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THE OPTIMAL PEBBLING NUMBER OF THE CATERPILLAR

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Abstract. Let G be a simple graph. If we place p pebbles on the vertices of G, then a pebbling move is taking two pebbles off one vertex and then placing one on an adjacent vertex. The optimal pebbling number of G, $f'(G)$, is the least positive integer p such that p pebbles are placed suitably on vertices of G and for any target vertex v of G, we can move one pebble to v by a sequence of pebbling moves. In this paper, we find the optimal pebbling number of the caterpillars.

1. Introduction

Suppose p pebbles are distributed onto the vertices of G; then we have a so-called distribution δ where we let $\delta(v)$ be the number of pebbles distributed to $v \in V(G)$ and $\delta(H) = \sum_{v \in V(H)} \delta(v)$ for each induced subgraph H of G. Note that now $\delta(G) = p$.

A pebbling move consists of moving two pebbles from one vertex and then placing one pebble at an adjacent vertex. If a distribution δ let us move, for any vertex v, at least one pebble to v by applying pebbling moves repeatedly(if necessary), then δ is called a *pebbling* of G. Suppose δ is a distribution of G and H is an induced subgraph of G and v is a vertex in H. Let $\delta(H, v)$ denote the maximum number of pebbles which can be moved to v by applying pebbling moves on H. Therefore, for each $v \in V(G)$ $\delta(G, v) > 0$ if δ is a pebbling of G. The *pebbling number* $f(G)$ of a graph G is defined as the minimum number of pebbles p such that any distribution with p pebbles is a pebbling of G. The problem of pebbling graph was first proposed by M. Saks and J. Lagarias [1] as a tool for solving a number theoretic problem posed by Lemke and Kleitman [6], and some excellent results have been obtained, see [1, 2, 3, 5, 8, 12].

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Motivated by the study of pebbling number, the notion of optimal pebbling was introduced later by L. Pachter et al. [10]. The *optimal pebbling number* of G, $f'(G)$, is $\min\{\delta(G)|\delta$ is a pebbling of G$\}$, and a distribution δ is an *optimal pebbling* of G if δ is a pebbling of G such that $\delta(G) = f'(G)$. First, L. Pachter et al. found the optimal pebbling number for the path.

Theorem 1.1. [10]. *Let P be a path of order 3t+r, i.e.,* $|V(P)| = 3t + r$, *where* $0 \leq r < 3$. *Then* $f'(P) = 2t + r$. Later, several results have been obtained.

Theorem 1.2. [11]. $f'(C_n) = f'(P_n)$.

Theorem 1.3. [11]. *For any graphs* G *and* H, $f'(G \times H) \leq f'(G)f'(H)$.

Theorem 1.4. [9] $f'(Q_n) = (\frac{4}{3})^{n+O(\log n)}$. Besides, Fu and Shiue devised a polynomial algorithm to determined the optimal pebbling number of the complete m-ary tree [4]. With regard to the complexity of determining the pebbling number, Milan and Clark showed that deciding whether $f'(G) \leq k$ for a graph G and an integer k is NP-complete [7]. This says that to determine the optimal pebbling number for a graph is difficult.

In order to find the optimal pebbling number of the caterpillar, we introduce the notion of *weighted pebbling*. Let α be a weighted function mapping from $V(G)$ into the set of positive integers. If δ is a distribution of G and $\delta(G, v) \geq \alpha(v)$ for every vertex v, then δ is called an α*-weighted pebbling* of G. In what follows, we call α a *pebbling type* of G and the *optimal* α*-weighted pebbling number* of G, $f'_\alpha(G)$, is $\min\{\delta(G)|\delta$ is an α-weighted pebbling of $G\}$. Clearly, if $\alpha(v) = 1$ for each $v \in V(G)$, then $f'_\alpha(G) = f'(G)$. In this paper, we shall determine the optimal pebbling number of the caterpillar via a special α-weighted pebbling of a path.

2. Main Result

A tree T is called a caterpillar if the deletion of all pendent vertices of the tree results in a path P'. For convenience, we shall call a path P with maximum length which contains P' a body of the caterpillar, and all the edges which are incident to pendent vertices are the *legs* of the caterpillar T. Furthermore, the vertex $v \in V(P)$ is a *joint* of T provided that $\deg_T(v) \geq 3$ or v is adjacent to the end vertices, see Figure 1 for an example.

