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ON THE C_0-SEMIGROUPS GENERATED BY SECOND ORDER DIFFERENTIAL OPERATORS ON THE REAL LINE

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Abstract. In this paper we deal with special classes of second order elliptic differential operators on the real line. We show that these operators are generators of positive C_0-semigroups on weighted spaces of continuous functions and we represent them as limits of iterates of integral-type operators.

By means of such representation, some qualitative properties of the semigroups are stated.

1. Introduction

In this paper we consider some classes of (possibly degenerate) elliptic second order differential operators defined on the whole real line and we study them in the setting of weighted spaces of continuous functions, by investigating both their generation properties and the constructive approximation of the semigroup generated by them, through iterates of suitable positive linear operators.

This kind of approach, which is very useful to the study of the qualitative properties of the solutions of diffusion equations, has been applied to a wide class of evolution problems in the setting of continuous function spaces on compact intervals [2-4, 13-15, 22-24], on the interval $[0, +\infty[$ ([6, 8-10, 12, 16, 19, 20] and on compact convex subsets of $\mathbb{R}^n$ (see [7, Chapter 6]). In the case of the whole real line this problem seems to be treated here for the first time.

The paper is organized as follows. In Section 2, for given functions $\alpha, \beta, \gamma \in C(\mathbb{R})$, γ bounded, we prove that the differential operator $Lu := \alpha u'' + \beta u' + \gamma u$, defined on a suitable subspace $D_m(L)$ of the weighted space $C_0^{w_m}(\mathbb{R}) := \Big\{f \in C(\mathbb{R}) \mid \lim\limits_{x\to\pm\infty} \dfrac{f(x)}{1+x^{2m}} = 0\Big\}$, generates a strongly continuous positive semigroup $(T_m(t))_{t\geq 0}$ on $C_0^{w_m}(\mathbb{R})$.

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Moreover we show that this semigroup is the transition semigroup associated with a suitable Markov process on $\mathbb{R}$.

We also investigate the existence of cores for $(L, D_m(L))$ and, in particular, we find conditions under which the space $K^2(\mathbb{R})$ of twice continuously differentiable functions with compact support is such a core.

In Section 3 we represent $(T_m(t))_{t\geq 0}$ as limit of iterates of integral-type operators introduced and studied in [11]. From the properties of these operators we obtain some information on the behavior of such semigroup on several classes of continuous function such as monotone, convex and Lipschitz continuous functions.

In the last section we analyse the particular case where $\beta = \gamma = 0$ and we state some weaker conditions under which similar generation and approximation results hold true.

2. The Differential Operator $(L, D_m(L))$

As usual we shall denote by $C(\mathbb{R})$ the space of all real valued continuous functions on $\mathbb{R}$ and with $C_b(\mathbb{R})$ (resp. $C_0(\mathbb{R})$) the Banach lattice of all bounded continuous functions (resp. continuous functions that vanish at infinity) endowed with the natural order and the uniform norm $\|\cdot\|_\infty$.

We shall also consider the closed subspace $C_*(\mathbb{R})$ of all functions $f \in C_b(\mathbb{R})$ such that $\lim\limits_{x\to\pm\infty} f(x) \in \mathbb{R}$.

The symbol $C^2(\mathbb{R})$ will stand for the space of all twice continuously differentiable functions on $\mathbb{R}$.

For $m \in \mathbb{N}$, setting

$$w_m(x) := \frac{1}{1+x^{2m}} \qquad (x \in \mathbb{R}),$$

we shall denote by $C_0^{w_m}(\mathbb{R})$ the Banach lattice of all functions $f \in C(\mathbb{R})$ such that $w_m f \in C_0(\mathbb{R})$, endowed with the natural order and the weighted norm $\|\cdot\|_m$ defined by $\|f\|_m := \|w_m f\|_\infty$ $(f \in C_0^{w_m}(\mathbb{R}))$.

Observe that $C_b(\mathbb{R}) \subset C_0^{w_m}(\mathbb{R}) \subset C_0^{w_{m+1}}(\mathbb{R})$, $\|\cdot\|_m \leq \|\cdot\|_\infty$ on $C_b(\mathbb{R})$ and $\|\cdot\|_{m+1} \leq \dfrac{m+2}{m+1}\|\cdot\|_m$ on $C_0^{w_m}(\mathbb{R})$.
Moreover $C_0(\mathbb{R})$ is dense in $C_0^{w_m}(\mathbb{R})$.

From now on we shall fix $\alpha, \beta \in C(\mathbb{R})$ and $\gamma \in C_b(\mathbb{R})$ such that

$$\alpha(x) > 0 \qquad \text{for every } x \in \mathbb{R}, \tag{2.1}$$

$$\alpha(x) = O(x^2) \quad \text{and} \quad \beta(x) = O(|x|) \qquad \text{as } x \to \pm\infty. \tag{2.2}$$

We shall also assume that

$$\text{(2.3)} \qquad \begin{aligned} &\text{the function } x \mapsto \int_{x_0}^{x} \frac{\beta(s)}{\alpha(s)} ds \text{ is locally bounded at } +\infty \\ &\text{or} \quad \beta(x) \geq 0 \text{ for } x \geq \delta, \end{aligned}$$

and

$$\text{(2.4)} \qquad \begin{aligned} &\text{the function } x \mapsto \int_{x_0}^{x} \frac{\beta(s)}{\alpha(s)} ds \text{ is locally bounded at } -\infty \\ &\text{or} \quad \beta(x) \leq 0 \text{ for } x \leq -\delta, \end{aligned}$$

for some $\delta > 0$ and $x_0 \in \mathbb{R}$.

Consider the differential operators

$$\text{(2.5)} \qquad Lu := \alpha u'' + \beta u' + \gamma u$$

defined on

$$\text{(2.6)} \qquad D_m(L) := \left\{ u \in C_0^{w_m}(\mathbb{R}) \cap C^2(\mathbb{R}) \mid \lim_{x \to \pm\infty} \frac{(\alpha u'' + \beta u')(x)}{1 + x^{2m}} = 0 \right\}$$

and

$$\text{(2.7)} \qquad \widetilde{L}u := Lu$$

defined on

$$\text{(2.8)} \qquad D(\widetilde{L}) := \left\{ u \in C_*(\mathbb{R}) \cap C^2(\mathbb{R}) \mid \lim_{x \to \pm\infty} (\alpha u'' + \beta u')(x) = 0 \right\}.$$

We can state the following result.

Theorem 2.1. *For every $m \geq 1$, the operator $(L, D_m(L))$ is the generator of a strongly continuous positive semigroup $(T_m(t))_{t\geq 0}$ on $C_0^{w_m}(\mathbb{R})$ such that $\|T_m(t)\| \leq e^{(\omega + \|\gamma\|_\infty)t}$ for every $t \geq 0$, where*

$$\omega := \sup_{x \in \mathbb{R}} \frac{\left|(4m^2 - 2m)\alpha(x)x^{2m-2} + 2m\beta(x)x^{2m-1}\right|}{1 + x^{2m}}.$$

Moreover, if $\gamma(x) \leq 0$ for every $x \in \mathbb{R}$, the restriction of $(T_m(t))_{t\geq 0}$ to $C_0(\mathbb{R})$ is a Feller semigroup on $C_0(\mathbb{R})$ whose generator is $(\widetilde{L}, D(\widetilde{L}) \cap C_0(\mathbb{R}))$.

Finally, if $\gamma \in C_*(\mathbb{R})$ and $\gamma(x) \leq 0$ for every $x \in \mathbb{R}$, then the restriction of $(T_m(t))_{t\geq 0}$ to $C_*(\mathbb{R})$ is a Feller semigroup on $C_*(\mathbb{R})$ whose generator is $(\widetilde{L}, D(\widetilde{L}))$.

