# CODERIVATIVES OF FRONTIER AND SOLUTION MAPS
# IN PARAMETRIC MULTIOBJECTIVE OPTIMIZATION

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**Abstract.** This paper concerns sensitivity analysis for general parametric constrained problems of multiobjective optimization in infinite-dimensional spaces by using advanced tools of modern variational analysis and generalized differentiation. We pay the main attention to computing and estimating coderivatives of frontier and efficient solution maps in parametric multiobjective problems with respect to generalized order optimality that include a vast majority of conventional multiobjective problems in the presence of geometric, operator, functional, and equilibrium constraints. The obtained results are new in both finite-dimensional and infinite-dimensional spaces.

## 1. INTRODUCTION

This paper is devoted to developing new applications of some advanced tools of modern variational analysis and generalized differentiation to parametric multiobjective optimization problems from the viewpoint of *coderivative approach* to *sensitivity* of optimal value/frontier and efficient solution maps under parameter perturbations.

Let $X$, $Y$, and $Z$ be Banach spaces, and let $\Theta \subset Z$ be an arbitrary nonempty subset (may not be convex and/or conic) that defines a *generalized ordering relation* on $Z$ in the sense precisely described in what follows. Given a single-valued *cost mapping* $f\colon X \times Y \to Z$ and a set-valued *constraint mapping* $G\colon X \rightrightarrows Y$, we consider the parametric family of *constrained multiobjective optimization problems*:

$$(1.1) \qquad \text{minimize}_\Theta \, f(x,y) \quad \text{subject to } y \in G(x),$$

where the "minimization" with respect to the decision variable $y$ is induced by the ordering set $\Theta$, and where $x$ is a perturbation parameter. Let us associate with the parametric problem (1.1) the corresponding *frontier map* $\mathcal{F} \colon X \rightrightarrows Z$ defined by

$$(1.2) \qquad \mathcal{F}(x) := \mathrm{Eff}\big(f(x, G(x)) \,|\, \Theta\big)$$

and the *optimal/efficient solution map* $S \colon X \rightrightarrows Y$ given by

$$(1.3) \qquad S(x) := \big\{y \in G(x) \,\big|\, f(x, y) \in \mathcal{F}(x)\big\}.$$

Note that the notion of optimality/efficiency employed in (1.2) and (1.3) is understood in the sense of the so-called *generalized order optimality* defined and discussed in Section 3; this includes and extends a vast majority if conventional notions broadly used in multiobjective optimization. Observe also that for the case of scalar optimization problems, the frontier map (1.2) reduces to the standard *marginal/optimal value function,* which plays a significant role in various aspects of optimization theory and its applications; see, e.g., [3, 20, 21, 28] with the discussions and references therein.

It has been well recognized that the *coderivative* of set-valued mappings introduced by Mordukhovich [17] is a convenient tool to study many important issues in variational analysis and optimization; we refer the reader to the recent books [4, 10, 20, 21, 28, 30] with their commentaries and bibliographies. In particular, the coderivative construction of [17] and its infinite-dimensional extensions allow us to fully characterized *robust Lipschitzian stability* of general set-valued mappings and their specifications frequently appeared in optimization frameworks; see [18, 20, 28] and also Remark 5.5 below for more details and references.

There are numerous applications of coderivatives to sensitivity analysis of scalar (single-objective) optimization problems; we refer the reader to [8, 12, 15, 19, 20, 23, 24, 25, 34] for just a few of them. Coderivatives have been also employed in, e.g., [1, 5, 6, 22, 21, 33, 32, 35] to derive necessary conditions in various multiobjective optimization problems. However, we are not familiar with any results on using coderivatives in sensitivity analysis for multiobjective problems (in particular, for computing/estimating coderivatives of frontier and solution maps in multiobjective optimization) although other *primal-type* generalized derivative constructions have been employed for these purposes in [9, 11, 14, 29, 31].

The *primary objective* of this paper is to obtain verifiable formulas for computing or upper estimating the coderivatives of frontier and solution maps in general parametric problems of constrained multiobjective optimization. The obtained results are new in both finite-dimensional and infinite-dimensional spaces not only for the general problems under consideration but also for their more conventional concretizations including those presented in the paper. Our approach is based on the coderivative calculus developed in [20] with utilizing specific structures of multiobjective problems and their frontier and optimal solution maps defined in (1.2)

and (1.3), respectively, with the notions of multiobjective optimality/efficiency made precise below.

The rest of the paper is organized as follows. In Section 2 we recall and discuss some *basic constructions* from variational analysis and generalized differentiation broadly employed in the formulations and proofs of the main results.

Section 3 is devoted to deriving *upper estimates* and *precise formulas* for computing coderivatives of the *frontier map* (1.2) with respect to the notion of efficiency induced by the concept of *generalized order optimality* from [21], which encompasses various conventional as well as extended notions of optimal/efficient solutions to multiobjective optimization and equilibrium problems. In Section 4 we specify general coderivative formulas for frontier maps in multiobjective problems involving explicit constraints of geometric, operator, functional, and equilibrium types particularly important in applications.

