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SOME PROPERTIES OF NEWTON'S METHOD FOR POLYNOMIALS WITH ALL REAL ZEROS

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Abstract. We prove an overshooting property of a multistep Newton method for polynomials with all real zeros, a special case of which is a classical result for the double-step Newton method. This result states, in essence, that a double Newton step from a point to the left of the smallest zero of a polynomial with all real zeros never overshoots the first critical point of the polynomial. Our result here states, in essence, that a Newton $(k+1)$-step from a point to the left of the smallest zero never overshoots the kth critical point of the polynomial, thereby generalizing the double-step result. Analogous results hold when starting from a point to the right of the largest zero.

We also derive a version of the aforementioned classical result that, unlike that result, takes into account the multiplicities of the first or last two zeros.

1. Introduction

Newton's method is a well-known iterative method for solving the equation $f(z) = 0$, defined by

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)} ,$$

for an appropriate z_0. We will consider an accelerated Newton method when $f(z)$ is a polynomial with all real zeros. There exists a classical result (see, e.g., [2, Theorem 5.5.9]), apparently originally proved by William Kahan in the early 1960's, which essentially states that a double-step Newton method, started from a point to the left (right) of the smallest (largest) zero of a polynomial with all real zeros, never overshoots the first (last) critical point of that polynomial. It also shows that if it overshoots that smallest (largest) zero, then one can proceed from such an

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overshoot without having to backtrack to a pre-overshoot iterate. This is a fairly curious property of Newton's method and we will show that it can be generalized for a Newton multistep method, where the Newton step is multiplied by any positive integer that is less than or equal to the degree of the polynomial.

Specifically (and in its simplest case), our generalization concerns a Newton $(k+1)$-step from a point to the left of the smallest zero. We will show that such a step never overshoots the kth critical point of the polynomial and that if it overshoots the kth zero, then the next Newton k-step is better than a Newton k-step from the original point. Our result is strengthened if information is available about the multiplicity of the $(k+1)$th zero: if that multiplicity is equal to q, then the same results hold for a Newton $(k+d)$-step, where $1 \leq d \leq \min\{k, q\}$. An analogous result holds for a starting point to the right of the largest zero. Our proof is quite different from the one in [2], which at first sight does not seem to allow for an easy generalization.

We will also show that this property can easily be specialized for the computation of the first zero if that zero's and the second zero's multiplicities are known with, once again, an analogue for the computation of the largest zero. It is intriguing that it is not only the multiplicity of the extreme zero that plays a role, but that of the next distinct zero as well. For comparison, let us consider an example where the first and second distinct zeros' multiplicities are 4 and 6, respectively. In that case, our result allows us to multiply the Newton step from a point to the left of the smallest zero by 8. By contrast, the result in [2] would only allow a doubling of that Newton step.

We point out that, as in [2, Theorem 5.5.9], we do not make any statement about the relative merits of the accelerated Newton method versus the regular one.

The paper is organized as follows. In Section 2, we present some notation and lemmas that will be needed in Section 3, which states and proves the main results.

2. Preliminaries

A polynomial with only real zeros is either convex or concave to the left and to the right of its smallest and largest zeros, respectively, which causes Newton's method to converge monotonically to the smallest or largest zero, when started from an initial point to the left or right of the smallest or largest zero, respectively.

The logarithmic derivative of a polynomial $p(x) = a_n \Pi_{j=1}^{n}(x - x_j)$ is $\phi(x) = p'(x)/p(x)$ and one can easily see that

$$\phi(x) = \frac{p'(x)}{p(x)} = \sum_{j=1}^{n} \frac{1}{x - x_j} \cdot$$

We will assume throughout that $x_1 \leq x_2 ... \leq x_n$ are all real. Denoting the zeros of $p'(x)$ by $x'_1 \leq x'_2 ... \leq x'_{n-1}$, which are then also all real, it is a direct consequence of Rolle's theorem that $x_1 \leq x'_1 \leq x_2 \leq x'_2 ... \leq x'_{n-1} \leq x_n$.

