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THE EXACT DISTRIBUTIONS OF XY AND X/Y FOR SOME ELLIPTICALLY SYMMETRIC DISTRIBUTIONS

Saralees Nadarajah

Abstract. The exact distributions of the product XY and the ratio X/Y are derived when (X, Y) has the elliptically symmetric Pearson type II distribution, elliptically symmetric Pearson type VII distribution and the elliptically symmetric Kotz type distribution.

1. INTRODUCTION

For a bivariate random vector (X, Y), the distributions of the product XY and the ratio X/Y are of interest in many areas of the sciences, engineering and medicine. Examples of XY include traditional portfolio selection models, relationship between attitudes and behavior, number of cancer cells in tumor biology and stream flow in hydrology. Examples of X/Y include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and safety factor in engineering.

In this paper, we study the exact distributions of XY and X/Y when (X, Y) has the

1. elliptically symmetric Pearson type II distribution given by the joint pdf

$$f(x,y) = \frac{N+1}{\pi\sqrt{1-\rho^2}}\left(1-\frac{(x-\alpha)^2+(y-\beta)^2-2\rho(x-\alpha)(y-\beta)}{1-\rho^2}\right)^N \tag{1}$$

for $\{(x-\alpha)2+(y-\beta)2-2\rho(x-\alpha)(y-\beta)\}/(1-\rho^2) < 1$, $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < \alpha < \infty$, $-\infty < \beta < \infty$, $N > -1$, and $-1 < \rho < 1$.

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2. elliptically symmetric Pearson type VII distribution given by the joint pdf

$$f(x,y)=\frac{N-1}{\pi m\sqrt{1-\rho^2}}\left(1+\frac{(x-\alpha)^2+(y-\beta)^2-2\rho(x-\alpha)(y-\beta)}{m\left(1-\rho^2\right)}\right)^{-N} \quad (2)$$

   for $-\infty<x<\infty$, $-\infty<y<\infty$, $-\infty<\alpha<\infty$, $-\infty<\beta<\infty$, $N>1$, $m>0$, and $-1<\rho<1$. The bivariate t-distribution and the bivariate Cauchy distribution are special cases of (2) for $N=(m+2)/2$ and $m=1$, $N=3/2$, respectively.

3. elliptically symmetric Kotz type distribution given by the joint pdf

$$\begin{aligned} f(x,y) &= \frac{sr^{N/s}\left\{(x-\alpha)^2+(y-\beta)^2-2\rho(x-\alpha)(y-\beta)\right\}^{N-1}}{\pi\Gamma\left(N/s\right)\left(1-\rho^2\right)^{N-1/2}} \\ &\quad \times \exp\left\{-r\left(\frac{(x-\alpha)^2+(y-\beta)^2-2\rho(x-\alpha)(y-\beta)}{1-\rho^2}\right)^s\right\} \end{aligned} \quad (3)$$

   for $-\infty<x<\infty$, $-\infty<y<\infty$, $-\infty<\alpha<\infty$, $-\infty<\beta<\infty$, $N>0$, $r>0$, $s>0$ and $-1<\rho<1$. When $s=1$, this is the original Kotz distribution introduced in Kotz (1975). When $N=1$, $s=1$ and $r=1/2$, (3) reduces to a bivariate normal density.

The parameter ρ is the correlation coefficient between the x and y components. For details on properties of these distributions see Johnson (1987), Fang *et al.* (1990), and Kotz and Nadarajah (2004).

The last two decades have seen a vigorous development of elliptically symmetric distributions as direct generalizations of the multivariate normal distribution which has dominated statistical theory and applications for almost a century. Elliptically symmetric distributions retain most of the attract properties of the multivariate normal distribution. The distributions mentioned above are three of the most popular elliptical symmetric distributions. For instance, since 1990, there has been a surge of activity relating to the elliptical symmetric Kotz type distribution.

It has attracted applications in areas such as Bayesian statistics, ecology, discriminant analysis, mathematical finance, repeated measurements, shape theory and signal processing. The elliptical symmetric Pearson type VII is becoming increasingly important in classical as well as in Bayesian statistical modeling. Its application is a very promising approach in multivariate analysis. Classical multivariate analysis is soundly and rigidly tilted toward the multivariate normal distribution while the Pearson type VII distribution offers a more viable alternative with respect to real-world data, particularly because its tails are more realistic. We have seen recently some unexpected applications in novel areas such as cluster analysis, discriminant analysis, missing data imputation, multiple regression, portfolio optimization, robust projection indices, security returns, and speech recognition.