Now, we are ready to prove the first lemma. First, we need the following facts. Since they are easy to see, we omit their proofs

Fact 1. Let T be a tree and δ be a distribution of T, and let v be a pendent vertex of T which is adjacent to u. If δ^* is a distribution of $T - v$ defined by

$\delta^*(u) = \delta(u) + \lfloor\frac{\delta(v)}{2}\rfloor$ and $\delta^*(w) = \delta(w)$ for each $w \in V(T) - \{u, v\}$, then $\delta^*(T - v, w) = \delta(T, w)$ for each $w \in V(T) - \{v\}$.

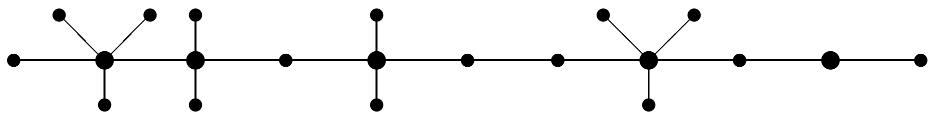

Fig. 1. A caterpillar with 5 joints.

Fact 2. Let G be a graph, and let δ_1 and δ_2 are two distributions of G. If $\delta_1(v) \geq \delta_2(v)$ for each $v \in V(G)$, then $\delta_1(G, v) \geq \delta_2(G, v)$ for each $v \in V(G)$.

The following lemma provides a recursive step to show that the problem of finding the optimal pebbling number of a caterpillar T is equivalent to the problem of finding the optimal α-weighted pebbling number of a body of T for some type α.

Lemma 2.1. *Let T be a tree, and let v be a pendent vertex of T which is adjacent to u. If α is a pebbling type of T satisfying $\alpha(v) = 1$ and β is a pebbling type of $T - v$, defined by $\beta(u) = \max\{2, \alpha(u)\}$ and $\beta(w) = \alpha(w)$ for $w \in V(T) - \{u, v\}$ then $f'_\alpha(T) = f'_\beta(T - v)$.*

Proof. Let δ^* be an optimal β-weighted pebbling of $T - v$. Then we choose a distribution δ of T such that $\delta(v) = 0$ and $\delta(w) = \delta^*(w)$ for each $w \in V(T) - \{v\}$. Clearly, $\delta(T) = \delta^*(T - v) = f'_\beta(T - v)$ and $\delta(T - v, w) = \delta^*(T - v, w) \geq \beta(w) = \alpha(w)$ for each $w \in V(T) - v$. Since $\delta(T - v, u) = \delta^*(T - v, u) \geq \beta(u) \geq 2$, we have $\delta(T, v) = \delta(v) + \lfloor\frac{\delta(T-v,u)}{2}\rfloor \geq \lfloor\frac{\beta(u)}{2}\rfloor \geq 1 = \alpha(v)$. This implies that δ is an α-weighted pebbling of T and it follows that $f'_\alpha(T) \leq f'_\beta(T - v)$.

To prove that $f'_\alpha(T) \geq f'_\beta(T - v)$, let δ be an optimal α-weighted pebbling of T and δ^* be a distribution of $T - v$ such that $\delta^*(u) = \delta(u) + \delta(v)$ and $\delta^*(w) = \delta(w)$ for each $w \in V(T - v) - \{u\}$. Clearly, $\delta^*(T - v) = \delta(T) = f'_\alpha(T)$. By Fact 1 and Fact 2, we have

$$\delta^*(T - v, w) \geq \delta(T, w) \geq \alpha(w) \text{ for each } w \in V(T - v). \qquad (*)$$

Since $\beta(w) = \alpha(w)$ for each $w \in V(T - v) - \{u\}$, δ^* will be a β-weighted pebbling as long as $\delta^*(T - v, u) \geq \beta(u)$.

First, if $\alpha(u) \geq 2$ then $\beta(u) = \alpha(u)$. By (*), $\delta^*(T - v, u) \geq \delta(T, u) \geq \alpha(u) = \beta(u)$. On the other hand, $\alpha(u)$ must be 1 and $\beta(u) = 2$ by the hypothesis. By the definition of δ^*, if $\delta(v) \geq 2$ or $\delta(T - v, u) \geq 2$ then $\delta^*(T - v, u) = \delta(T - v, u) + \delta(v) \geq 2 = \beta(u)$. Now, if $\delta(T - v, u) = 0$, then there are at least two pebbles on v in the distribution δ, i.e., $\delta(v) \geq 2$. Finally, if $\delta(T - v, u) = 1$

then $\delta(v) \geq 1$. For otherwise, there is no way to move a pebble to v by using the distribution δ and pebbling moves. Again, by the definition of δ^*, we have $\delta^*(T-v,u) = \delta(T-v,u) + \delta(v) \geq 2 = \beta(u)$. This concludes the proof. ■