The function $\phi(x)$ is singular at each zero of $p(x)$ and zero at each zero of $p'(x)$ that does not coincide with a zero of $p(x)$. It is strictly decreasing on intervals between different zeros of $p(x)$. If $x_k < x_{k+1}$, then the graph of $\phi(x)$ on $[x_k, x_{k+1}]$ looks as in Figure 1.

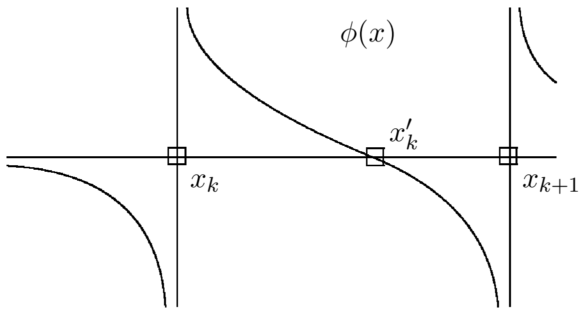

Fig. 1. The graph of $\phi(x) = \dfrac{p'(x)}{p(x)}$.

The following lemma states that taking a Newton s-step, i.e., a Newton step multiplied by a positive integer $s \leq n$, for a polynomial $p(x) = a_n(x - x_1)(x - x_2)...(x - x_n)$ with all real zeros $x_1 \leq x_2 \leq ... \leq x_n$ at a point $\bar{x} < x_1$, will never overshoot the sth zero of the polynomial, counting multiplicities.

Lemma 2.1. *Let $p(x)$ be a polynomial of degree $n \geq 2$ with all real zeros $x_1 \leq x_2 \leq ... \leq x_n$ and let $s \leq n - 1$ be a positive integer. Then for any $\bar{x} < x_1$, one has that $\bar{x} - s\dfrac{p(\bar{x})}{p'(\bar{x})} < x_s$. If $p(x)$ has no zero of multiplicity n, then the lemma holds with $s \leq n$.*

Proof. We first consider the case $s \leq n - 1$. Since

$$\sum_{j=1}^{s} \frac{1}{x_j - \bar{x}} < \sum_{j=1}^{n} \frac{1}{x_j - \bar{x}} = \left| \frac{p'(\bar{x})}{p(\bar{x})} \right| , \quad \text{we have that}$$

$$\left(\frac{1}{s} \sum_{j=1}^{s} \frac{1}{x_j - \bar{x}} \right)^{-1} > s \left| \frac{p(\bar{x})}{p'(\bar{x})} \right| .$$

The left-hand side in this inequality is the harmonic mean of the positive numbers $(x_1 - \bar{x}), (x_2 - \bar{x}), ..., (x_s - \bar{x})$, and is therefore always less than or equal to the largest of these numbers, which is $x_s - \bar{x}$. This means that $s\left|\dfrac{p(\bar{x})}{p'(\bar{x})}\right| < x_s - \bar{x}$, which concludes the proof for $s \leq n-1$. For $s = n$, the proof proceeds as in the case $s \leq n-1$, except that the strict inequality is now caused by the fact that the harmonic mean is always strictly less than the largest number, unless all numbers are equal. But that is made impossible by the requirement that no zero be of multiplicity n. ∎

Remarks. For a zero of multiplicty n, we have $p(x) = a_n(x - x_1)^n$ and it is easy to show that in this case $\bar{x} - n\dfrac{p(\bar{x})}{p'(\bar{x})} = x_1$. Analogous results to the ones in this lemma are obtained for $\bar{x} > x_n$.

The following optimization problem will play an important role in our main result in the next section.

Lemma 2.2. *An optimal solution of the optimization problem*

$$\textit{max}\left\{\sum_{j=1}^{m} \frac{\alpha_j}{1 - aw_j} \;:\; \sum_{j=1}^{m} \alpha_j w_j = b\,;\; w_1 \geq w_j\ (\forall j \neq 1)\,;\; w_j \geq 0\ (\forall j)\right\},$$

where m and the α_j's are fixed positive integers, and a and b are fixed constants with $a > 0$ and $b < \alpha_1/a$, is obtained for $w_1 = b/\alpha_1$ and $w_j = 0$ $(j \geq 2)$.