The explicit expressions for the distributions of XY and X/Y are given in Sections 2, 3 and 4. The calculations involve the complementary incomplete gamma function defined by

$$\Gamma(a,x) = \int_x^\infty t^{a-1} \exp(-t)\, dt$$

and the Gauss hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!},$$

where $(c)_k = c(c+1) \cdots (c+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. Pearson Type II Distribution

Theorem 1 derives an explicit expression for the pdf of $Z = XY$ in terms of the Gauss hypergeometric function.

Theorem 1. *Suppose X and Y are jointly distributed according to* (1). *Then, the pdf of $Z = XY$ can be expressed as*

$$f(z) = (N+1) B(N+1, N+1) \frac{(C-A)^{2N+1}}{A^{N+1}} \, {}_2F_1\left(N+1, N+1; 2N+2; 1-\frac{C}{A}\right) \quad (4)$$

for $(\rho-1)/2 + \mid \rho \mid (\mid \rho \mid - \rho)/2 \leq z \leq (\rho+1)/2 - \mid \rho \mid (\mid \rho \mid + \rho)/2$*, where*

$$A = \frac{1}{2}\left\{1 - \rho^2 + 2\rho z - \sqrt{1-\rho^2}\sqrt{1-(2z-\rho)^2}\right\} \quad (5)$$

and

$$C = \frac{1}{2}\left\{1 - \rho^2 + 2\rho z + \sqrt{1-\rho^2}\sqrt{1-(2z-\rho)^2}\right\}. \quad (6)$$

Proof. Set $(X, Y) = (X, Z/X)$. Under this transformation, the Jacobian is $1/ \mid X \mid$ and so one can express the joint pdf of (X, Z) as

$$
\begin{aligned}
f(x, z) &= \frac{N+1}{\pi\sqrt{1-\rho^2}} \left(1 - \frac{x^2 + z^2/x^2 - 2\rho z}{1-\rho^2}\right)^N \\
&= \frac{(N+1)\left(C - x^2\right)^N \left(x^2 - A\right)^N}{\pi\left(1-\rho^2\right)^{N+1/2} \mid x \mid^{2N+1}},
\end{aligned}
\tag{7}
$$

where A and C are given by (5) and (6), respectively. Thus, the marginal pdf of Z can be written as

$$
\begin{aligned}
f(z) &= \frac{2(N+1)}{\pi\left(1-\rho^2\right)^{N+1/2}} \int_{\sqrt{A}}^{\sqrt{C}} \frac{\left(C - x^2\right)^N \left(x^2 - A\right)^N}{\mid x \mid^{2N+1}} dx \\
&= \frac{N+1}{\pi\left(1-\rho^2\right)^{N+1/2}} \int_{A}^{C} \frac{(C-y)^N (y-A)^N}{y^{N+1}} dy,
\end{aligned}
\tag{8}
$$

where the last step follows by substituting $y = x^2$. Note that $A \geq 0$ and $C \geq 0$ over the range of values of z stated. The result of the theorem follows by directly applying equation $(2.2.6.1)$ in Prudnikov *et al.* $(1986$, Volume $1)$ to calculate the integral in (8). ∎

Using special properties of the Gauss hypergeometric function, one can derive simpler forms of (4) for integer values of N. This is illustrated in the corollary below.

Corollary 1.1. *Suppose X and Y are jointly distributed according to (1) with $N \geq 0$ taking integer values. Then, the pdf of $Z = XY$ can be expressed as*

$$
\begin{aligned}
f(z) = {} & \frac{2(N+1)}{\pi\left(1-\rho^2\right)^{N+1/2}} \sum_{j=0}^{N} \sum_{k=0}^{N} (-1)^{N+j-k} \binom{N}{j} \binom{N}{k} \\
& C^{N-j} A^{N-k} \delta(2j + 2k - 2N - 1)
\end{aligned}
\tag{9}
$$

for $(\rho - 1)/2 + \mid \rho \mid (\mid \rho \mid - \rho)/2 \leq z \leq (\rho + 1)/2 - \mid \rho \mid (\mid \rho \mid + \rho)/2$, where A and C are given by (5) and (6), respectively, and

$$
\delta(m) = \begin{cases}
\dfrac{C^{(m+1)/2} - A^{(m+1)/2}}{m+1}, & \textit{if} \quad m \geq 0, \\
\log\sqrt{C} - \log\sqrt{A}, & \textit{if} \quad m = -1, \\
\dfrac{A^{(m+1)/2} - C^{(m+1)/2}}{m+1}, & \textit{if} \quad m \leq -2.
\end{cases}
$$

Theorem 2 derives an explicit elementary expression for the pdf of $Z = X/Y$. Note that the resulting pdf depends only on ρ.