Proof. We shall infer our result by using Theorem 3.1 in [6] and, to this end, we shall verify the assumptions made there by adopting the same notation.

Set
$$W(x) := \exp\left(-\int_{x_0}^{x} \frac{\beta(s)}{\alpha(s)} ds\right) \qquad (x \in \mathbb{R}),$$
and
$$\omega_1 := \sup_{x\in\mathbb{R}} \frac{|\alpha(x)(2w_m'(x)^2 - w_m(x)w_m''(x)) - \beta(x)w_m(x)w_m'(x)|}{w_m(x)^2}.$$

We shall prove that $\omega_1 \in \mathbb{R}$ and

$$\int_{x_0}^{+\infty} W(x) \int_{x}^{+\infty} \frac{dtdx}{\alpha(t)W(t)} = \int_{-\infty}^{x_0} W(x) \int_{-\infty}^{x} \frac{dtdx}{\alpha(t)W(t)} dtdx = +\infty. \tag{2.9}$$

Actually, it is easy to check that $\omega_1 = \omega$.

Now assume that the function $x \mapsto \displaystyle\int_{x_0}^{x} \frac{\beta(s)}{\alpha(s)} ds$ is locally bounded at $+\infty$ so that there exist $\delta, m, M > 0$ such that $m \le W(x) \le M$ for every $x \ge \delta$.

On the other hand, by (2.2), there exist $\delta_1 \ge \max(|x_0|, \delta)$ and $C_\alpha > 0$ such that

$$\alpha(x) \le C_\alpha x^2 \quad \text{for } |x| \ge \delta_1. \tag{2.10}$$

Then, for $x \ge \delta_1$
$$W(x) \int_{x}^{+\infty} \frac{dt}{\alpha(t)W(t)} \ge m \int_{\delta_1}^{+\infty} \frac{dt}{\alpha(t)W(t)} \ge \frac{m}{MC_\alpha} \int_{\delta_1}^{+\infty} \frac{dt}{t^2} = \frac{m}{\delta_1 MC_\alpha}$$
and hence (2.9) follows.

If we assume that there exists $\delta > 0$ such that $\beta(x) \ge 0$ for every $x \ge \delta$, then W is decreasing in $[\delta, +\infty[$ and, setting again $\delta_1 \ge \max(|x_0|, \delta)$, satisfying (2.10), we get
$$\begin{aligned}
&\int_{x_0}^{+\infty} W(x) \int_{x}^{+\infty} \frac{dtdx}{\alpha(t)W(t)} \ge \int_{\delta_1}^{+\infty} W(x) \int_{x}^{+\infty} \frac{dtdx}{\alpha(t)W(t)} \\
&= \int_{\delta_1}^{+\infty} dt \int_{\delta_1}^{t} \frac{W(x)}{\alpha(t)W(t)} dx \ge \int_{\delta_1}^{+\infty} \frac{t-\delta_1}{\alpha(t)} dt \ge \frac{1}{C_\alpha} \int_{\delta_1}^{+\infty} \frac{t-\delta_1}{t^2} dt = +\infty
\end{aligned}$$
so that (2.9) follows.

A similar reasoning can be done by replacing $+\infty$ with $-\infty$ and hence the proof is complete. ∎

Remark 2.2. In the light of Proposition 2.2 and Theorem 2.3 of [6], the same proof as above actually shows that, under the same assumptions of Theorem 2.1, the operators $(\widetilde{L}-\gamma I, D(\widetilde{L})\cap C_0(\mathbb{R}))$ and $(L-\gamma I, D_m(L))$ generate strongly continuous positive semigroups on $C_0(\mathbb{R})$ and $C_0^{w_m}(\mathbb{R})$, respectively. Here I denotes the identity operator.

Consider now the evolution problem associated with the differential operator $(L, D_m(L))$, i.e.,

$$(2.11)\quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t) \\ \quad +\beta(x)\dfrac{\partial u}{\partial x}(x,t) + \gamma(x)u(x,t) \quad x\in\mathbb{R},\ t>0, \\ u(x,0) = u_0(x) \qquad x\in\mathbb{R}, \\ \lim\limits_{x\to\pm\infty}\dfrac{u(x,t)}{1+x^{2m}} = 0 \qquad t\geq 0, \\ \lim\limits_{x\to\pm\infty}\dfrac{1}{1+x^{2m}}\left(\alpha(x)\dfrac{\partial^2 u}{\partial x^2}(x,t)+\beta(x)\dfrac{\partial u}{\partial x}(x,t)\right) = 0 \qquad t\geq 0. \end{cases}$$

From a well known result in semigroup theory (see, e.g., [17, Proposition 6.2]) it follows that, for every $m\geq 1$ and $u_0\in D_m(L)$, there exists a unique continuous solution $u:\mathbb{R}\times[0,+\infty[\,\to\mathbb{R}$ of (2.11), expressed by the semigroup $(T_m(t))_{t\geq 0}$ as

$$u(x,t) = T_m(t)(u_0)(x) \qquad (x\in\mathbb{R}, t\geq 0)$$

and such that for every $t\geq 0$ and $x\in\mathbb{R}$

(1) $u(\cdot,t)\in C^2(\mathbb{R})\cap C_0^{w_m}(\mathbb{R})$;

(2) $|u(x,t)|\leq (1+x^{2m})e^{(\omega+\|\gamma\|_\infty)t}\|u_0\|_m$;

(3) if u_0 is positive, then $u(\cdot,t)$ is positive.

By the next theorem, which is a direct consequence of Theorem 3.3 in [6], we shall establish the existence of a Markov process generated by L and, at the same time, by means of such process, we shall obtain a first representation formula of the semigroup $(T_m(t))_{t\geq 0}$.

Theorem 2.3. *Under the same assumptions of Theorem 2.1 suppose further that $\gamma\in C_*(\mathbb{R})$ and $\gamma(x)\leq 0$ for every $x\in\mathbb{R}$. Then there exists a Markov process $(\Omega,\mathcal{F},(P^x)_{x\in[-\infty,+\infty]},(Z_t)_{0\leq t\leq+\infty})$ with state space $[-\infty,+\infty]$, whose paths are continuous almost surely, such that for every $t\geq 0$ and $x\in\mathbb{R}$*

(i) $P^x\{Z_t=-\infty\} = P^x\{Z_t=+\infty\} = 0$;

(ii) *the distribution* $P^x_{Z_t}$ *of the random variable* Z_t *with respect to* P^x *possesses finite moments of order up to* $2m$;

(iii) $T_m(t)f(x) = \displaystyle\int_\Omega f^*(Z_t)dP^x$ *for every* $f \in C_0^{w_m}(\mathbb{R})$,

where f^* *denotes the extension of* f *to* $[-\infty, +\infty]$*, vanishing at* $-\infty$ *and* $+\infty$.

Remark 2.4.

1. Since (ii) holds true for every $m \geq 1$, the distributions $P^x_{Z_t}$ have finite moments of every order, so their characteristic functions are infinitely many times continuously differentiable.
2. For every $m \geq 1$ and $t \geq 0$, by (iii), we have that $T_m(t) = T_{m+1}(t)$ on $C_0^{w_m}(\mathbb{R})$.

In Section 3 we shall present a further representation formula of $(T_m(t))_{t\geq 0}$ as limit of iterates of suitable positive linear operators on $C_0^{w_m}(\mathbb{R})$. To this purpose it is important to find a core for $(L, D_m(L))$.