Proposition 2.2. *Let T be a caterpillar of order $n \geq 3$ and P be a body of T. If α is a pebbling type of P defined by $\alpha(v) = 2$ provided that v is a joint of T and $\alpha(v) = 1$ otherwise. Then $f'(T) = f'_\alpha(P)$.*

Proof. This is a direct consequence of Lemma 2.1, by adding legs to P repeatedly. ■

In what follows, we try to find an explicit formula for finding the optimal α-weighted pebbling of a path P with $\alpha(v) = 1$ or 2 for each $v \in V(P)$. Therefore, the optimal pebbling of a caterpillar can be obtained accordingly.

Throughout the rest of this paper, T is a caterpillar, P is a body of T and α is a pebbling type of P which is defined as in Proposition 2.2. Moreover, we let $S_1 = \{v \in V(P) | \delta(v) = 0 \text{ and } \delta(P,v) = 1\}$ where δ is a distribution of P. Then the following fact is obvious.

Fact 3. Let δ be a distribution of P. If $v \in S_1$ then there exists exactly one vertex u adjacent to v which satisfies the inequalities $2 \leq \delta(P,u) \leq 3$.

We start with finding a good lower bound for $f'_\alpha(P)$.

Proposition 2.3. *If δ is an α-weighted pebbling of P, then $\delta(P) \geq |V(P)| - \lfloor \frac{1}{2}|S_1| \rfloor$.*

Proof. Since $\alpha(v) \geq 1$, it is easy to see that the lemma holds for $|V(P)| \leq 3$. Let δ be an α-weighted pebbling of P, and let $S_0 = \{v \in V(P) | \delta(v) = 0\}$. Note that $\delta(P,v) \geq 2$ for each $v \in S_0 - S_1$. Hence, we have

$$\sum_{v \in S_0} \delta(P,v) \geq |S_1| + 2|S_0 - S_1| = 2|S_0| - S_1. \tag{1}$$

Let $P = v_1 v_2 \cdots v_n$ where $n > 3$. For $i = 1, 2, \cdots n$, we define the subpath $L_i = v_1 v_2 \cdots v_i$ and the subpath $R_i = v_i v_{i+1} \cdots v_n$. For convenience, we denote $\delta(L_i, v_i)$ by $\ell(v_i)$ and $\delta(R_i, v_i)$ by $r(v_i)$ for $1 \leq i \leq n$. Then it is easy to see that $\ell(v_1) = \delta(v_1)$, $r(v_n) = \delta(v_n)$, $\ell(v_i) = \delta(v_i) + \lfloor \frac{\ell(v_{i-1})}{2} \rfloor$ for $2 \leq i \leq n$, and $r(v_i) = \delta(v_i) + \lfloor \frac{r(v_{i+1})}{2} \rfloor$ for $1 \leq i \leq n-1$. This implies that $\delta(P,v) = \ell(v) + r(v)$ if $\delta(v) = 0$. So, we have

$$\sum_{v \in S_0} \delta(P,v) = \sum_{v \in S_0} \ell(v) + \sum_{v \in S_0} r(v).$$

Now, we will prove that $\sum_{v \in S_0} \ell(v) \leq \delta(P) - |V(P)| + |S_0|$. Let s be a positive number. We define $\phi_0(s) = s$ and $\phi_i(s) = \lfloor \frac{\phi_{i-1}(s)}{2} \rfloor$ for each positive integer i. Clearly,

$$\sum_{i=1}^{t} \phi_i(s) \leq s - 1 \text{ for any positiveinteger } t. \tag{2}$$

Consider the subpath $P'' = v_{k+1}v_{k+2}\cdots v_{k+\ell}v_{k+\ell+1}\cdots v_{k+\ell+m}$, which satifies that $\delta(v_{k+i}) \geq 1$ for $1 \leq i \leq \ell$ and $v_{k+\ell+j} \in S_0$ for $1 \leq j \leq m$. Since $\delta(v_{k+\ell+j}) = 0$ for $1 \leq j \leq m$, we have

$$\ell(v_{k+\ell+j}) = \phi_j(\ell(v_{k+\ell})) \text{ for } 1 \leq j \leq m. \tag{3}$$