Theorem 2. *If X and Y are jointly distributed according to (1) then the pdf of $Z = X/Y$ is*

$$f(z) = \frac{2g(\arctan(z))}{1+z^2} \tag{10}$$

for $-\infty < z < \infty$, where

$$g(\theta) = \frac{\sqrt{1-\rho^2}}{2\pi\left\{1-\rho\sin(2\theta)\right\}}.$$

Proof. Set $(X,Y) = (\alpha + R\sin\Theta, \beta + R\cos\theta)$. Under this transformation, the Jacobian is R and so one can express the joint pdf of (R,Θ) as

$$g(r,\theta) = \frac{(N+1)\left\{1-\rho\sin(2\theta)\right\}^N}{\pi\left(1-\rho^2\right)^{N+1/2}} r\left(W-r^2\right)^N, \tag{11}$$

where $W = (1-\rho^2)/\{1-\rho\sin(2\theta)\}$. The region of integration $\{(x-\alpha)2 + (y-\beta)2 - 2\rho(x-\alpha)(y-\beta)\}/(1-\rho^2) < 1$ reduces to $r^2 \leq W$ or equivalently $r \leq \sqrt{W}$. Thus, the marginal pdf of Θ can be obtained as

$$\begin{aligned} g(\theta) &= \frac{(N+1)\left\{1-\rho\sin(2\theta)\right\}^N}{\pi\left(1-\rho^2\right)^{N+1/2}} \int_0^{\sqrt{W}} r\left(W-r^2\right)^N dr \\ &= \frac{\left\{1-\rho\sin(2\theta)\right\}^N}{2\pi\left(1-\rho^2\right)^{N+1/2}} W^{N+1} \\ &= \frac{\sqrt{1-\rho^2}}{2\pi\left\{1-\rho\sin(2\theta)\right\}}. \end{aligned}$$

The result of the theorem follows by noting that the pdf of $Z = \tan\Theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1+z^2} \tag{12}$$

and that $g(\arctan(\theta)) = g(\pi + \arctan(\theta))$ for the form for $g(\cdot)$. ∎

3. Pearson Type vii Distribution

Theorem 3 derives an explicit expression for the pdf of $Z = XY$ in terms of the Gauss hypergeometric function.

Theorem 3. *Suppose X and Y are jointly distributed according to (2). Then, the pdf of $Z = XY$ can be expressed as*

$$
\begin{aligned}
f(z) = {} & \frac{N-1}{\pi} m^{N-1} B\left(N+\frac{1}{2}, N-\frac{1}{2}\right)\left(1-\rho^2\right)^{N-1/2} z^{1/2-N} \\
& \times {}_2F_1\left(\frac{N}{2}+\frac{1}{4}, \frac{N}{2}-\frac{1}{4}; N+\frac{1}{2}; 1-\frac{\left\{m\left(1-\rho^2\right)-2\rho z\right\}^2}{4z^2}\right)
\end{aligned}
\tag{13}
$$

for $-\infty < z < \infty$, provided that $\{m\left(1-\rho^2\right)-2\rho z\}^2 < 4z^2$.

Proof. Set $(X, Y) = (X, Z/X)$.

Under this transformation, the Jacobian is $1/\mid X \mid$ and so one can express the joint pdf of (X, Z) as

$$
\begin{aligned}
f(x, z) & = \frac{N-1}{\pi m\sqrt{1-\rho^2} \mid x \mid}\left(1+\frac{x^2+z^2/x^2-2\rho z}{m\left(1-\rho^2\right)}\right)^{-N} \\
& = \frac{(N-1) m^{N-1}\left(1-\rho^2\right)^{N-1/2} \mid x \mid^{2N-1}}{\pi\left[x^4+\left\{m\left(1-\rho^2\right)-2\rho z\right\} x^2+z^2\right]^N}.
\end{aligned}
\tag{14}
$$

Thus, the marginal pdf of Z can be written as

$$
\begin{aligned}
f(z) & = \frac{(N-1) m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{\pi} \int_{-\infty}^{\infty} \frac{\mid x \mid^{2N-1}}{\left[x^4+\left\{m\left(1-\rho^2\right)-2\rho z\right\} x^2+z^2\right]^N} dx \\
& = \frac{2(N-1) m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{\pi} \int_0^{\infty} \frac{x^{2N-1}}{\left[x^4+\left\{m\left(1-\rho^2\right)-2\rho z\right\} x^2+z^2\right]^N} dx \\
& = \frac{(N-1) m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{\pi} \int_0^{\infty} \frac{y^{N-1/2}}{\left[y^2+\left\{m\left(1-\rho^2\right)-2\rho z\right\} y+z^2\right]^N} dy,
\end{aligned}
\tag{15}
$$

where the last step follows by substituting $y = x^2$. The result of the theorem follows by direct application of equation (2.2.9.7) in Prudnikov *et al.* (1986, volume 1) to calculate the integral in (15). ∎

Using special properties of the Gauss hypergeometric function, one can derive simpler forms of (13) for integer and half integer values of N. This is illustrated in the two corollaries below.