Let us recall that a core for a linear operator $(B, D(B))$ on a Banach space E is a subspace D which is dense in $D(B)$ with respect to the graph norm $\|u\|_B := \|u\| + \|Bu\| \quad (u \in D(B))$.
If $(B, D(B))$ is closed and if $\lambda I - B$ is invertible for some $\lambda \in \mathbb{C}$, then a sub space D of $D(B)$ is a core for $(B, D(B))$ if and only if $(\lambda I - B)(D)$ is dense in E, where I stands for the identity operator on E.

We also point out that, since $\gamma \in C_b(\mathbb{R})$, a subspace D is a core for $(L, D_m(L))$ if and only if it is a core for the operator $(L - \gamma I, D_m(L))$.

The next result allows us to investigate the existence of cores for $(L, D_m(L))$, by simply determining cores for the operator $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$. Denote by S the space of all functions $f \in C^2(\mathbb{R})$ which are constant on some neighborhoods of $+\infty$ and $-\infty$.

Proposition 2.5. *Let* D *be a subspace of* $D(\widetilde{L}) \cap C_0(\mathbb{R})$*. If* D *is a core for* $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$ *then* D *is a core for* $(L, D_m(L))$.

Moreover the linear subspace $\widetilde{D}$ *generated by* $D \cup S$ *is a core for* $(\widetilde{L}, D(\widetilde{L}))$.

Proof. Since $(L - \gamma I, D_m(L))$ and $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$ generate strongly continuous semigroups in $C_0^{w_m}(\mathbb{R})$ and $C_0(\mathbb{R})$, respectively (see Remark 2.2), there exists $\lambda \in \mathbb{R}$ such that $\lambda I - (L - \gamma I)$ is invertible from $D_m(L)$ into $C_0^{w_m}(\mathbb{R})$ and $\lambda I - (\widetilde{L} - \gamma I)$ is invertible from $D(\widetilde{L}) \cap C_0(\mathbb{R})$ into $C_0(\mathbb{R})$.
So, if D is a core for $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$, then $(\lambda I - (\widetilde{L} - \gamma I))(D)$ is dense in $(C_0(\mathbb{R}), \|\cdot\|_\infty)$. Therefore $(\lambda I - (L - \gamma I))(D) = (\lambda I - (\widetilde{L} - \gamma I))(D)$ is dense in $(C_0^{w_m}(\mathbb{R}), \|\cdot\|_m)$ and hence D is a core for $(L - \gamma I, D_m(L))$, that is D is a core for $(L, D_m(L))$.

Now let $f \in D(\widetilde{L})$ and $\varepsilon > 0$; consider a function $g \in S$ such that

$$g(x) = \begin{cases} \lim\limits_{t \to +\infty} f(t) & \text{if } x \geq 1, \\ \lim\limits_{t \to -\infty} f(t) & \text{if } x \leq -1. \end{cases}$$

Clearly $f - g \in D(\widetilde{L}) \cap C_0(\mathbb{R})$ and hence, since D is a core for $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$, there exists $h \in D$ such that

$$\|f - (g + h)\|_{\widetilde{L} - \gamma I} = \|(f - g) - h\|_{\widetilde{L} - \gamma I} \leq \varepsilon$$

and $g + h \in \widetilde{D}$. ∎

Consider now the following subspaces of $D(\widetilde{L}) \cap C_0(\mathbb{R})$:

$$D_1 := \left\{ u \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) \mid \begin{array}{l} u = 0 \text{ on a neighborhood of} + \infty \\ \text{and } \lim\limits_{x \to -\infty} (\alpha u'' + \beta u')(x) = 0 \end{array} \right\} \tag{2.12}$$

and

$$D_2 := \left\{ u \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) \mid \begin{array}{l} u = 0 \text{ on a neighborhood of } - \infty \\ \text{and } \lim\limits_{x \to +\infty} (\alpha u'' + \beta u')(x) = 0 \end{array} \right\} \tag{2.13}$$

In the sequel we shall denote by $K^2(\mathbb{R})$ the subspace of all functions $f \in C^2(\mathbb{R})$ having compact support.

Proposition 2.6. *The following statements hold true:*

(i) *If*

$$\lim_{x \to +\infty} \frac{\alpha(x) u'(x)}{x} = 0 \quad \textit{for every } u \in D(\widetilde{L}) \cap C_0(\mathbb{R}), \tag{2.14}$$

then D_1 *is a core for* $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$.

(ii) *If*

$$\lim_{x \to -\infty} \frac{\alpha(x) u'(x)}{x} = 0 \quad \textit{for every } u \in D(\widetilde{L}) \cap C_0(\mathbb{R}), \tag{2.15}$$

then D_2 *is a core for* $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$.

(iii) *If both conditions* (2.14) *and* (2.15) *are satisfied, then* $K^2(\mathbb{R})$ *is a core for* $(\widetilde{L} - \gamma I, D(\widetilde{L}) \cap C_0(\mathbb{R}))$.

Proof. Let $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$ satisfying condition (2.14). Adapting the methods of [1, Theorem 3.1], we shall construct a sequence $(u_n)_{n\geq 1}$ in D_1, tending to u with respect to the norm $\|\cdot\|_{\widetilde{L}-\gamma I}$.

Consider $\varphi \in K^2(\mathbb{R})$ such that

$$\text{(i) } 0 \leq \varphi \leq 1,$$
$$\text{(ii) } \varphi(x) = 1 \quad \text{for } |x| \leq 1,$$
$$\text{(iii) } \varphi(x) = 0 \quad \text{for } |x| \geq 2,$$

and, for $n \geq 1$ and $x \in \mathbb{R}$, set

$$u_n(x) := \begin{cases} u(x)\,\varphi\left(\dfrac{x}{n}\right) & \text{if } x > n, \\ u(x) & \text{if } x \leq n. \end{cases} \tag{2.16}$$

Clearly $u_n \in D_1$.

From (2.2) and (2.14) it follows that, for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \geq \delta$

$$|u(x)| \leq \varepsilon, \qquad \left|\alpha(x)\,u''(x) + \beta(x)\,u'(x)\right| \leq \frac{\varepsilon}{4},$$

$$\left|\frac{\alpha(x)\,u'(x)}{x}\right| \leq \frac{\varepsilon}{16\,\|\varphi'\|_\infty}, \qquad \left|\frac{\beta(x)\,u(x)}{x}\right| \leq \frac{\varepsilon}{8\,\|\varphi'\|_\infty},$$

$$\left|\frac{\alpha(x)\,u(x)}{x^2}\right| \leq \frac{\varepsilon}{16\,\|\varphi''\|_\infty}.$$

Then, for $n \geq \delta$ and $x \geq 2n$, we obtain

$$|u(x) - u_n(x)| = \left|\left(1 - \varphi\left(\frac{x}{n}\right)\right) u(x)\right| = |u(x)| \leq \varepsilon,$$

$$|(L-\gamma I)(u)(x) - (L-\gamma I)(u_n)(x)| = |(L-\gamma I)(u)(x)| \leq \varepsilon.$$