By (2), we also have

$$\ell(v_{k+j}) = \phi_1(\ell(v_{k+j-1})) + \delta(v_{k+j}) \leq \ell(v_{k+j-1}) - 1 + \delta(v_{k+j}) \text{ for } 2 \leq j \leq \ell. \tag{4}$$

By (3), we have

$$\phi_{m+1}(\ell(v_{k+\ell})) = \phi_1(\phi_m(\ell(v_{k+\ell})) = \phi_1(\ell(v_{k+\ell+m})). \tag{5}$$

By combining (2) and (4), we obtain

$$\begin{aligned}
\sum_{j=1}^{m+1} \phi_j(\ell(v_{k+\ell})) &\leq \ell(v_{k+\ell}) - 1 \\
&\leq \ell(v_{k+\ell-1}) - 1 + \delta(v_{k+\ell}) - 1 \\
&= \ell(v_{k+\ell-2}) - 1 + \delta(v_{k+\ell-1}) - 1 + \delta(v_{k+\ell}) - 1 \\
&\quad \vdots \\
&\leq (\ell(v_{k+1}) - 1) + \sum_{j=2}^{\ell} (\delta(v_{k+j}) - 1).
\end{aligned}$$

Also, by Combining (3) and (5), we have

$$\begin{aligned}
\sum_{v \in V(P'') \bigcap S_0} \ell(v) &= \sum_{j=1}^{m} \ell(v_{k+\ell+j}) \\
&= \sum_{j=1}^{m} \phi_j(\ell(v_{k+\ell})) = \sum_{i=1}^{m+1} \phi_j(\ell(v_{k+\ell})) - \phi_{m+1}(\ell(v_{k+\ell})) \\
&\leq (\ell(v_{k+1}) - 1) - \phi_1(\ell(v_{k+\ell+m})) + \sum_{j=2}^{\ell} (\delta(v_{k+j}) - 1).
\end{aligned}$$

If $k=0$ then $\ell(v_{k+1})=\ell(v_1)=\delta(v_1)=\delta(v_{k+1})$, and so we have

$$\begin{aligned}\sum_{v\in V(P'')\bigcap S_0}\ell(v) &\le -\phi_1(\ell(v_{k+\ell+m}))+\sum_{j=1}^{\ell}(\delta(v_{k+\ell})-1)\\ &= -\phi_1(\ell(v_{k+\ell+m}))+\sum_{v\in V(P'')-S_0}(\delta(v)-1).\end{aligned} \tag{6}$$

Otherwise, $k\ge 1$ then $\ell(v_{k+1})=\phi_1(\ell(v_k)+\delta(v_{k+1}))$. This implies that

$$\sum_{v\in V(P'')\bigcap S_0}\ell(v)=\phi_1(\ell(v_k))-\phi_1(\ell(v_{k+\ell+m}))+\sum_{v\in V(P'')-S_0}(\delta(v)-1). \tag{6'}$$

Let $P=P_0\sim P_1'\sim P_1\sim P_2'\sim\cdots\sim P_{m-1}\sim P_m'\sim P_m$ where P_i' is the maximal subpath such that for each vertex $v\in V(P_i')$, $\delta(v)\ge 1$ for $1\le i\le m$. Note that $P_0=\emptyset$ if $v_1\in V(P_1')$ and $P_m=\emptyset$ if $v_n\in V(P_m')$. We also let u_i be the rightmost vertex of P_i for $1\le i\le m$. Obviously, if $P_0\ne\emptyset$, then we have

(i) $\ell(u_0)=0$ and $\sum_{v\in V(P_0)}\ell(v)=0$;

and by combining (6), (6′) and (i), we obtain

(ii) $\sum_{v\in V(P_1)}\ell(v)\le -\phi_1(\ell(u_1))+\sum_{v\in V(P_i')}(\delta(v)-1)$ and $\sum_{v\in V(P_i)}\ell(v)\le \phi_1(\ell(u_{i-1}))$ $-\phi_1(\ell(u_i))+\sum_{v\in V(P_i')}(\delta(v)-1)$ for $2\le i\le m-1$; and

(iii) if $P_m=\emptyset$ then $\sum_{v\in V(P_m)}\ell(v)=0\le\phi_1(\ell(u_{m-1}))+\sum_{v\in V(P_m')}(\delta(v)-1)$.