Corollary 2. *Suppose X and Y are jointly distributed according to (2) with $N \geq 2$ taking integer values. Then, the pdf of $Z = XY$ can be expressed as*

$$f(z) = \frac{(2N-3)!!m^{N-1}(1+\rho)^{N-1/2}}{2^{N-1}(N-2)!\left\{m(1+\rho)+2z\right\}^{N-1/2}} \tag{16}$$

for $-\infty < z < \infty$, provided that $\left\{m\left(1-\rho^2\right)-2\rho z\right\}^2 < 4z^2$ for $N = 2, 3, 4, \ldots$ or that $\left\{m\left(1-\rho^2\right)-2\rho z\right\}^2 > 4z^2$ for $N = 3, 5, 7, \ldots$. Here, $(2i+1)!!$ is defined as $1 \cdot 3 \cdots (2i+1)$.

Corollary 3. *Suppose X and Y are jointly distributed according to (2) with $N > 1$ taking half integer values. Then, the pdf of $Z = XY$ can be expressed as*

$$\begin{aligned} f(z) = {} & \frac{(N-1)(N-3/2)!m^{N-1}2^{2N-4}\left(1-\rho^2\right)^{N-1/2}}{\pi(2N-2)!z} \\ & \left\{m\left(1-\rho^2\right)+2z(\rho+1)\right\}^{1/2-N} \end{aligned}$$

for $-\infty < z < \infty$, provided that $m\left(1-\rho^2\right) > 2\rho z$.

Theorem 4 derives an explicit expression for the pdf of $Z = X/Y$ in terms of the Gauss hypergeometric function.

Theorem 4. *If X and Y are jointly distributed according to (2) and let*

$$D(\theta) = 1 - \rho\sin(2\theta), \tag{17}$$

$$E(\theta) = (\rho\beta - \alpha)\cos\theta + (\rho\alpha - \beta)\sin\theta, \tag{18}$$

and

$$F = \alpha^2 + \beta^2 - 2\rho\alpha\beta + m\left(1-\rho^2\right). \tag{19}$$

Then, provided that $E^2(\arctan(z)) < FD(\arctan(z))$, the pdf of $Z = X/Y$ can be expressed as

$$f(z) = \frac{2g(\arctan(z))}{1+z^2} \tag{20}$$

for $-\infty < z < \infty$, where

$$\begin{aligned} g(\theta) = {} & \frac{F^{1-N}\Gamma(2N-2)(N-1)m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{D\Gamma(2N)\pi} \\ & {}_2F_1\left(1, N-1; N+\frac{1}{2}; 1-\frac{E^2}{DF}\right). \end{aligned} \tag{21}$$

Proof. Set $(X, Y) = (R \sin \Theta, R \cos \theta)$. Under this transformation, the Jacobian is R and so one can express the joint pdf of (R, Θ) as

$$g(r, \theta) = \frac{(N-1) m^{N-1} \left(1-\rho^2\right)^{N-1/2} r}{\pi \left(D r^2 + 2 E r + F\right)^N},$$

where D, E and F are given by (17), (18) and (19), respectively. Thus, the marginal pdf of Θ can be obtained as

$$g(\theta) = \frac{(N-1) m^{N-1} \left(1-\rho^2\right)^{N-1/2}}{\pi} \int_0^\infty \frac{r}{\left(D r^2 + 2 E r + F\right)^N} dr \tag{22}$$

$$= \frac{F^{1-N} \Gamma(2N-2)(N-1) m^{N-1} \left(1-\rho^2\right)^{N-1/2}}{D \Gamma(2N) \pi} {}_2F_1 \left(1, N-1; N+\frac{1}{2}; 1-\frac{E^2}{DF}\right), \tag{23}$$

which follows by a direct application of equation (2.2.6.1) in Prudnikov *et al.* (1986, volume 1). The result of the theorem follows by noting that the pdf of $Z = \tan \Theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1 + z^2}$$

and that $g(\arctan(\theta)) = g(\pi + \arctan(\theta))$ for the form for $g(\cdot)$. ■

Remark 1. Several particular cases can be considered where (21) reduces to elementary forms. For instance, if $\alpha = \beta = 0$ then (21) reduces to $g(\theta) = \sqrt{1-\rho^2}/[2\pi\{1-\rho\sin(2\theta)\}]$. Two other cases are considered below in Theorems 5 and 6.