On the other hand, for $n \leq x \leq 2n$, we have

$$|u(x) - u_n(x)| = \left|\left(1 - \varphi\left(\frac{x}{n}\right)\right) u(x)\right| \leq |u(x)| \leq \varepsilon$$

and

$$\begin{aligned}
&|(L-\gamma I)(u)(x) - (L-\gamma I)(u_n)(x)| \\
= &\left|\alpha(x)\,u''(x) + \beta(x)\,u'(x) - \alpha(x)\,u''(x)\,\varphi\left(\frac{x}{n}\right) - 2\alpha(x)\,\frac{u'(x)}{n}\varphi'\left(\frac{x}{n}\right)\right. \\
&\left. - \alpha(x)\,\frac{u'(x)}{n^2}\varphi''\left(\frac{x}{n}\right) - \beta(x)\,u'(x)\,\varphi\left(\frac{x}{n}\right) - \beta(x)\,\frac{u(x)}{n}\varphi'\left(\frac{x}{n}\right)\right|
\end{aligned}$$

$$\leq \left|\left(\alpha(x)u''(x)+\beta(x)u'(x)\right)\left(1-\varphi\left(\frac{x}{n}\right)\right)\right| + 2\frac{\|\varphi'\|_\infty}{n}\left|\alpha(x)u'(x)\right|$$
$$+\frac{\|\varphi''\|_\infty}{n^2}\left|\alpha(x)u(x)\right| + \frac{\|\varphi'\|_\infty}{n}\left|\beta(x)u(x)\right|$$
$$\leq \frac{\varepsilon}{4} + 2\frac{\|\varphi'\|_\infty}{n}\frac{\varepsilon}{16\|\varphi'\|_\infty}|x| + \frac{\|\varphi''\|_\infty}{n^2}\frac{\varepsilon}{16\|\varphi''\|_\infty}x^2 + \frac{\|\varphi'\|_\infty}{n}\frac{\varepsilon}{8\|\varphi'\|_\infty}|x|$$
$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{8n}2n + \frac{\varepsilon}{16n^2}4n^2 + \frac{\varepsilon}{8n}2n = \varepsilon.$$

Therefore, for $n \geq \delta$

$$\|u-u_n\|_{\widetilde{L}-\gamma I} = \sup_{x\geq n}|u-u_n| + \sup_{x\geq n}|(L-\gamma I)(u-u_n)| \leq 2\varepsilon.$$

In the same manner we can prove statements (ii) and (iii) by replacing the sequence in (2.16) with

$$v_n(x) := \begin{cases} u(x)\varphi\left(\dfrac{x}{n}\right) & \text{if } x < -n, \\ u(x) & \text{if } x \geq -n, \end{cases}$$

in the case (ii) and, in the case (iii), with

$$w_n(x) := \begin{cases} u(x)\varphi\left(\dfrac{x}{n}\right) & \text{if } |x| > n, \\ u(x) & \text{if } |x| \leq n. \end{cases}$$

The proof is now complete. ■

Here we present some criteria to deduce some information on the growth at infinity of the first derivative of functions belonging to $D(\widetilde{L}) \cap C_0(\mathbb{R})$.

Proposition 2.7. *Assume that* $\alpha \in C^2(]\delta,+\infty[)$ *and* $\beta \in C^1(]\delta,+\infty[)$ *for some* $\delta > 0$. *Further assume that:*

(i) $(\alpha' - \beta)(x) = O(x)$ *as* $x \to +\infty$,

(ii) $(\alpha'' - \beta')(x) = O(\mathbf{1})$ *as* $x \to +\infty$.

Then $\lim\limits_{x\to+\infty}\dfrac{\alpha(x)u'(x)}{x} = 0$ *for every* $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$.

Proof. Let $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$ and $\varepsilon > 0$; then there exists $\delta_1 > \delta$ such that, for $x \geq \delta_1$, we have $|u(x)| \leq \varepsilon$ and $|(\alpha u'' + \beta u')(x)| \leq \varepsilon$. Since

$$\begin{aligned}\int_{\delta_1}^{x}\left(\alpha u''+\beta u'\right)(s)\,ds &= \alpha(x)u'(x) - \alpha(\delta_1)u'(\delta_1) \\ &\quad - \left(\alpha'(x)-\beta(x)\right)u(x) + \left(\alpha'(\delta_1)-\beta(\delta_1)\right)u(\delta_1) \\ &\quad + \int_{\delta_1}^{x}\left(\alpha''(s)-\beta'(s)\right)u(s)\,ds,\end{aligned}$$

we get

$$\begin{aligned}
|\alpha(x)\,u'(x)| \le & \left|\int_{\delta_1}^{x} \left(\alpha u'' + \beta u'\right)(s)\,ds\right| \\
& + \left|\alpha'(x) - \beta(x)\right| |u(x)| + \left|\left(\alpha'(\delta_1) - \beta(\delta_1)\right) u(\delta_1)\right. \\
& \left. - \alpha(\delta_1)\,u'(\delta_1)\right| + \left\|\alpha'' - \beta'\right\|_{\infty} \left|\int_{\delta_1}^{x} u(s)\,ds\right| \\
\le & \ \varepsilon\,|x-\delta_1| + \varepsilon \left\|\alpha' - \beta\right\|_{\infty} + \left|\left(\alpha'(\delta_1) - \beta(\delta_1)\right) u(\delta_1)\right. \\
& \left. - \alpha(\delta_1)\,u'(\delta_1)\right| + \varepsilon \left\|\alpha'' - \beta'\right\|_{\infty} |x-\delta_1|,
\end{aligned}$$

which proves the statement. ■

Proposition 2.8. *Let $\delta > 0$ and let $\varphi : [\delta, +\infty[\ \to \mathbb{R}$ be a differentiable function such that $\varphi(x) \neq 0$, for every $x \geq \delta$. Define again*

$$W(x) := \exp\left(-\int_{\delta}^{x} \frac{\beta(s)}{\alpha(s)} ds\right) \quad (x \in \mathbb{R}) \tag{2.17}$$

and suppose that $\dfrac{1}{\alpha W} \in L^1([\delta, +\infty[)$.

Assume that one of the following conditions is satisfied:

(a) $\lim\limits_{x\to+\infty} \varphi(x)\,W(x) = 0$*;*

(b) $\lim\limits_{x\to+\infty} \varphi(x)\,W(x) \in \mathbb{R}\setminus\{0\}$ *and* $\dfrac{1}{\varphi} \notin L^1([\delta, +\infty[)$*;*

(c) (i) $\lim\limits_{x\to+\infty} \varphi(x)\,W(x) \in \{-\infty, +\infty\}$,

(ii) $W \notin L^1([\delta, +\infty[)$, $(\varphi\beta - \varphi'\alpha)(x) \neq 0$ *for* $x \geq \delta$ *and* $\dfrac{\varphi^2}{\varphi\beta - \varphi'\alpha} = O(1)$ *as* $x \to +\infty$.

Then $\lim\limits_{x\to+\infty} \varphi(x)\,u'(x) = 0$*, for every* $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$.

Proof. Observe that, for every $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\left(\frac{u'}{W}\right)'(x) = \frac{\alpha(x)\,u''(x) + \beta(x)\,u'(x)}{\alpha(x)\,W(x)}$$

so that, for $x \geq \delta$,

$$\varphi(x)\,u'(x) = \varphi(x)\,W(x)\left[u'(\delta) + \int_{\delta}^{x} \frac{(\alpha u'' + \beta u')(s)}{\alpha(s)\,W(s)} ds\right]. \tag{2.18}$$

On the other hand, since $\alpha u'' + \beta u'$ is bounded and $\frac{1}{\alpha W} \in L^1(\delta, +\infty)$, we have that

$$\int_{\delta}^{+\infty} \frac{(\alpha u'' + \beta u')(s)}{\alpha(s) W(s)} ds \in \mathbb{R}, \tag{2.19}$$

that is

$$\lim_{x\to+\infty} \frac{u'(x)}{W(x)} = u'(\delta) + \int_{\delta}^{+\infty} \frac{(\alpha u'' + \beta u')(s)}{\alpha(s) W(s)} ds \tag{2.20}$$

is finite. Therefore, in case (a), the statement immediately follows.