Otherwise, $P_m\ne\emptyset$ which implies

$$\begin{aligned}&\sum_{v\in V(P_m)}\ell(v)\le\phi_1(\ell(u_{m-1}))-\phi_1(\ell(u_m))+\sum_{v\in V(P_m')}(\delta(v)-1)\\ &\le\phi_1(\ell(u_{m-1}))+\sum_{v\in V(P_m')}(\delta(v)-1).\end{aligned}$$

By combining (i), (ii) and (iii), we have

$$\sum_{v\in S_0}\ell(v)=\sum_{i=0}^{m}\sum_{v\in V(P_i)}\ell(v)\le\sum_{i=1}^{m}\sum_{v\in V(P_i')}(\delta(v)-1).$$

Since $\bigcup_{i=1}^{m}V(P_i')=V(P)-S_0$ and $\delta(P)=\sum_{v\in V(P)-S_0}\delta(v)$, we obtain

$$\sum_{v\in S_0}\ell(v)\le\sum_{v\in V(P)-S_0}(\delta(v)-1)=\sum_{v\in V(P)-S_0}\delta(v)-(|V(P)|-|S_0|)$$
$$=\delta(P)-|V(P)|+|S_0|.$$

By a similar argument as above, we also have $\sum_{v\in S_0} r(v)\le\delta(P)-|V(P)|+|S_0|$. Hence,

$$\sum_{v\in S_0}\delta(P,v)=\sum_{v\in S_0}\ell(v)+\sum_{v\in S_0}r(v)\le 2(\delta(P)-|V(P)|+|S_0|). \tag{7}$$

By combining (1) and (7), we have $2|S_0|-|S_1|\le\sum_{v\in S_0}\delta(P,v)\le 2(\delta(P)-|V(P)|+|S_0|)$, and the proof is complete. ■

In order to determine $f'_\alpha(P)$, we also need the following notions. A subpath Q of P is said to be 1-*maximal* with respect to α, if Q is a maximal connected subgraph of P such that for each $v\in V(Q)$, $\alpha(v)=1$ and for each vertex u which is adjacent to v, $\alpha(u)=1$; and Q is 2-*maximal* with respect to α, if Q is a maximal connected subgraph of P such that for each adjacent pair u and w in $V(Q)$, $\alpha(u)=1$ implies $\alpha(w)=2$ or $\alpha(u)=\alpha(w)=2$. For clearness, we give an example in Figure 2.

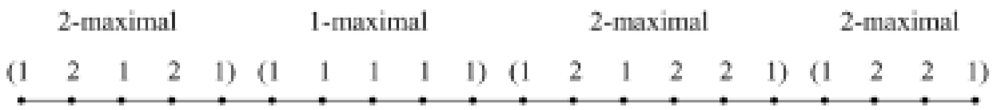

Fig. 2. $\alpha(P)$.

The following result can be derived from Theorem 1.1.

Lemma 2.4. *Let Q be a* 1*-maximal subpath of P with respective to α. Then* $f'_\alpha(Q)=|V(Q)|-\left\lfloor\frac{|V(Q)|}{3}\right\rfloor$.

Lemma 2.5. *Let $Q=v_1v_2\cdots v_k$, $k\ge 3$ be a* 2*-maximal subpath of P with respective to α. Then $f'_\alpha(Q)=k-1$.*

Proof. Let δ be an optimal α-weighted pebbling of P. Then by Fact 3, $v_i\notin S_1$ for $2\le i\le k-1$. This implies that $|S_1|\le 2$. By Proposition 2.3, we have $f'_\alpha(P)\ge k-1$. Now, by letting δ be the distribution satisfying $\delta(v_1)=\delta(v_k)=0$, $\delta(v_2)=2$ and $\delta(v_i)=1$ for $3\le i\le k-1$, we have $f'_\alpha(P)\le k-1$. This concludes the proof. ■

Lemma 2.6. *Let* $P = v_1v_2\cdots v_n$ *and* $Q = v_iv_{i+1}\cdots v_{i+k+1}$ *be a subpath of* P *where* $2 \le i < i+k+1 \le n-1$*, and let* δ *be a* α*-weighted pebbling of* P*. If* $\alpha(v_{i-1}) = \alpha(v_{i+k+2}) = 2$ *and* $\alpha(v_{i+j}) = 1$ *for* $j = 0, 2, \cdots k+1$*, then* $|V(Q)\bigcap S_1| \le 2\lfloor \frac{k}{3} \rfloor + 2$.