Theorem 5. *Suppose X and Y are jointly distributed according to (2). Assume $N \geq 2$ is an integer and $E > 0$, where D, E and F are given by (17), (18) and (19), respectively. Then, the pdf of $Z = X/Y$ can be expressed as*

$$f(z) = \frac{2 g(\arctan(z))}{1 + z^2} \tag{24}$$

for $-\infty < z < \infty$, where

$$g(\theta) = \begin{cases} \dfrac{m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{2(-1)^{N-1}\pi(N-2)!}\dfrac{\partial^{N-2}}{\partial F^{N-2}}\left(\dfrac{E}{\left(DF-E^2\right)^{3/2}}\right. \\ \qquad \left. \times \arctan\dfrac{E}{\sqrt{DF-E^2}} - \dfrac{1}{DF-E^2}\right), \textit{ if } DF > E^2, \\ \dfrac{m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{2(-1)^{N-1}\pi(N-2)!}\dfrac{\partial^{N-2}}{\partial F^{N-2}}\left(\dfrac{E}{\left(E^2-DF\right)^{3/2}}\right. \\ \qquad \left. \times \ln\dfrac{\sqrt{DF}}{E+\sqrt{E^2-DF}} - \dfrac{1}{DF-E^2}\right), \textit{ if } DF < E^2, \\ \dfrac{D^{N-2}m^{N-1}\left(1-\rho^2\right)^{N-1/2}}{2\pi(2N-1)E^{2N-2}}, \textit{ if } DF = E^2. \end{cases} \tag{25}$$

Proof. Apply equation (2.2.9.14) in Prudnikov *et al.* (1986, volume 1) to the integral in (22). ■

Theorem 6. *Suppose X and Y are jointly distributed according to (2). Assume $N = M + 1/2$, $M \geq 2$ is an integer and $E > 0$, where D, E and F are given by (17), (18) and (19), respectively. Then, the pdf of $Z = X/Y$ can be expressed as*

$$f(z) = \frac{2g(\arctan(z))}{1+z^2} \tag{26}$$

for $-\infty < z < \infty$, where

$$g(\theta) = \frac{(M-1/2)\, m^{M-1/2}(-2)^M \left(1-\rho^2\right)^M}{2\pi(2M-1)!!}\frac{\partial^{M-2}}{\partial F^{M-2}} \left(\frac{1}{\sqrt{F}\left(\sqrt{DF}+E\right)^2}\right). \tag{27}$$

Proof. Apply equation (2.2.9.20) in Prudnikov *et al.* (1986, volume 1) to the integral in (22). ■

4. Kotz Type Distribution

Theorem 7 derives an explicit expression for the pdf of $Z = XY$ in terms of the complementary incomplete gamma function for the case $\rho \neq 0$. The pdf for $\rho = 0$ given in Corollary 4 can be obtained as the limiting case for $\rho \to 0$.

Theorem 7. *Suppose X and Y are jointly distributed according to (3) with $\rho \neq 0$. Then, the pdf of $Z = XY$ can be expressed as*

$$f(z) = \frac{2}{\pi \Gamma(N/s)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-1-2j}{k} \left\{ g_1(j,k,z) + g_2(j,k,z) \right\} \quad (28)$$

for $-\infty < z < \infty$, where $G = 2 \mid z \mid -2\rho z$, $H = 2 \mid \rho z \mid$,

$$g_1(j,k,z) = \frac{(-1)^j \left(1 - 1\rho^2\right)^{k+1}}{(2\rho z)^{1+k} r^{k/s}} \left\{ \Gamma\left(\frac{N+k}{s}, \frac{rG^s}{\left(1-\rho^2\right)^s} \right) - \Gamma\left(\frac{N+k}{s}, \frac{rH^s}{\left(1-\rho^2\right)^s} \right) \right\},$$

and

$$g_2(j,k,z) = \frac{(-1)^j (2z)^{2j+k}}{\left(1-\rho^2\right)^{2j+k} r^{(1-2j-k)/s}} \Gamma\left(\frac{N-2j-k-1}{s}, \frac{rH^s}{\left(1-\rho^2\right)^s} \right).$$

Proof. Set $(X, Y) = (X, Z/X)$. Under this transformation, the Jacobian is $1/ \mid X \mid$ and so one can express the joint pdf of (X, Z) as

$$f(x,z) = \frac{s r^{N/s} \left(x^2 + z^2/x^2 - 2\rho z\right)^{N-1}}{\pi \Gamma\left(N/s\right) \left(1-\rho^2\right)^{N-1/2} \mid x \mid} \exp\left\{ -r \left(\frac{x^2 + z^2/x^2 - 2\rho z}{1-\rho^2} \right)^s \right\} \quad (29)$$