In case (b), again by (2.19), there exists $\lim_{x\to+\infty} \varphi(x) u'(x) =: l \in \mathbb{R}$. Suppose that $l > 0$. Then there exists $\delta_1 > \delta$ such that, for $x \geq \delta_1$

$$u'(x) > \frac{l}{2\varphi(x)}$$

and hence

$$u(x) - u(\delta_1) = \int_{\delta_1}^{x} u'(s)\, ds > \frac{l}{2} \int_{\delta_1}^{x} \frac{1}{\varphi(s)} ds. \tag{2.21}$$

Since $\int_{\delta_1}^{+\infty} \frac{1}{\varphi(s)} ds = +\infty$, from (2.21) it follows that $\lim_{x\to+\infty} u(x) = +\infty$ which contradicts the assumption $u \in C_0(\mathbb{R})$. The conclusion is the same if $l < 0$ and hence the limit is necessarily equal to zero.

Finally, suppose that the assumptions of case (c) are satisfied. First note that $\lim_{x\to+\infty} \frac{u'(x)}{W(x)} = 0$. This follows by case (b) because, setting $\eta := \frac{1}{W}$, we have

$$\lim_{x\to+\infty} \eta(x) W(x) = 1 \text{ and } \frac{1}{\eta} \notin L^1(\delta, +\infty).$$

Hence by De L'Hôpital's rule and by condition (iii) we obtain

$$\begin{aligned}
\lim_{x\to+\infty} \varphi(x) u'(x) &= \lim_{x\to+\infty} \frac{u'(x) W(x)^{-1}}{(\varphi(x) W(x))^{-1}} \\
&= \lim_{x\to+\infty} \frac{\left(u''W + u'\frac{\beta}{\alpha}W\right)(x) (W(x))^{-2}}{\left(-\varphi'W + \varphi W\frac{\beta}{\alpha}\right)(x) (\varphi(x) W(x))^{-2}} \\
&= \lim_{x\to+\infty} \frac{\varphi(x)^2 (\alpha u'' + \beta u')(x)}{-\varphi'(x)\alpha(x) + \varphi(x)\beta(x)} = 0.
\end{aligned}$$

So the statement is completely proved. ■

In what follows the hypothesis on the integrability of the function $\frac{1}{\alpha W}$ is dropped.

Proposition 2.9. *Let $\delta > 0$ and let $\varphi : [\delta, +\infty[\to \mathbb{R}$ be a differentiable function such that $\varphi(x) \neq 0$, for every $x \geq \delta$. Define W as in* (2.17) *and assume that*

(i) $\lim_{x\to+\infty} \varphi(x) W(x) = 0$,

(ii) $(\varphi\beta - \varphi'\alpha)(x) \neq 0$ *for* $x \geq \delta$ *and* $\frac{\varphi^2}{\varphi\beta - \varphi'\alpha} = O(1)$ *as* $x \to +\infty$.

Then $\lim_{x\to+\infty} \varphi(x) u'(x) = 0$, *for every* $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$.

Proof. By making use of formula (2.18), the result immediately follows if $\int_\delta^{+\infty} \frac{|\alpha u'' + \beta u'|(s)}{\alpha(s) W(s)} ds < +\infty$. If $\int_\delta^{+\infty} \frac{|\alpha u'' + \beta u'|(s)}{\alpha(s) W(s)} ds = +\infty$ then, again by De L'Hôpital's rule and by (ii),

$$\begin{aligned}
&\lim_{x\to+\infty} \varphi(x) W(x) \int_\delta^x \frac{|\alpha u'' + \beta u'|(s)}{\alpha(s) W(s)} ds \\
&= \lim_{x\to+\infty} \frac{|\alpha u'' + \beta u'|(x)}{\alpha(x) W(x)} \frac{-\varphi(x)^2 W(x)^2}{\left(\varphi' W - \varphi W \frac{\beta}{\alpha}\right)(x)} \\
&= \lim_{x\to+\infty} \frac{\varphi(x)^2 |\alpha u'' + \beta u'|(x)}{-\varphi'(x)\alpha(x) + \varphi(x)\beta(x)} = 0.
\end{aligned}$$

On the other hand, by (2.18), for every $x \geq \delta$

$$|\varphi(x) u'(x)| \leq |\varphi(x)| W(x) \left(|u'(\delta)| + \int_\delta^x \frac{|\alpha u'' + \beta u'|(s)}{\alpha(s) W(s)} ds \right)$$

and hence $\lim_{x\to+\infty} \varphi(x) u'(x) = 0$. ■

A similar reasoning may be used to study the behavior of u' at $-\infty$. Therefore, combining Propositions 2.5-2.9 we obtain the next result.

Theorem 2.10. *Let $\delta > 0$. In each of the following cases:*

(1) (i) $\alpha \in C^2(\mathbb{R}\setminus[-\delta, \delta])$ *and* $\beta \in C^1(\mathbb{R}\setminus[-\delta, \delta])$,

(ii) $(\alpha' - \beta)(x) = O(x)$ *and* $(\alpha'' - \beta')(x) = O(\mathbf{1})$ *as* $x \to \pm\infty$;

(2) (i) *$\alpha \in C^1(\mathbb{R}\backslash[-\delta,\delta])$,*

(ii) *$\dfrac{1}{\alpha W} \in L^1(]-\infty,-\delta]) \cap L^1([\delta,+\infty[)$,*

(iii) *$\lim\limits_{x\to\pm\infty} \dfrac{\alpha(x)\,W(x)}{x} \in \{-\infty,+\infty\}$,*

(iv) *$W \notin L^1([\delta,+\infty[)$ and $W \notin L^1(]-\infty,-\delta])$,*

(v) *$x(\beta-\alpha')(x)+\alpha(x) \neq 0$ for $|x| \geq \delta$ and*
$$\frac{\alpha(x)}{x(\beta-\alpha')(x)+\alpha(x)} = O(1) \quad \textit{as } x \to \pm\infty;$$

(3) (i) *$\alpha \in C^1(\mathbb{R}\backslash[-\delta,\delta])$,*

(ii) *$\lim\limits_{x\to\pm\infty} \dfrac{\alpha(x)\,W(x)}{x} = 0$,*

(iii) *$x(\beta-\alpha')(x)+\alpha(x) \neq 0$ for $|x| \geq \delta$ and*
$$\frac{\alpha(x)}{x(\beta-\alpha')(x)+\alpha(x)} = O(1) \quad \textit{as } x \to \pm\infty;$$

the space $K^2(\mathbb{R})$ is a core for $(L, D_m(L))$, for every $m \geq 1$. Moreover the space of all functions which are twice continuously differentiable and constant at infinity is a core for $(\widetilde{L}, D(\widetilde{L}))$.

Proof. By Proposition 2.6, we have to prove that, for every $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$,
$$\lim_{x\to\pm\infty} \frac{\alpha(x)\,u'(x)}{x} = 0.$$
In case (1) such limits hold true because of Proposition 2.7.

In the cases (2) and (3) it is enough to apply Propositions 2.8-(c) and 2.9, with $\varphi(x) := \dfrac{\alpha(x)}{x}$ $(|x| \geq \delta)$. Thus the statements follows from Proposition 2.5. ■

3. On the Semigroup $(T_m(t))_{t\geq 0}$

As in [11] we shall denote by $E(\mathbb{R})$ the space of all functions $f \in C(\mathbb{R})$ such that

$$\int_{-\infty}^{+\infty} |f(ay+b)|\, e^{-y^2/2} dy < +\infty \quad \text{for every } a \geq 0 \text{ and } b \in \mathbb{R}. \tag{3.1}$$

It is easy to check that $C_0^{w_m}(\mathbb{R}) \subset E(\mathbb{R})$, for every $m \geq 1$.