Proof. Let $S_2 = \{v \in V(Q) | \delta(P, v) \ge 2\}$ and $|S_2| = x$. By the hypothsis and Fact 3, we conclude that

$$\text{if } v_{i+1} \in S_2 \text{ then } v_i \notin S_1 \text{ and } if\ v_{i+k} \in S_2 \text{ then } v_{i+k+1} \notin S_1. \tag{A}$$

By Fact 3, each vertex v in $(V(Q) \setminus \{v_i, v_{i+k+1}\}) \bigcap S_1$ is adjacent to exactly one vertex in S_2. Also, since Q is a subpath, each vertex in S_2 is adjacent to at most two vertices in $(V(Q) \setminus \{v_i, v_{i+k+1}\}) \bigcap S_1$. Hence, we have $|V(Q) \bigcap S_1| \le 2x+2$. This implies

$$x \ge |V(Q) \bigcap S_1|/2 - 1. \tag{B}$$

Since $V(Q) \bigcap S_1 \subseteq V(Q) - S_2$, we have

$$|V(Q) \bigcap S_1| \le |V(Q)| - |S_2| = |V(Q)| - x \le k + 3 - |V(Q) \bigcap S_1|/2.$$

It follows that $|V(Q) \bigcap S_1| \le \lfloor \frac{2k}{3} \rfloor + 2$. Let $k = 3t + r$ and $0 \le r \le 2$. Then $t = \lfloor \frac{k}{3} \rfloor$.

Case 1. $r = 0$ or 1. We have $|V(Q) \bigcap S_1| \le 2t + \lfloor \frac{2r}{3} \rfloor + 2 = 2t + 2 = 2\lfloor \frac{k}{3} \rfloor + 2$.

Case 2. $r = 2$. We have $|V(Q) \bigcap S_1| \le 2t + \lfloor \frac{2\times 2}{3} \rfloor + 2 = 2t + 3$. Suppose that $|V(Q) \bigcap S_1| = 2t+3$. Then $x \le |V(Q)| - |V(Q) \bigcap S_1| = (3t+2+2) - (2t+3) = t+1$. But, we have $x \ge (2t+3)/2 - 1 = t + 1/2$ by (B) and thus $x = t+1$. This implies that $|S_2| + |V(Q) \bigcap S_1| = 3t + 4 = k + 2 = |V(Q)|$. Therefore, for each $v \in V(Q)$, either $v \in S_1$ or $v \in S_2$. By (A), Fact 3 and the hypothesis, if $v_{i+j} \in S_2$ and $v_{i+j+1} \in S_1$ then $v_{i+j+2} \in S_1$, and if $v_{i+j} \in S_1$ and $v_{i+j+1} \in S_1$ then $v_{i+j+2} \in S_2$ for $0 \le j \le k-1$. This implies that $|V(Q) \bigcap S_1|$ is even. It is a contradiction to our assumption. Therefore, we have $|V(Q) \bigcap S_1| \le 2t + 2 = 2\lfloor \frac{k}{3} \rfloor + 2$. ■

Now, we are ready for the main theorem.

Theorem 2.7. *Let* T *be a caterpillar with* P *a body of* T *and* $|V(P)| = n \ge 3$*. Let* $\alpha(v) = 2$ *if* v *is a joint of* T *and* $\alpha(v) = 1$ *otherwise. Let* $P'_1, P'_2, \cdots, P'_m$ *be* 2*-maximal subpaths of* P *with respect to* α *and* P_i *be a subpath between* P'_i *and* P'_{i+1} *for* $i = 1, 2, \cdots, m-1$*. Then* $f'(T) = n - m - \displaystyle\sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor$.