Proof. First, on the feasible set we have $w_1 \leq b/\alpha_1 < 1/a$ and therefore also $w_j < 1/a$ $(j \geq 2)$ because $w_1 \geq w_j$ $(j \geq 2)$. As a consequence, $1 - aw_j > 0$ for all j so that the objective function is continuous on the feasible region, which forms a compact set. A continuous function on a compact set achieves its maximum on this set and to find the solution it is therefore sufficient to examine the points which satisfy the optimality conditions. To satisfy the first-order optimality conditions (see, e.g., [1]), we need to find Lagrange multipliers λ, $\{\nu_j\}_{j=2}^{m}$, and $\{\mu_j\}_{j=1}^{m}$, such that

$$\frac{a\alpha_1}{(1 - aw_1)^2} - \alpha_1\lambda - \sum_{j=2}^{m} \nu_j - \mu_1 = 0 \tag{1}$$

$$\frac{a\alpha_j}{(1 - aw_j)^2} - \alpha_j\lambda + \nu_j - \mu_j = 0 \qquad (j \geq 2)\,, \tag{2}$$

with $\nu_j \geq 0$ $(j \geq 2)$, $\mu_j \geq 0$ $(\forall j)$, $(w_j - w_1)\nu_j = 0$ $(j \geq 2)$, and $w_j\mu_j = 0$ $(\forall j)$. We must have that $w_1 > 0$ because otherwise the constraints $w_1 \geq w_j \geq 0$ $(j \neq 1)$ would also make all other variables vanish, thereby violating the equality constraint. This implies that $\mu_1 = 0$. (In fact, we could have simply left out the constraint $w_1 \geq 0$ from the beginning since it is implied by the other constraints.)

We will now show that at the optimal solution it is not possible to have nonzero variables w_j ($j \neq 1$) with a different (and necessarily smaller) value than the value of w_1. We proceed by contradiction: assume that there exist nonzero w_j's that have values that are different from the value of w_1. For such w_j's we have that the corresponding $\nu_j = \mu_j = 0$ and therefore that

$$\frac{a\alpha_j}{(1-aw_j)^2} - \alpha_j\lambda = 0\ , \text{ which implies } \frac{a}{(1-aw_j)^2} = \lambda\ . \tag{3}$$

Since $1 - aw_j > 0$ for all j, it follows that all such w_j's must have the same value. We denote that value by $\bar{w}$ and we define $J_1 = \{j : w_j = w_1\}$. If $J_1 \neq \{1\}$, then for $\ell \in J_1 \setminus \{1\}$, and using the expression for λ from (3) in (2), we have

$$\frac{a\alpha_\ell}{(1-aw_1)^2} - \alpha_\ell\lambda + \nu_\ell = 0 \implies \frac{a\alpha_\ell}{(1-aw_1)^2} - \frac{a\alpha_\ell}{(1-a\bar{w})^2} = -\nu_\ell.$$

But $w_1 > \bar{w}$ so that the left-hand side in the last equation is strictly positive. However, the right-hand side is nonpositive and we have arrived at a contradiction when $J_1 \neq \{1\}$. If $J_1 = \{1\}$, then $\nu_j = 0$ for all $j \geq 2$ and we have from (1)

$$\frac{a\alpha_1}{(1-aw_1)^2} - \alpha_1\lambda = 0 \implies \frac{a\alpha_1}{(1-aw_1)^2} = \frac{a\alpha_1}{(1-a\bar{w})^2}\ .$$