Finally, Section 5 contains results on computing/estimating coderivatives of the *efficient solution map* (1.3) to the multiobjective optimization problem (1.1) via the corresponding coderivatives of the frontier map (1.2). These results are based on the extended coderivative analysis of solution maps to parameterized *generalized equations* with moving constraints on decision variables and *parameter-independent fields* (see below); they are certainly of their own interest being important for other applications.

Throughout the paper we use the standard *notation* of variational analysis and generalized differentiation; see, e.g., [20, 21]. Unless otherwise stated, all the spaces are assumed to be *Banach* with the norm $\| \cdot \|$ and the canonical pairing between the space in question and its topological dual. For a dual space $X^*$, the notation $x_k^* \to x^*$ stands for the *norm convergence* of the sequence $\{x_k^*\}$, $k \in \mathbb{N} := \{1, 2, \ldots\}$, while $x_k^* \xrightarrow{w^*} x^*$ indicates the convergence of $\{x_k^*\}$ in the *weak\* topology* of $X^*$. Furthermore, the symbol $x \xrightarrow{\Omega} \bar{x}$ for a set $\Omega \subset X$ means that $x \to \bar{x}$ with $x \in \Omega$. Given a set-valued mapping $F \colon X \rightrightarrows X^*$, we denote by

$$(1.4) \quad \operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \left\{ x^* \in X^* \ \middle| \ \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \right.$$
$$\left. \text{with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N} \right\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* of the mapping $F$ as $x \to \bar{x}$ with respect to the norm topology of $X$ and the weak\* topology of $X^*$. As usual, $\mathbb{B}$ stands for the *closed unit ball* in the space in question,

$$B_\eta(x) = B(x; \eta) := x + \eta \mathbb{B},$$

and $A^* \colon Y^* \to X^*$ denotes the *adjoint operator*— to a linear continuous operator $A \colon X \to Y$ between Banach spaces.

## 2. Basic Definitions and Preliminaries

In this section we recall some basic definitions and preliminary material on variational analysis and generalized differentiation taken from [20] and broadly used in the paper. In [20], the reader can find proofs as well as related results, discussions, and commentaries.

Given $\Omega \subset X$ and $\varepsilon \geq 0$, we define the collection of $\varepsilon$-*normals* to $\Omega$ at $x \in \Omega$ by

$$(2.1) \qquad \widehat{N}_\varepsilon(x; \Omega) := \Big\{ x^* \in X^* \Big| \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq \varepsilon \Big\}.$$

When $\varepsilon = 0$, observe that $\widehat{N}(x; \Omega) := \widehat{N}_0(x; \Omega)$ in (2.1) is a cone called the *prenormal cone* or the *Fréchet normal cone* to $\Omega$ at $x$. We further let for convenience $\widehat{N}_\varepsilon(x; \Omega) := \emptyset$ if $x \notin \Omega$. The sequential outer limit (1.4) of $\widehat{N}_\varepsilon(x; \Omega)$ as $x \to \bar{x} \in \Omega$ and $\varepsilon \downarrow 0$ defined by

$$(2.2) \qquad N(\bar{x}; \Omega) := \underset{\substack{x \to \bar{x} \\ \varepsilon \downarrow 0}}{\mathrm{Lim\,sup}}\ \widehat{N}_\varepsilon(x; \Omega)$$

is known as *limiting*, or *basic*, or *Mordukhovich normal cone* to $\Omega$ at $\bar{x}$ introduced in [16] in finite-dimensional spaces. We say that $\Omega$ is *normally regular* at $\bar{x} \in \Omega$ if $N(\bar{x}; \Omega) = \widehat{N}(\bar{x}; \Omega)$. Besides convex sets and those with smooth boundaries, the class of normal regularity includes a range of "nice" sets and is stable with respect to various operations under appropriate qualification conditions; see [20, 21] for numerous results and discussions.

It is possible (while quite nontrivial) to equivalently put $\varepsilon = 0$ in (2.2), i.e., to replace $\varepsilon$-normals in (2.2) by Fréchet normals at $x \in \Omega$ provided that $\Omega$ is locally closed around $\bar{x}$ and that the space $X$ is *Asplund*. The latter remarkable subclass of Banach spaces has been comprehensively investigated in the geometric theory of functional analysis; it can be described as a collection of Banach spaces whose every separable subspace has a separable dual. It is well known that the class of Asplund spaces is sufficient large containing, in particular, all reflexive spaces and all spaces with separable duals. It has been broadly applied in variational analysis; see [20, 21] with numerous references and discussions on Asplund spaces and their applications.

