z})$ *is a weakly efficient solution of (MDE).*

Proof. By Theorem 2.2, there exist $\bar{\lambda}\in\mathbb{R}^p,\ \bar{y}\in\mathbb{R}^m,\ \bar{z}\in\mathbb{R}^q$ and $\bar{w}_i\in D_i,\ i=1,\cdots,p$, such that $\sum_{i=1}^{p}\bar{\lambda}_i(\nabla f_i(\bar{x})+\bar{w}_i)-\sum_{j=1}^{m}\bar{y}_j\nabla g_j(\bar{x})+\sum_{k=1}^{q}\nu_k\nabla l_k(\bar{x})=0,\ \bar{y}_jg_j(\bar{x})=0,\ j=1,\cdots,m$, and $\bar{w}_i\in D_i,\ i=1,\cdots,p$. Thus $(\bar{x},\bar{\lambda},\bar{w},\bar{y},\bar{z})$ is a feasible for (MDE) and $\bar{x}^T\bar{w}_i=s(\bar{x}|D_i),\ i=1,\cdots,p$. Notice that $f_i(\bar{x})+s(\bar{x}|D_i)=f_i(\bar{x})+\bar{x}^T\bar{w}_i=f_i(\bar{x})+\bar{x}^T\bar{w}_i-\sum_{i\in I_0}\bar{y}_ig_i(\bar{x})+z^Tl(\bar{x})$. By Theorem 3.1, we obtain that the following cannot hold:

$$(f_1(\bar{x})+s(\bar{x}|D_1),\cdots,f_p(\bar{x})+s(\bar{x}|D_p))$$
$$<(f_1(u)+u^Tw_1+\sum_{i\in I_0}y_ig_i(u)+z^Tl(u)$$
$$,\cdots,f_p(u)+u^Tw_p-\sum_{i\in I_0}y_ig_i(u)+z^Tl(u))$$

where (u, λ, w, y, z) is any feasible solution of (MDE). Since $\bar{x}^T \bar{w}_i = s(\bar{x}|D_i)$, we have that the following cannot hold:

$$\begin{aligned}&(f_1(\bar{x}) + \bar{x}^T \bar{w}_1 - \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) + z^T l(\bar{x})\\ &\qquad , \cdots , f_p(\bar{x}) + \bar{x}^T \bar{w}_p - \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) + z^T l(\bar{x}))\\ &< (f_1(u) + u^T w_1 - \sum_{i \in I_0} y_i g_i(u) + z^T l(u)\\ &\qquad , \cdots , f_p(u) + u^T w_p - \sum_{i \in I_0} y_i g_i(u) + z^T l(u)).\end{aligned}$$

Since $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is a feasible solution for (MDE), $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is a weakly efficient solution of (MDE). Hence the result holds. ■

Theorem 3.4. (Strong Duality). *If $\bar{x} \in S$ is an efficient solution of (MPE), and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle > 0,\ \forall j \in I(\bar{x})$, $\langle \nabla l_k(\bar{x}), z^* \rangle = 0,\ k = 1, \cdots, q$, the vector $\nabla l_k(\bar{x}),\ k = 1, \cdots, q$ are linearly independent. Then there exist $\bar{\lambda} \in \mathbb{R}^p,\ \bar{w}_i \in D_i,\ i = 1, \cdots, p,\ \bar{y} \in \mathbb{R}^m,\ \bar{z} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is feasible for (MDE) and $\bar{x}^T \bar{w}_i = s(\bar{x}|D_i),\ i = 1, \cdots, p$. Moreover, if the assumptions of Theorem 3.2 are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is an efficient solution of (MDE).*

The proof is similar to the one used for the previous strong duality theorem.

4. Special Cases

We give some special cases of our duality results.

(1) If $l = 0$, then the primal problem (MPE) and the dual problem (MDE) become the primal problem (VP) and the dual problem (VD) considered in Yang *et al.* [14] respectively.