But this is impossible because $w_1 > \bar{w}$ so that we have arrived at a contradiction in this case as well. This means that, at the optimal solution, for $j \geq 2$, we either have $w_j = w_1$ or $w_j = 0$. Without loss of generality, let us assume that $\{w_j\}_{j=2}^k$ ($k \leq m$) are the variables equal to w_1 with the remainder of the variables, if there are any, equal to zero. The equality constraint then gives

$$w_1 = \frac{b}{\sum_{j=1}^k \alpha_j}\ .$$

We now show that the solution with $k = 1$ is optimal. Substituting our candidate solution into the objective function and using the fact that $ab < \alpha_1$, we obtain

$$\begin{aligned}
&\left(\sum_{j=1}^m \alpha_j - \sum_{j=1}^k \alpha_j\right) + \frac{\sum_{j=1}^k \alpha_j}{1 - ab/\sum_{j=1}^k \alpha_j} \\
&= \left(\sum_{j=1}^m \alpha_j\right) + \frac{ab}{1 - ab/\sum_{j=1}^k \alpha_j} \leq \left(\sum_{j=1}^m \alpha_j\right) + \frac{ab}{1 - ab/\alpha_1}\ .
\end{aligned}$$

Since

$$\left(\sum_{j=1}^m \alpha_j\right) + \frac{ab}{1 - ab/\alpha_1} = \left(\sum_{j=2}^m \alpha_j\right) + \frac{\alpha_1}{1 - ab/\alpha_1}\ ,$$

this upper bound on the objective function corresponds to the value obtained for the feasible solution $w_1 = b/\alpha_1$ and $w_j = 0$ for $j \geq 2$. ∎

This lemma immediately leads to the following corollary.

Corollary 2.1. *An optimal solution of the optimization problem*

$$\textit{max} \left\{ \sum_{j=1}^{m} \frac{\alpha_j}{1 - a w_j} \; : \; \sum_{j=1}^{m} \alpha_j w_j = b \, ; \, w_1 \geq w_2 \geq \ldots \geq w_m \geq 0 \right\} ,$$

where m and the α_j's are fixed positive integers, and a and b are fixed constants with $a > 0$ and $b < \alpha_1/a$, is obtained for $w_1 = b/\alpha_1$ and $w_j = 0$ $(j \geq 2)$.

Proof. The feasible set of this optimization problem is contained in the feasible set of the optimization problem in Lemma 2.2 and its objective function is the same. The maximum value obtained in Lemma 2.2 must therefore be an upper bound on the maximum value that is obtainable here. Because the solution of the problem in Lemma 2.2 also satisfies the constraints in the current problem, that solution must then also be optimal for this problem. ∎

3. Overshooting Theorems

In this section we generalize Theorem 5.5.9 in [2] by considering the overshooting properties of a Newton $(k+1)$-step from a point to the left of the smallest zero, as we explained in the introduction. The result is strengthened if information about the multiplicity of the $(k+1)$th zero is available. We subsequently use this result to derive a version of Theorem 5.5.9 in [2], which, like that theorem, concerns the first two (or last two) zeros of the polynomial, but also, unlike that theorem, takes into account the multiplicities of those zeros.

Here is the first result.

Theorem 3.1. *Let $p(x)$ be a polynomial of degree $n \geq 2$ with all real zeros $x_1 \leq x_2 \leq \ldots \leq x_n$, let $x_1' \leq x_2' \leq \ldots \leq x_{n-1}'$ be the zeros of $p'(x)$, and let q, with $1 \leq q \leq n-k$, be the largest positive integer such that $x_{k+1} = x_{k+q}$ for $1 \leq k \leq n-1$. Set $y = \bar{x} - (k+d)\dfrac{p(\bar{x})}{p'(\bar{x})}$ for a point $\bar{x} < x_1$ and for any positive integer d, satisfying $1 \leq d \leq \min\{k, q\}$. Require also that when $k = n - q$ and $d = q$, $p(x)$ cannot have a zero of multiplicity n.*

Then $y < x_k'$ and if $y \geq x_k$, then $x_k < x_{k+1}$, and

$$\bar{x} - k\frac{p(\bar{x})}{p'(\bar{x})} \leq y - k\frac{p(y)}{p'(y)} \leq x_k \, , \tag{4}$$

where the rightmost inequality is strict when $y > x_k$.