Proof. By Proposition 2.2, it suffices to prove $f'_\alpha(P) = n-m-\sum_{i=1}^{m-1}\left\lfloor \frac{|V(P_i)|}{3} \right\rfloor$. First, if $m = 1$, the proof follows by Lemma 2.5. Now, we let $m > 1$. Clearly, for each $i = 1, 2, \ldots, m-1$, P_i is either an empty graph or a 1-maximal subpath of P with respect to α. If P_i is an empty graph then $|V(P_i)| - \lfloor \frac{|V(P_i)|}{3} \rfloor = 0$. Combining with Lemma 2.4 and Lemma 2.5, it is easy to see

$$\begin{aligned} f'_\alpha(P) &\leq \sum_{i=1}^{m}(|V(P'_i)|-1) + \sum_{i=1}^{m-1}\left(|V(P_i)| - \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor\right) \\ &= \left(\sum_{i=1}^{m}|V(P'_i)| + \sum_{i=1}^{m-1}|V(P_i)|\right) - m - \sum_{i=1}^{m-1}\left\lfloor \frac{|V(P_i)|}{3} \right\rfloor \\ &= n - m - \sum_{i=1}^{m-1}\left\lfloor \frac{|V(P_i)|}{3} \right\rfloor . \end{aligned}$$

Therefore, it is left to show that the above upper bound is also a lower bound. By Proposition 2.3, we have $f'_\alpha(P) \geq |V(P)| - \left\lfloor \frac{1}{2}|S_1| \right\rfloor = n - \left\lfloor \frac{1}{2}|S_1| \right\rfloor$. Hence, it suffices to show $|S_1| \leq 2m + \sum_{i=1}^{m-1} 2\left\lfloor \frac{|V(P_i)|}{3} \right\rfloor$.

For $1 \leq i \leq m$, let u_i and w_i be the leftmost vertex and the rightmost vertex of P'_i respectively. Note that w_i and u_{i+1} are adjacent to the left end vertex and the right end vertex of P_i respectively. We denote the subpath induced by $\{w_i, u_{i+1}\} \bigcup V(P_i)$ by $w_i \sim P_i \sim u_{i+1}$. By Fact 3, we have $\{V(P'_i) - \{u_i, w_i\}\} \bigcap S_1 = \emptyset$ for $1 \leq i \leq m$. This implies that

$$S_1 = V(P) \bigcap S_1 = \{\{u_1, w_m\} \bigcap S_1\} \bigcup \{ \bigcup_{i=1}^{m-1} [V(w_i \sim P_i \sim u_{i+1}) \bigcap S_1]\}.$$

Hene, we have

$$|S_1| \leq |\{u_1, w_m\}| + \sum_{i=1}^{m-1} |V(w_i \sim P_i \sim u_{i+1}) \bigcap S_1|.$$

By Lemma 2.6, we have

$$|V(w_i \sim P_i \sim u_{i+1}) \bigcap S_1| \leq 2\left\lfloor \frac{|V(P_i)|}{3} \right\rfloor + 2 \text{ for } 1 \leq i \leq m-1.$$

Therefore, we obtain

$$|S_1| \leq 2 + \sum_{i=1}^{m-1}(2\left\lfloor \frac{|V(P_i)|}{3} \right\rfloor + 2) = 2m + \sum_{i=1}^{m-1} 2\left\lfloor \frac{|V(P_i)|}{3} \right\rfloor .$$

This concludes the proof. ∎

Before we finish this paper, we give an example to clarify the idea used in this paper.

Example. Let T be a caterpillar in Figure 3. Here, $n = 25$, $m = 4$, $\sum_{i=1}^{m-1} \left\lfloor \frac{|V(P_i)|}{3} \right\rfloor = 2$ and $f'(T) = 25 - 4 - 2 = 19$.

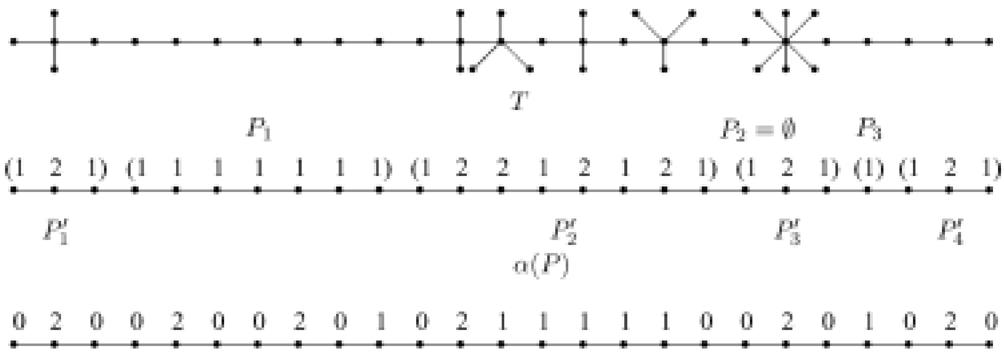

Fig. 3. An optimal α-weighted pebbling of P.

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