For every $n \geq 1$, consider the positive linear operator G_n^* defined by setting

$$G_n^*(f) := G_n\left(\left(1+\frac{\gamma}{n}\right) f\right) \qquad (f \in E(\mathbb{R})), \tag{3.2}$$

where, for every $f \in E(\mathbb{R})$ and $x \in \mathbb{R}$,

$$(3.3) \qquad G_n(f)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f\left(\sqrt{\frac{2\alpha(x)}{n}}y + x + \frac{\beta(x)}{n}\right) e^{-y^2/2} dy.$$

In [11, Section 5] we proved that, for every $m \geq 1$, each G_n^* maps continuously $C_0^{w_m}(\mathbb{R})$ into itself and there exists a constant $M_m > 0$, independent of n, such that

$$(3.4) \qquad \|G_n^*\|_{C_0^{w_m}(\mathbb{R})} \leq \left(1 + \frac{M_m}{n}\right)\left(1 + \frac{\|\gamma\|_\infty}{n}\right)$$

(see formulas (2.10) and (5.2) in [11]).

Moreover, the sequence $(G_n^*)_{n\geq 1}$ is an approximation process on $C_0^{w_m}(\mathbb{R})$ and, denoted by $UC_b^2(\mathbb{R})$ the space of all twice differentiable functions on $\mathbb{R}$ with uniformly continuous and bounded second derivative, then, for $m \geq 2$ and $f \in UC_b^2(\mathbb{R})$, the following asymptotic formula is satisfied

$$(3.5) \qquad \lim_{n\to\infty} n\left(G_n^*(f) - f\right) = \alpha f'' + \beta f' + \gamma f = L(f) \qquad \text{in } C_0^{w_m}(\mathbb{R}).$$

Therefore, as a consequence of a Trotter's theorem (see [25], Theorem 5.3) we can obtain the next result.

Theorem 3.1. *Let $m \geq 2$. Assume that the functions α, β satisfy conditions $(2.1) - (2.4)$ as well as one of conditions $(1) - (3)$ of Theorem 2.10. Denoted by $(T_m(t))_{t\geq 0}$ the semigroup generated by $(L, D_m(L))$, then, for every $f \in C_0^{w_m}(\mathbb{R})$ and $t \geq 0$*

$$(3.6) \qquad T_m(t) f = \lim_{n\to+\infty} (G_n^*)^{k(n)} f \qquad \textit{in } C_0^{w_m}(\mathbb{R}),$$

where $(k(n))_{n\geq 1}$ is an arbitrary sequence of positive integers such that $k(n)/n \to t$ and $(G_n^*)^{k(n)}$ denotes the iterate of order $k(n)$ of G_n^*.

In particular the limit holds uniformly on compact subsets of $\mathbb{R}$.

Proof. From (3.4) we infer that

$$\|(G_n^*)^p\|_{C_0^{w_m}(\mathbb{R})} \leq \exp\left((M_m + \|\gamma\|_\infty)\frac{p}{n}\right)$$

for every $n, p \geq 1$.

Moreover, since $K^2(\mathbb{R})$ is a core for the generator $(L, D_m(L))$, $(\lambda I - L)(K^2(\mathbb{R}))$ is dense in $C_0^{w_m}(\mathbb{R})$ for $\lambda > 0$ sufficiently large.

Finally, as we previously pointed out, if $u \in K^2(\mathbb{R}) \subset UC_b^2(\mathbb{R})$ then $\lim_{n\to\infty} n\,(G_n^*(u) - u) = Lu$ in $C_0^{w_m}(\mathbb{R})$. Hence the result follows from a consequence of a

Trotter's theorem ([25, Theorem 5.3], see also [17, Corollary 5.3] or [21, Chapter 3, Theorem 6.7]) ■

Remark 3.2. If $\gamma \in \mathbb{R}$, by (3.6) we can obtain an estimate for the norm of $T_m(t)$ sharper than the one given in Theorem 2.1. Indeed, denoted by $(S(t))_{t\geq 0}$ the semigroup generated by $(L - \gamma I, D_m(L))$, we have that for every $t \geq 0$ and $f \in C_0^{w_m}(\mathbb{R})$

$$T_m(t)f = \lim_{n\to\infty} (G_n^*)^{k(n)} f = \lim_{n\to\infty} \left(1 + \frac{\gamma}{n}\right)^{k(n)} (G_n)^{k(n)} f = e^{\gamma t} S(t) f$$

whence $\|T_m(t)\| \leq e^{\gamma t} \|S(t)\| \leq e^{(\omega+\gamma)t}$.

From formula (3.6) and from some shape preserving properties of the operators G_n^* we can deduce several qualitative information on $(T_m(t))_{t\geq 0}$ and hence on the solution of the differential problem (2.11).

As usual, for $k \geq 0$ and $M > 0$, the symbol $Lip(k, M)$ will stand for the class of all functions $f \in C(\mathbb{R})$ such that

$$|f(x) - f(y)| \leq M |x - y|^k \qquad \text{for every } x, y \in \mathbb{R}.$$

Moreover we set $e_m(x) := x^m$ $(x \in \mathbb{R})$, for every $m \geq 1$.

If $a \in \mathbb{R} \cup \{-\infty, +\infty\}$, we shall set

$$D_a(\mathbb{R}) := \begin{cases} \{f \in C_0^{w_m}(\mathbb{R}) \mid f \text{ is convex and increasing on } \mathbb{R}\} & \text{if } a = -\infty, \\ \Big\{f \in C_0^{w_m}(\mathbb{R}) \mid f \text{ is convex, increasing on} & \\ \qquad [a, +\infty[\text{and decreasing on }]-\infty, a]\Big\} & \text{if } a \in \mathbb{R}, \\ \{f \in C_0^{w_m}(\mathbb{R}) \mid f \text{ is convex and decreasing on } \mathbb{R}\} & \text{if } a = +\infty. \end{cases}$$

Corollary 3.3. *Let $a \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $\nu \geq 1$. Under the same assumptions of the Theorem 3.1, the following statements hold true.*

(1) *Assume that $\left(1 + \frac{\gamma}{n}\right) f \in D_a(\mathbb{R})$ for every convex function $f \in D_a(\mathbb{R})$ and $n \geq \upsilon$. If β is positive on $]a, +\infty[$ and negative on $]-\infty, a[$, then*

$$T_m(t)f(x) \geq e^{\gamma(x)t} f(x)$$

for every $t \geq 0$, $f \in D_a(\mathbb{R})$ and $x \in \mathbb{R}$.

(2) *Assume that β is affine and γ is constant. Then each $T_m(t)$ maps affine functions into affine functions.*

In particular, if $\beta = 0$, $T_m(t)f(x) = e^{\gamma t} f(x)$, for every affine function f and for every $x \in \mathbb{R}$.

(3) *Assume that α is constant, β is affine and $\left(1+\frac{\gamma}{n}\right) f$ is convex, for every convex function $f \in C_0^{w_m}(\mathbb{R})$ and for every $n \geq \nu$.*

Then each $T_m(t)$ maps convex functions into convex functions.

(4) *Assume that α is constant, β is convex, $e_1 + \frac{\beta}{n}$ is increasing and $\left(1+\frac{\gamma}{n}\right) f$ is increasing and convex, for every increasing convex function $f \in C_0^{w_m}(\mathbb{R})$ and for every $n \geq \nu$.*

Then each $T_m(t)$ maps increasing convex functions into increasing convex functions.

(5) *Assume that α is constant, $e_1 + \frac{\beta}{n}$ is increasing and $\left(1+\frac{\gamma}{n}\right) f$ is increasing, for every increasing function $f \in C_0^{w_m}(\mathbb{R})$ and for every $n \geq \nu$.*

Then each $T_m(t)$ maps increasing functions into increasing functions.