When $k = 1$, *then if* $y < x_k$:

$$\bar{x} - \frac{p(\bar{x})}{p'(\bar{x})} < y - \frac{p(y)}{p'(y)} < x_1 \ . \tag{5}$$

If there is a zero of multiplicity n *when* $k = n - q$ *and* $d = q$, *then* $y = x_1 = x_2 = \ldots = x_n = x'_{n-1}$.

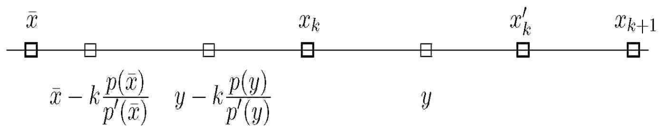

Fig. 2. Positioning of relevant points in Theorem 3.1 when $y > x_k$.

Proof. Throughout the proof we will use the fact that, because $\bar{x} < x_1$, $\phi(\bar{x}) = p(\bar{x})/p'(\bar{x}) < 0$. We begin by observing that $x_k \le x'_k$ so that if $y < x_k$, then $y < x'_k$.

If $y \ge x_k$, then with Lemma 2.1 we have that $x_k \le y < x_{k+d} = x_{k+1}$ so that $x_k < x_{k+1}$. We note that we are able to use Lemma 2.1 because, unless $d = q$, $k + d < k + q \le n$, so that $k + d \le n - 1$. If $d = q$, then we can use the lemma because of the requirement that there be no zero of multiplicity n in this case when $k + q = n$; if there were such a zero when $k = n - q$ and $d = q$, then the remark following Lemma 2.1 implies that $y = x_1 = x_2 = \ldots = x_n$, so that $y = x'_{n-1}$. Having established that $x_k < x_{k+1}$, we obtain as an immediate consequence that $x_k < x'_k$. Therefore, when $y = x_k$, $y < x'_k$. Also, the inequalities in (4) when $y = x_k$ follow directly from Lemma 2.1.

We are left with the case $y > x_k$ for which we already know that $x_k < x_{k+1}$. For convenience we define $\gamma_j = (|\phi(\bar{x})|(x_j - \bar{x}))^{-1}$, where $\phi(x)$ is the same as before. With this definition of γ_j, we have that $\gamma_1 \ge \gamma_2 \ge \ldots \ge \gamma_n > 0$ and also that $\sum_{j=1}^{n} \gamma_j = 1$. We now examine $\phi(y)$:

$$\begin{aligned}
\phi(y) &= \sum_{j=1}^{n} \frac{1}{\bar{x} - (k+d)/\phi(\bar{x}) - x_j} \\
&= \sum_{j=1}^{n} \frac{\phi(\bar{x})}{(\bar{x} - x_j)\phi(\bar{x}) - (k+d)} \\
&= |\phi(\bar{x})| \sum_{j=1}^{n} \frac{1}{(k+d) - (x_j - \bar{x})|\phi(\bar{x})|}
\end{aligned}$$

$$
\begin{aligned}
&= |\phi(\bar{x})| \sum_{j=1}^{n} \frac{1}{(k+d) - \gamma_j^{-1}} \\
&= \frac{|\phi(\bar{x})|}{k+d} \left(n - \sum_{j=1}^{n} \frac{1}{1-(k+d)\gamma_j} \right) \\
&\geq \frac{|\phi(\bar{x})|}{k+d} \left(n - \sum_{j=1}^{k} \frac{1}{1-(k+d)\gamma_j} - \max \left\{ \frac{d}{1-(k+d)\gamma_{k+1}} \right.\right. \\
&\qquad \left.\left. + \sum_{j=k+d+1}^{n} \frac{1}{1-(k+d)\gamma_j} \right\} \right) .
\end{aligned}
$$