Set $y(x) = x^2 + z^2/x^2 - 2\rho z$ and note that

$$\frac{dy(x)}{dx} = \frac{2\left(x^4 - z^2\right)}{x^3}.$$

It follows that $y(x)$ decreases over $(-\infty, \sqrt{\mid z \mid})$, increases over $(\sqrt{\mid z \mid}, \infty)$, and is symmetric around $x = \sqrt{\mid z \mid}$ with $y(\sqrt{\mid z \mid}) = G$. Expressing x in terms of y, one can write

$$\frac{1}{\mid x \mid} \frac{dx}{dy(x)} = \pm \frac{1}{2\sqrt{(y + 2\rho z)^2 - 4z^2}}$$

which, by using

$$(1+w)^{-a} = \sum_{m=0}^{\infty} \binom{-a}{m} w^m, \quad (30)$$

can be expanded as

$$\frac{1}{\mid x \mid} \frac{dx}{dy(x)} = \pm \frac{1}{2} \sum_{j=0}^{\infty} \binom{-1/2}{j} (-1)^j \left(4z^2\right)^j (y + 2\rho z)^{-(1+2j)}. \quad (31)$$

The last term within the sum of (31) can be expanded further by using (30). One has to consider the cases $y < H$ and $y > H$: if $y < H$ then

$$\begin{aligned} & \frac{1}{\mid x \mid} \frac{dx}{dy(x)} \qquad (32) \\ & = \pm \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-(1+2j)}{k} (-1)^j \left(4z^2\right)^j (2\rho z)^{-(1+2j+k)} y^k \end{aligned}$$

and if $y > H$ then

$$\begin{aligned} & \frac{1}{\mid x \mid} \frac{dx}{dy(x)} \qquad (33) \\ & = \pm \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-(1+2j)}{k} (-1)^j \left(4z^2\right)^j (2\rho z)^k y^{-(1+2j+k)}. \qquad (33) \end{aligned}$$

Combining (29), (32) and (33), the marginal pdf of Z can be written as

$$\begin{aligned} f(z) = & \frac{2sr^{N/s}}{\pi \Gamma(N/s) \left(1-\rho^2\right)^{N-1}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{-1/2}{j} \binom{-(1+2j)}{k} (-1)^j \left(4z^2\right)^j \\ & \times \left\{ (2\rho z)^{-(1+2j+k)} I_1 + (2\rho z)^k I_2 \right\}, \qquad (34) \end{aligned}$$

where

$$I_1 = \int_G^H y^{N+k-1} \exp\left\{ -\frac{ry^s}{\left(1-\rho^2\right)^s} \right\} dy$$

and

$$I_2 = \int_H^{\infty} y^{N-2-2j-k} \exp\left\{ -\frac{ry^s}{\left(1-\rho^2\right)^s} \right\} dy.$$

By using the definition of the complementary incomplete gamma function, one can express

$$I_1 = \frac{\left(1-\rho^2\right)^{N+k}}{sr^{(N+k)/s}} \left\{ \Gamma\left(\frac{N+k}{s}, \frac{rG^s}{\left(1-\rho^2\right)^s} \right) - \Gamma\left(\frac{N+k}{s}, \frac{rH^s}{\left(1-\rho^2\right)^s} \right) \right\} \qquad (35)$$

and

$$I_2 = \frac{\left(1-\rho^2\right)^{N-2j-k-1}}{sr^{(N-2j-k-1)/s}} \Gamma\left(\frac{N-2j-k-1}{s}, \frac{rH^s}{\left(1-\rho^2\right)^s} \right). \qquad (36)$$

The result in (28) follows by substituting (35) and (36) into (34). ■

Corollary 4. *Suppose X and Y are jointly distributed according to (3) with $\rho = 0$. Then, the pdf of $Z = XY$ can be expressed as*

$$f(z) = \frac{2}{\pi \Gamma(N/s)} \sum_{j=0}^{\infty} \binom{-1/2}{j} \frac{(-1)^j (2z)^{2j}}{r^{(1-2j)/s}} \Gamma\left(\frac{N-2j-1}{s}, rG^s\right) \quad (37)$$

for $-\infty < z < \infty$, where $G = 2 \mid z \mid$.

Theorem 8 derives an explicit expression for the pdf of $Z = X/Y$ in terms of the complementary incomplete gamma function.