(VP) Minimize $(f_1(x) + s(x|D_1), \cdots, f_p(x) + s(x|D_p))$

subject to $g(x) \geqq 0$,

(VD) Maximize

$$(f_1(u) + u^T w_1 - \sum_{i \in I_0} y_i g_i(u), \cdots, f_p(u) + u^T w_p - \sum_{i \in I_0} y_i g_i(u))$$

subject to
$$\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) - y^T \nabla g(u) = 0,$$
$$\sum_{i \in I_\alpha} y_i g_i(u) \leqq 0,\ \alpha = 1, \cdots, r,$$

$$y \geqq 0, \quad w_i \in D_i, \quad i = 1, \cdots, p,$$
$$\lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+.$$

(2) Let $D_i = \{B_i w : w^T B_i w \leqq 1, \}$. Then $s(x|D_i) = (x^T B_i x)^{1/2}$ and the sets $D_i,\ i = 1, \cdots, p$, are compact and convex. If $l = 0$, $I_0 = M$ and $I_\alpha = \emptyset$, $\alpha = 1, \cdots, r$, then (MPE) and (MDE) reduce to (VP) and $(\text{VDP})_1$ in Lal *et al.* [7], respectively. If $l = 0$, $I_0 = \emptyset$, $I_1 = M$ and $I_\alpha = \emptyset$, $\alpha = 2, \cdots, r$, then (MPE) and (MDE) reduce to (VP) and $(\text{VDP})_2$ in Lal *et al.* [7], respectively.

References

1. F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983, pp. 235-236.

2. V. Jeyakumar, Equivalence of saddle-points and optima, and duality for a class of nonsmooth non-convex problems, *Journal of Mathematical Analysis and Applications*, **130** (1988), 334-343.

3. V. Jeyakumar and B. Mond, On generalized convex mathematical programming, *Journal of the Australian Mathematical Society*, **34B** (1992), 43-53.

4. S. J. Kim, M. H. Kim and D. S. Kim, Optimality and Duality for a Class of Nondifferentiable Multiobjective Fractional Programming Problems, *Journal of Optimization Theory and Applications*, **129(1)** (2006), 131-146.

5. H. Kuk, G. M. Lee and D. S. Kim, Nonsmooth multiobjective programs with V-ρ-invexity, *Indian Journal of Pure and Applied Mathematics*, **29** (1998), 405-412.

6. Z. A. Liang, H. X. Huang and P. M. Pardalos, Optimality conditions and duality for a class of nonlinear fractional programming problems, *Journal of Optimization Theory and Applications*, **110(3)** (2001), 611-619.

7. S. N. Lal, B. Nath and A. Kumar, Duality for some nondifferentiable static multiobjective programming problems, *Journal of Mathematical Analysis and Applications*, **186** (1994), 862-867.

8. H. J. Lee, Y. J. Lee and D. S. Kim, Generalized second order symmetric duality in nondifferentiable multiobjective programming, *Taiwanese Journal of Mathematics*, **11(3)** (2007), 745-764.

9. O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969, pp. 66-67.

10. O. L. Mangasarian, and S. Fromovitz, The Fritz John Necessary Optimality Conditions in the Presence of Equality and Inequality Constraints, *Journal of Mathematical Analysis and Application*, **17** (1967), 37-47.

11. S. K. Mishra and R. N. Mukherjee, On generalized convex multi-objective nonsmooth programming, *Journal of the Australian Mathematical Society*, **38B** (1996), 140-148.

12. B. Mond and M. Schechter, Non-differentiable symmetric duality, *Bulletin of the Australian Mathematical Society*, **53** (1996), 177-187.

13. V. Preda, On efficiency and duality for multiobjective programs, *Journal of Mathematical Analysis and Applications*, **166** (1992), 365-377.

14. X. M. Yang, K. L. Teo and X. Q. Yang, Duality for a class of nondifferentiable multiobjective programming problem, *Journal of Mathematical Analysis and Applications*, **252** (2000), 999-1005.

Do Sang Kim and Kwan Deok Bae
Division of Mathematical Sciences,
Pukyong National University,
Busan 608-737, Republic of Korea
E-mail: dskim@pknu.ac.kr