Because $\sum_{j=1}^{n} \gamma_j = 1$, we impose the constraints for the maximization problem as $d\gamma_{k+1} + \sum_{j=k+d+1}^{n} \gamma_j = 1 - \sum_{j=1}^{k} \gamma_j$ and $\gamma_{k+1} \geq \gamma_{k+d+1} \geq ... \geq \gamma_n \geq 0$. Our assumption that $\bar{x} - (k+d)/\phi(\bar{x}) > x_k$ implies that $\gamma_k > 1/(k+d)$ and therefore also that $\gamma_j > 1/(k+d)$ for $1 \leq j \leq k$. As a consequence,

$$
1 - \sum_{j=1}^{k} \gamma_j < 1 - \frac{k}{d+k} = \frac{d}{d+k} .
$$

The solution of the maximization problem then follows directly from Corollary 2.1 of Lemma 2.2 with $m = n-(k+d)+1$, $\alpha_1 = d$, $\alpha_j = 1$ ($j \geq 2$), $b = 1 - \sum_{j=1}^{k} \gamma_j$, and $a = k+d$, which yields

$$
\begin{aligned}
&\phi(y) \\
&\geq \frac{|\phi(\bar{x})|}{k+d} \left(n - \sum_{j=1}^{k} \frac{1}{1-(k+d)\gamma_j} - \frac{d}{1-(k+d)\frac{1}{d}\left(1-\sum_{j=1}^{k}\gamma_j\right)} - (n-(k+d)) \right) \\
&= \frac{|\phi(\bar{x})|}{k+d} \left(k+d + \sum_{j=1}^{k} \frac{1}{(k+d)\gamma_j - 1} - \frac{d^2}{\sum_{j=1}^{k}((k+d)\gamma_j - 1)} \right) . \qquad (6)
\end{aligned}
$$

Since $(k+d)\gamma_j - 1 > 0$ for $1 \leq j \leq k$, we can use the harmonic arithmetic means inequality to obtain

$$
\frac{1}{\frac{1}{k}\sum_{j=1}^{k}((k+d)\gamma_j - 1)} \leq \frac{1}{k} \sum_{j=1}^{k} \frac{1}{(k+d)\gamma_j - 1} ,
$$

$$
\text{or} \quad \frac{k^2}{\sum_{j=1}^{k}((k+d)\gamma_j - 1)} \leq \sum_{j=1}^{k} \frac{1}{(k+d)\gamma_j - 1} .
$$

Using this inequality in (6), we obtain

$$\phi(y) \geq \frac{|\phi(\bar{x})|}{k+d}\left(k+d+\frac{k^2}{\sum_{j=1}^{k}((k+d)\gamma_j-1)}-\frac{d^2}{\sum_{j=1}^{k}((k+d)\gamma_j-1)}\right)$$

$$= \frac{|\phi(\bar{x})|}{k+d}\left(k+d+\frac{k^2-d^2}{(k+d)\sum_{j=1}^{k}\gamma_j-k}\right)$$

$$\geq \frac{|\phi(\bar{x})|}{k+d}\left(k+d+\frac{k^2-d^2}{(k+d)\sum_{j=1}^{n}\gamma_j-k}\right)$$

$$= \frac{|\phi(\bar{x})|}{k+d}\left(k+d+\frac{(k-d)(k+d)}{(k+d)-k}\right)$$

$$= \frac{k}{d}|\phi(\bar{x})|\,, \tag{7}$$

so that $\phi(y) > 0$. Since $x_k < y < x_{k+1}$, Figure 1 allows us to conclude that $y < x'_k$.

From (7) we also have

$$y - \frac{k}{\phi(y)} = \bar{x} + \frac{k+d}{|\phi(\bar{x})|} - \frac{k}{\phi(y)} \geq \bar{x} + \frac{k+d}{|\phi(\bar{x})|} - \frac{kd}{k|\phi(\bar{x})|} = \bar{x} - \frac{k}{\phi(\bar{x})}\,.$$