Theorem 8. *Suppose X and Y are jointly distributed according to (3) and let*

$$D(\theta) = 1 - \rho \sin(2\theta), \quad (38)$$

$$E(\theta) = (\rho\beta - \alpha)\cos\theta + (\rho\alpha - \beta)\sin\theta, \quad (39)$$

$$U = \alpha^2 + \beta^2 - 2\rho\alpha\beta, \quad (40)$$

and

$$V(\theta) = \left(1 - \rho^2\right)(\alpha \sin\theta - \beta\cos\theta)^2 / D(\theta). \quad (41)$$

Furthermore, define

$$\begin{aligned} g_1(\theta) = \frac{\sqrt{1-\rho^2}}{2D\pi\Gamma(N/s)} \Bigg\{ & \Gamma\left(\frac{N}{s}, \frac{rU^s}{\left(1-\rho^2\right)^s}\right) \\ & - \frac{E}{\sqrt{D}} \sum_{k=0}^{\infty} \left(\frac{E^2}{D} - U\right)^k \binom{-1/2}{k} \left(\frac{r^{1/s}}{1-\rho^2}\right)^{k+1/2} \Delta(U) \Bigg\} \end{aligned} \quad (42)$$

and

$$\begin{aligned} g_2(\theta) = & \frac{E\sqrt{1-\rho^2}}{D^{3/2}\pi\Gamma(N/s)} \\ & \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(\frac{r^{1/s}}{1-\rho^2}\right)^{k+1/2} \left(\frac{E^2}{D} - U\right)^k \left\{\Delta(V) - \Delta(U)\right\}, \end{aligned} \quad (43)$$

where

$$\Delta(a) = \Gamma\left(\frac{N-k-1/2}{s}, \frac{ra^s}{\left(1-\rho^2\right)^s}\right).$$

If $E(\arctan(z)) \geq 0$ *and* $E(\pi + \arctan(z)) \geq 0$ *then the pdf of* $Z = X/Y$ *can be expressed as*

$$f(z) = \frac{g_1(\arctan(z)) + g_1(\pi + \arctan(z))}{1 + z^2} \tag{44}$$

for $-\infty < z < \infty$. *If* $E(\arctan(z)) \geq 0$ *and* $E(\pi + \arctan(z)) < 0$ *then the pdf of* $Z = X/Y$ *can be expressed as*

$$f(z) = \frac{g_1(\arctan(z)) + g_1(\pi + \arctan(z)) + g_2(\pi + \arctan(z))}{1 + z^2} \tag{45}$$

for $-\infty < z < \infty$. *If* $E(\arctan(z)) < 0$ *and* $E(\pi + \arctan(z)) \geq 0$ *then the pdf of* $Z = X/Y$ *can be expressed as*

$$f(z) = \frac{g_1(\arctan(z)) + g_2(\arctan(z)) + g_1(\pi + \arctan(z))}{1 + z^2} \tag{46}$$

for $-\infty < z < \infty$. *Finally, if* $E(\arctan(z)) < 0$ *and* $E(\pi + \arctan(z)) < 0$ *then the pdf of* $Z = X/Y$ *can be expressed as*

$$f(z) = \frac{g_1(\arctan(z)) + g_2(\arctan(z)) + g_1(\pi + \arctan(z)) + g_2(\pi + \arctan(z))}{1 + z^2} \tag{47}$$

for $-\infty < z < \infty$.

Proof. Set $(X, Y) = (T \cos \Theta, T \sin \Theta)$. Under this transformation, the Jacobian is T and so one can express the joint pdf of (T, Θ) as

$$g(t, \theta) = \frac{str^{N/s} \left(Dt^2 + 2Et + U\right)^{N-1}}{\pi \Gamma(N/s) \left(1 - \rho^2\right)^{N-1/2}} \exp\left\{ -\frac{r \left(Dt^2 + 2Et + U\right)^s}{\left(1 - \rho^2\right)^s} \right\}, \tag{48}$$

where D, E and U are given by (38), (39) and (40), respectively. Set $z(t) = Dt^2 + 2Et + U$ and, solving the quadratic equation, note that $t = (-E \pm \sqrt{E^2 - D(U - z)}) / D$ for all possible values of z. We can write

$$\begin{aligned} \frac{dz(t)}{dt} &= 2(Dt + E) \\ &= \pm 2\sqrt{E^2 - D(U - z)}. \end{aligned}$$

Note further that $z(t)$ is an increasing function of t with $z(0) = U$ if $E \geq 0$. On the other hand, if $E < 0$ then $z(t)$ decreases between $0 \leq t \leq z^{-1}(V)$ before increasing for all $t \geq z^{-1}(V)$, where V is given by (41). Thus, the marginal pdf of Θ can be expressed as $g(\theta) = g_1(\theta)$ if $E \geq 0$ and as $g(\theta) = g_l(\theta) + g_2(\theta)$ if $E < 0$, where

$$g_1(\theta) = \frac{sr^{N/s}}{2D\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \int_U^\infty z^{N-1} \exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\} \left\{1 - \frac{E}{\sqrt{E^2 - D(U-z)}}\right\} dz \quad (49)$$

and

$$g_2(\theta) = -\frac{Esr^{N/s}}{D\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \int_V^U \frac{z^{N-1}\exp\left\{-rz^s/\left(1-\rho^2\right)^s\right\}}{\sqrt{E^2-D(U-z)}} dz. \quad (50)$$