(6) *Assume that α and γ are constant and $\beta \in Lip\,(1, M_\beta)$, then $T_m\,(t)\,f \in Lip(k, M\exp((kM_\beta + |\gamma|)t))$, for every $f \in Lip\,(k, M) \cap C_0^{w_m}(\mathbb{R})$ and $t \geq 0$.*

Proof. Given $f \in D_a(\mathbb{R})$, from formula (4.1) in [11] it follows that, for every $x \in \mathbb{R}$

$$\begin{aligned} G_n^*(f)(x) &= G_n\left(\left(1+\frac{\gamma}{n}\right) f\right)(x) \geq \left(\left(1+\frac{\gamma}{n}\right) f\right)\left(x + \frac{\beta\,(x)}{n}\right) \\ &\geq \left(\left(1+\frac{\gamma}{n}\right) f\right)(x) = \left(1+\frac{\gamma(x)}{n}\right) f(x), \end{aligned}$$

whence, by the positivity of G_n^*, for every $k \in \mathbb{N}$

$$(G_n^*)^k(f)(x) \geq \left(1+\frac{\gamma(x)}{n}\right)^k f(x).$$

Therefore, if $t \geq 0$ and $(k(n))_{n\geq 1}$ is a sequence of positive integers such that $k(n)/n \to t$, by Theorem 3.1 we obtain

$$T_m\,(t)\,(f)\,(x) = \lim_{n\to\infty} (G_n^*)^{k(n)}(f)(x) \geq e^{\gamma(x)t} f(x),$$

so (1) is fulfilled.

Properties (2)-(5) follow directly from statements (1)-(4) of Theorem 5.4 in [11] and Theorem 3.1. Analogously, as regards (6), by statement (5) of Theorem 5.4 in [11], we get

$$(G_n^*)^{k(n)}\,(Lip\,(k, M)) \subset Lip\left(k, M\left|1+\frac{\gamma}{n}\right|^{k(n)} \exp\left(\frac{k\,(n)\,kM_\beta}{n}\right)\right);$$

thus, if $f \in Lip\,(k, M) \cap C_0^w(\mathbb{R})$, we have

$$\begin{aligned}|T_m(t)(f)(x) - T_m(t)(f)(y)| &= \lim_{n\to\infty} \left|(G_n^*)^{k(n)}(f)(x) - (G_n^*)^{k(n)}(f)(y)\right| \\ &\leq M \exp((kM_\beta + |\gamma|)t)\,|x-y|^k\,.\end{aligned}$$

The proof is now complete. ■

By adapting the proof of Proposition 9.1 in [19] and by applying Corollary 3.6 in [11] we deduce the next result.

Corollary 3.3. *Let $\gamma = 0$. Under the same assumptions of Theorem* **??***, for every $f \in C_0^{w_m}(\mathbb{R})$, the following statements are equivalent.*

(a) $\|G_n(f) - f\|_m = o(1/n)$ *as $n \to \infty$;*

(b) $\|T_m(t)(f) - f\|_m = o(t)$ *as $t \to 0^+$;*

(c) *there exist $c_1, c_2 \in \mathbb{R}$ such that*

$$f(x) = c_1 \int_0^x \exp\left(-\int_0^t \frac{\beta(s)}{\alpha(s)}ds\right)dt + c_2 \qquad (x \in \mathbb{R}).$$

4. The Case $\beta = \gamma = 0$

If $\beta = \gamma = 0$, then conditions (2.3) and (2.4) are obviously satisfied. Moreover, in order to determine a core for the operator $(L, D_m(L))$, we may require not so much regularity for the coefficient α as in the general case investigated in the last sections; indeed it is enough that there exist $\delta, C_1, C_2 > 0$ and $p \in C^1([\delta, +\infty[)$ such that

$$C_1 p(x) \leq \alpha(x) \leq C_2 p(x) \qquad \text{for } x \geq \delta. \tag{4.1}$$

Consider the following subspace D_1 of $C_0(\mathbb{R})$ defined by (2.12), i.e.

$$D_1 = \left\{ u \in C_0(\mathbb{R}) \cap C^2(\mathbb{R}) \mid \begin{array}{l} u = 0 \text{ on a neighborhood of } +\infty \\ \text{and } \lim\limits_{x\to-\infty} \alpha(x)u''(x) = 0 \end{array} \right\}.$$

Moreover denote by $\widetilde{D}_1$ the linear subspace generated by $D_1 \cup S$ (see Proposition 2.5)

Theorem 4.1. *Suppose that conditions* $(2.1), (2.2)$ *and* (4.1) *are satisfied. Then, in each of the following cases:*

(1) $p \in C^2([\delta, +\infty[)$ *and* $p^{(r)}(x) = O(x^{2-r})$ *as* $x \to +\infty$ *for* $r = 0, 1, 2$*;*

(2) (i) $\dfrac{1}{p} \in L^1([\delta, +\infty[)$ *and* $\lim\limits_{x\to+\infty} \dfrac{p(x)}{x} = +\infty$

or $\lim\limits_{x\to+\infty} \dfrac{p(x)}{x} = 0$*;*

(ii) $p \in C^1([\delta, +\infty[)$*,* $xp'(x) - p(x) \neq 0$ *for* $x \geq \delta$ *and*

$\dfrac{p(x)}{xp'(x) - p(x)} = O(1)$ *as* $x \to +\infty$*;*

the space D_1 *is a core for* $(L, D_m(L))$*, for every* $m \geq 1$*. Moreover* $\widetilde{D}_1$ *is a core for* $(\widetilde{L}, D(\widetilde{L}))$.

Proof. By virtue of Proposition 2.6 , D_1 is a core for $(\widetilde{L}, D(\widetilde{L}) \cap C_0(\mathbb{R}))$ if, for every $u \in D(\widetilde{L}) \cap C_0(\mathbb{R})$,

$$\lim_{x\to+\infty} \frac{\alpha(x)\, u'(x)}{x} = 0.$$

Then, taking (4.1) into account, we have to prove that $\lim\limits_{x\to+\infty} \dfrac{p(x)\, u'(x)}{x} = 0$.

In case (1), such limit relation follows immediately from Proposition 2.7.

As regards case (2), it follows from Proposition 2.8-(c) and 2.9, where $\varphi(x) := \dfrac{p(x)}{x}$ $(x \geq \delta)$. By Proposition 2.5 we also deduce the final assertion. ■

Assume now that there exists $\alpha_0 > 0$ such that

$$\alpha(x) \geq \alpha_0 \qquad \text{for } x \leq -\delta. \tag{4.2}$$

Then $u'' \in C_0(\mathbb{R})$, for every $u \in D_1$ and hence $D_1 \subset UC_b^2(\mathbb{R})$.
Therefore, from (3.5) we also get

$$\lim_{n\to\infty} n(G_n(u) - u) = \alpha u'' = Lu \qquad \text{in } C_0^{w_m}(\mathbb{R})$$

for every $u \in D_1$ and $m \geq 2$.

At this point, by reasoning as in the proof of Theorem 3.1, we can represent the semigroup $(T_m(t))_{t\geq 0}$ in terms of iterates of the operators G_n, which in this case we can write as

$$G_n(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f\left(\sqrt{\frac{2\alpha(x)}{n}}y + x\right) e^{-y^2/2} dy \qquad (f \in E(\mathbb{R}), x \in \mathbb{R}).$$

Theorem 4.2. *Let $m \geq 2$. Assume that α satisfies conditions* (2.1)*,* (2.2)*,* (4.1) *and* (4.2) *as well as on of conditions* (1) *or* (2) *of Theorem 4.1. Then, for every $f \in C_0^{w_m}(\mathbb{R})$ and $t \geq 0$*

$$T_m(t)f = \lim_{n\to\infty} G_n^{k(n)} f \qquad \text{in } C_0^{w_m}(\mathbb{R}),$$

where $(k(n))_{n\geq 1}$ is an arbitrary sequence of positive integers such that $k(n)/n \to t$ and ${G_n}^{k(n)}$ denotes the iterate of order $k(n)$ of G_n^*.