Next, we recall that $x_k < y < x_{k+1}$ and that $\phi(y) > 0$. From this follows that

$$y - \frac{k}{\phi(y)} = y - \frac{k}{\displaystyle\sum_{j=1}^{k}\frac{1}{y-x_j} + \sum_{j=k+1}^{n}\frac{1}{y-x_j}} < y - \frac{k}{\displaystyle\sum_{j=1}^{k}\frac{1}{y-x_j}}$$

$$< y - \frac{k}{\dfrac{k}{y-x_k}} = x_k\,.$$

The only thing left to prove is that (5) holds. If $y < x_1$, then we have immediately from Lemma 2.1 (or from the fact that a polynomial is either strictly convex or strictly concave to the left of its smallest zero) that $y - \dfrac{p(y)}{p'(y)} < x_1$ and therefore that

$$\bar{x} - \frac{p(\bar{x})}{p'(\bar{x})} < y < y - \frac{p(y)}{p'(y)} < x_1\,.$$

This concludes the proof. ■

Remarks. If $x_k < x_{k+1}$, then q in the statement of the theorem is the multiplicity of x_{k+1}. If q is unknown, then one can simply apply the theorem with $d = 1$.

An analogous theorem can be obtained for a point $\bar{x}$ to the right of the largest zero, which for $k = 1$ reduces to Theorem 5.5.9 in [2].

The following theorem is a direct consequence of the first theorem as applied to the first (or last) two zeros of the polynomial. It provides a version of Theorem 5.5.9 in [2] that takes into account the multiplicities of those zeros.

Theorem 3.2. *Let $p(x)$ be a polynomial of degree $n \geq 2$ with all real zeros $x_1 \leq x_2 \leq \ldots \leq x_n$, let $x_1' \leq x_2' \leq \ldots \leq x_{n-1}'$ be the zeros of $p'(x)$, let $x_1 = x_2 = \ldots = x_s$, where s is a positive integer, let q be the largest positive integer such that $x_{s+1} = x_{s+2} = \ldots = x_{s+q}$ and let the multiplicity of x_1 be strictly less than n. Set $y = \bar{x} - (s+d)\dfrac{p(\bar{x})}{p'(\bar{x})}$ for a point $\bar{x} < x_1$ and for any positive integer d, satisfying $1 \leq d \leq \min\{s, q\}$. Then $y < x_s'$ and*

$$\bar{x} - s\frac{p(\bar{x})}{p'(\bar{x})} \leq y - s\frac{p(y)}{p'(y)} \leq x_1. \tag{8}$$

The inequalities in (8) are strict when $y < x_1$ and the rightmost inequality is strict when $y > x_1$.

If the multiplicity of x_1 is equal to n, then $s = n$ and $y = \bar{x} - n\dfrac{p(\bar{x})}{p'(\bar{x})} = x_1 = x_s'$.

Proof. We first recall that $p(\bar{x})/p'(\bar{x}) < 0$. If $y = \bar{x} - (s+d)\dfrac{p(\bar{x})}{p'(\bar{x})} < x_1 = x_s$, then, because there are no zeros of multiplicity n, we have from Lemma 2.1 that $y - s\dfrac{p(y)}{p'(y)} < x_s$ and therefore that

$$\bar{x} - s\frac{p(\bar{x})}{p'(\bar{x})} < y < y - s\frac{p(y)}{p'(y)} < x_s = x_1 \leq x_s' \ ,$$

so that both $y < x_s'$ and (8) are satisfied. For $y \geq x_1 = x_s$ we have, with Lemma 2.1, that $x_1 = x_s \leq y < x_{s+d} = x_{s+1}$, so that $x_s < x_{s+1}$. The rest of the proof then follows directly from Theorem 3.1. with $k = s$. ∎

Remarks. If $x_s < x_{s+1}$ then s and q are the multiplicities of x_s and x_{s+1}, respectively. If one does not know q, then one can simply set $d = 1$. If s is also unknown, then our theorem reduces to Theorem 5.5.9 in [2]. Once again, analogous results hold for a point $\bar{x}$ to the right of the largest zero of the polynomial.

References

1. A. V. Fiacco and G. P. McCormick, *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, J. Wiley and Sons, New York, 1968.

2. J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980.

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