These two expressions actually reduce to those given by (42) and (43), respectively, as shown below. Consider (49). Note that $\mid (E^2 - DU)/(Dz) \mid \leq 1$ for all $z \geq U$. Thus, using the series expansion

$$(1+x)^{-1/2} = \sum_{k=0}^\infty \binom{-1/2}{k} x^k,$$

one can expand (49) as

$$\begin{aligned}
g_1(\theta) &= \frac{sr^{N/s}}{2D\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \int_U^\infty z^{N-1}\exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\} \\
&\quad \times \left\{1 - \frac{E}{\sqrt{Dz}} \sum_{k=0}^\infty \binom{-1/2}{k} \left(\frac{E^2 - DU}{Dz}\right)^k\right\} dz \\
&= \frac{sr^{N/s}}{2D\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \left[\int_U^\infty z^{N-1}\exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\} dz \right. \\
&\quad \left. -\frac{E}{\sqrt{D}} \sum_{k=0}^\infty \binom{-1/2}{k} \frac{\left(E^2 - DU\right)^k}{D^k} \int_U^\infty z^{N-k-3/2} \exp\left\{-\frac{rz^s}{\left(1-\rho^2\right)^s}\right\} dz\right] \\
&= \frac{sr^{N/s}}{2D\pi\Gamma(N/s)\left(1-\rho^2\right)^{N-1/2}} \left[\frac{\left(1-\rho^2\right)^N}{sr^{N/s}} \Gamma\left(\frac{N}{s}, \frac{rU^s}{\left(1-\rho^2\right)^s}\right) \right. \\
&\quad -\frac{E}{\sqrt{D}} \sum_{k=0}^\infty \binom{-1/2}{k} \frac{\left(E^2 - DU\right)^k}{D^k} \frac{\left(1-\rho^2\right)^{N-k-1/2}}{sr^{(N-k-1/2)/s}} \\
&\quad \left. \Gamma\left(\frac{N-k-1/2}{s}, \frac{rU^s}{\left(1-\rho^2\right)^s}\right)\right],
\end{aligned}$$

where the last step follows from the definition of the complementary incomplete gamma function. One can similarly show that (50) reduces to the form given in (43). The result of the theorem follows by noting that the pdf of $Z = \tan\Theta$ can be expressed as

$$f(z) = \frac{g(\arctan(z)) + g(\pi + \arctan(z))}{1 + z^2}$$

and that $g(\theta) = g_1(\theta)$ if $E \geq 0$ and $g(\theta) = g_1(\theta) + g_2(\theta)$ if $E < 0$. ∎

Remark 2. If $\alpha = \beta = 0$ then the pdf of $Z = X/Y$ reduces to the simple form

$$f(z) = \frac{2g(\arctan(z))}{1 + z^2} \tag{51}$$

for $-\infty < z < \infty$, where

$$g(\theta) = \frac{\sqrt{1-\rho^2}}{2\pi\left\{1 - \rho\sin(2\theta)\right\}}.$$

If $N = 1$, $s = 1$ and $r = 1/2$ (the bivariate normal case) then the pdf of Z is given by (44)–(47) with $g_1(\cdot)$ and $g_2(\cdot)$ taking the simple forms

$$\begin{aligned} g_1(\theta) = & \frac{1}{\sqrt{2}D\pi}\exp\left\{-\frac{U}{2(1-\rho^2)}\right\}\Bigg[\sqrt{2(1-\rho^2)} \\ & -\frac{E}{\sqrt{D}}\exp\left\{\frac{E^2}{2D(1-\rho^2)}\right\}\Gamma\left(\frac{1}{2}, \frac{E^2}{2D(1-\rho^2)}\right)\Bigg] \end{aligned}$$

and

$$g_2(\theta) = \frac{E}{\sqrt{2D^3}\pi}\exp\left\{\frac{E^2 - DU}{2D(1-\rho^2)}\right\}\left[\Gamma\left(\frac{1}{2}, \frac{U^2}{2D(1-\rho^2)}\right) - \Gamma\left(\frac{1}{2}, \frac{V^2}{2D(1-\rho^2)}\right)\right],$$

compare with Marsaglia (1965) and Hinkley (1969).

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Saralees Nadarajah
School of Mathematics,
University of Manchester,
Manchester M60 1QD,
U.K.
E-mail: saralees.nadarajah@manchester.ac.uk