In particular the limit holds uniformly on compact subsets of $\mathbb{R}$.

Remark 4.3. Consider the Markov process associated with the semigroup $(T_m(t))_{t\geq 0}$, as stated in Theorem 2.3. Then the mean value $E_x(Z_t)$ of the random variable Z_t with respect to P^x is equal to x, for every $x \in \mathbb{R}$ and $t \geq 0$. Moreover there exist $C > 0$ and $x_0 > 0$ such that the variance $V_x(Z_t) \leq (e^{Ct} - 1)x^2$, for $|x| \geq x_0$.

Indeed, since $G_n(e_1) = e_1$ for every $n \geq 1$, by applying Theorem 4.2 for a given $m \geq 2$ we obtain

$$E_x(Z_t) = T_m(t)(e_1)(x) = \lim_{n\to\infty} (G_n)^{k(n)}(e_1)(x) = x,$$

where $(k(n))_{n\geq 1}$ is a sequence of positive integer such that $k(n)/n \to t$.

From (2.2) it follows that there exist $C > 0$ and $x_0 > 0$ such that $2\alpha(x) \leq Cx^2$, for $|x| \geq x_0$, therefore, since

$$G_n(e_2)(x) = \frac{2\alpha(x)}{n} + x^2 \leq \left(\frac{C}{n} + 1\right) x^2$$

we have

$$T_m(t)(e_2)(x) = \lim_{n\to\infty} (G_n)^{k(n)}(e_2)(x) \leq x^2 \lim_{n\to\infty} \left(\frac{C}{n} + 1\right)^{k(n)} = e^{Ct}x^2$$

which implies

$$V_x(Z_t) = E_x((Z_t)^2) - (E_x(Z_t))^2 = T_m(t)(e_2)(x) - x^2 \leq (e^{Ct} - 1)x^2.$$

Similar estimates have been obtained in [12, Remark 3.6] for Markov processes with state space $[0, +\infty]$.

By adapting the proof of Theorem 3.1 in [10] and applying Theorem 4.1 and 4.2 in [11] and Theorem 4.2, we obtain the following result which complements the ones stated in Corollaries 3.3 and 3.4. The detailed verification is left to the reader.

Corollary 4.4. *Under the same assumptions of Theorem 4.2, for every $f \in C_0^{w_m}(\mathbb{R})$ the following statements are equivalent.*

(a) *f is convex;*

(b) $G_{n+1}(f)(x) \leq G_n(f)(x)$, *for every* $x \in \mathbb{R}$ *and* $n \geq 1$;

(c) $f(x) \leq G_n(f)(x)$, *for every* $x \in \mathbb{R}$ *and* $n \geq 1$;

(d) $f(x) \leq T_m(t)(f)(x)$, *for every* $x \in \mathbb{R}$ *and* $t \geq 0$.

Moreover, if α *is constant, then the statements (a)-(d) are equivalent to*

(e) $T(t)(f)$ *is convex.*

Finally, if one of the statements (a)-(d) is satisfied, then for every $x \in \mathbb{R}$ *and* $0 \leq s \leq t$

$$T_m(s)(f)(x) \leq T_m(t)(f)(x)$$

and hence there exists $\lim_{t \to +\infty} T_m(t)(f)(x) \in \mathbb{R} \cup \{+\infty\}$.

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References

1. A. Albanese and E. M. Mangino, Cores of second order differential linear operators with unbounded coefficients on $\mathbb{R}^n$, *Semigroup Forum*, **70(2)** (2005), 278-295.

2. F. Altomare, *Lototsky-Schnabl operators on the unit interval and degenerate diffusion equations*, Progress in Functional Analysis, K. D. Bierstedt, J. Bonet, J. Horváth and M. Maestre (eds.), Proceedings of the International Functional Analysis Meeting on Occasion of the Sixtieth Birthday of Professor M. Valdivia (Peñiscola, Spain, October, 21-27, 1990), North-Holland Mathematics Studies 170, North-Holland, Amsterdam, 1992, 259-277.

3. F. Altomare, Lototsky-Schnabl operators on the unit interval, *C.R. Acad. Sci. Paris.*, **313** Serie I (1991), 371-375.

4. F. Altomare, *Approximation theory methods for the study of diffusion equations*, Approximation Theory, Proc. IDOMAT '95, Mathematical Research, Vol. 86, 9-26, Akademie Verlag, Berlin, 1995.

5. F. Altomare and A. Attalienti, Degenerate evolution equations in weighted continuous function spaces, Markov processes and the Black-Scholes equation-Part I, *Result. Math.*, **42** (2002), 193-211.

6. F. Altomare and A. Attalienti, Degenerate evolution equations in weighted continuous function spaces, Markov processes and the Black-Scholes equation-Part II, *Result. Math.*, **42** (2002) 212-228.

7. F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Application*, Vol. 17, de Gruyter Studies in Mathematics, de Gruyter, Berlin-New York, 1994.

8. F. Altomare and I. Carbone, On some degenerate differential operators on weighted function spaces, *J. Math. Anal. & Appl.*, **213** (1997), 308-333.

9. F. Altomare and E. M. Mangino, On a generalization of Baskakov operators, *Rev. Roumaine Math. Pures Appl.*, **44(5-6)** (1999), 683-705.

10. F. Altomare and E. M. Mangino, On a class of elliptic-parabolic equations on unbounded intervals, *Positivity*, **5** (2001), 239-257.

11. F. Altomare and S. Milella, Integral-type operators on weighted spaces of continuous function spaces on the real line, *J. Approx. Theory*, **152** (2006), 107-124.

12. F. Altomare and I. Raşa, On a class of exponential-type operators and their limit semigroups, *J. Approx. Theory*, **135** (2005), 258-275.

13. F. Altomare and I. Raşa, *On some classes of diffusion equations and related approximation problems*, ISNM, Vol. 151, Birkhäuser Verlag, Basel, 2005, pp. 13-26.

14. A. Attalienti, *Generalized Bernstein-Durrmeyer operators and the associated limit semigroup*, J. Approx. Theory 99 (1999), 289-309.

15. A. Attalienti and M. Campiti, Degenerate evolution problems and Beta-type operators, *Studia Math.*, **140** (2000), 117-139.

16. I. Carbone, Trotter's type theorems for generalized Szász-Mirakjan operators in polynomial weighted function spaces, *Indian J. Math.*, **44** (2002), 1-20.

17. K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics 194, Springer-Verlag, Berlin, 2000.

18. W. Feller, The parabolic differential equations and the associated semi-groups of transformations, *Ann. of Math.*, **55** (1952), 468-519.

19. E. M. Mangino, Differential operators with second order degeneracy and positive approximation processes, *Constr. Approx.*, **18** (2002), 443-466.

20. E. M. Mangino, A positive approximation sequence related to Black and Scholes equation, *Suppl. Rend. Circ. Mat. Palermo Serie II, Suppl.*, **68** (2002), 657-670.

21. A. Pazy, *Semigroups of Linear Operators and Application to Partial Differential Equations*, Appl. Math. Sci. Vol. 44, Springer-Verlag, 1983.

22. I. Raşa, *Semigroups associated to Mache operators*, Advanced Problems in Constructive Approximation (M. D. Buhmann and D. H. Mache, eds.), ISNM, Vol. 142, Birkhäuser-Verlag, Basel, 2002, pp. 143-152.

23. I. Raşa, *Semigroups associated to Mache operators (II)*, ISNM, Vol. 51, Birkhäuser Verlag, Basel, 2005, pp. 225-228.

24. I. Raşa, One-dimensional diffusion and approximation, *Mediterr. J. Math.*, **2** (2005), 153-169.

25. H. Trotter, Approximation of semigroups of operators, *Pacific J. Math.*, **8** (1958), 887-919.

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