Note that the basic normal cone (2.2) is often *nonconvex*, even for simple sets in $\mathbb{R}^2$ as, e.g., for the graph of the function $\varphi(x) = |x|$ and for the epigraph of the function $\varphi(x) = -|x|$ at $\bar{x} = 0$. Therefore this normal cone *cannot* be tangentially generated by a polarity/duality correspondence, since polarity always implies convexity. In spite of (actually due to) nonconvexity, the normal cone

(2.2) and the associated with it coderivatives of set-valued mappings and subdifferentials of extended-real-valued functions enjoy comprehensive calculus rules and other nice properties, especially in the Asplund space setting; see [20, 21] for an extended theory and various applications. This is mainly based on the powerful *variational/extremal principles* of modern variational analysis.

Let $F\colon X \rightrightarrows Y$ be a set-valued mapping between Banach spaces with the graph

$$\operatorname{gph} F := \big\{(x,y) \in X \times Y \,\big|\, y \in F(x)\big\}.$$

The *normal coderivative $D_N^* F(\bar{x}, \bar{y})\colon Y^* \rightrightarrows X^*$ of $F$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is defined by

$$(2.3) \qquad D_N^* F(\bar{x}, \bar{y})(y^*) := \big\{x^* \in X^* \,\big|\, (x^*, -y^*) \in N((\bar{x}, \bar{y}); \operatorname{gph} F)\big\},$$

which by (2.2) can be equivalently described as

$$D_N^* F(\bar{x}, \bar{y})(y^*)$$
$$(2.4) \quad = \Big\{x^* \in X^* \,\Big|\, \exists\, \varepsilon_k \downarrow 0,\ (x_k, y_k) \to (\bar{x}, \bar{y}),\ (x_k^*, y_k^*) \xrightarrow{w^*} (x^*, y^*)$$
$$\text{with}\ (x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \operatorname{gph} F\big)\Big\}.$$

The *mixed coderivative $D_M^* F(\bar{x}, \bar{y})\colon Y^* \rightrightarrows X^*$ of $F$ at $(\bar{x}, \bar{y})$ is defined by

$$D_M^* F(\bar{x}, \bar{y})(y^*):$$
$$(2.5) \qquad = \Big\{x^* \in X^* \,\Big|\, \exists\, \varepsilon_k \downarrow 0,\ (x_k, y_k) \to (\bar{x}, \bar{y}),\ x_k^* \xrightarrow{w^*} x^*,$$
$$y_k^* \to y^*\ \text{with}\ (x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \operatorname{gph} F\big)\Big\}.$$

We omit $\bar{y} = f(\bar{x})$ in the above coderivative notion if $F = f\colon X \to Y$ is single-valued. Note also that $\varepsilon_k$ can be dropped ($\equiv 0$) in the limiting expressions (2.4) and (2.5) if the graph of $F$ is locally closed around $(\bar{x}, \bar{y})$ and if both spaces $X$ and $Y$ are Asplund–in this case the product space $X \times Y$ is Asplund as well.

We can see from (2.4) and (2.5) that the only difference between the normal and mixed coderivatives is that the weak* convergence of $y_k^* \xrightarrow{w^*} y^*$ in (2.4) is replaced by the strong (in the norm topology of $Y^*$) convergence of $y_k^* \to y^*$ in (2.3). Thus these coderivatives agree if $\dim Y < \infty$, where they both reduce to the coderivative $D^* F(\bar{x}, \bar{y})$ originally introduced in [17]. In general we obviously have the inclusion

$$(2.6) \qquad D_M^* F(\bar{x}, \bar{y})(y^*) \subset D_N^* F(\bar{x}, \bar{y})(y^*)\ \text{for all}\ y^* \in Y^*,$$

where the equality holds in various settings with $\dim Y = \infty$, which are partially listed in [20, Proposition 4.9], while the inclusion in (2.6) may be strict even for

single-valued Lipschitz continuous mappings with values in Hilbert spaces; see [20, Example 1.35].

We say that $F$ is *M-regular* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if

$$(2.7) \qquad D_M^* F(\bar{x}, \bar{y})(y^*) = \left\{ x^* \in X^* \,\middle|\, (x^*, -y^*) \in \widehat{N}\big((\bar{x}, \bar{y}); \operatorname{gph} F\big) \right\}$$
$$\text{for all } y^* \in Y^*.$$

By (2.6) this property is generally less restrictive in comparison with its *N-regularity* counterpart when $D_M^* F(\bar{x}, \bar{y})$ is replaced by $D_N^* F(\bar{x}, \bar{y})$ in (2.7). Both there graphical regularity properties hold, in particular, if either the graph of $F$ is *convex* or $F = f \colon X \to Y$ is single-valued and *strictly differentiable* at $\bar{x}$ in the sense that

$$\lim_{x, u \to \bar{x}} \frac{f(x) - f(u) - \langle \nabla f(\bar{x}), x - u \rangle}{\|x - u\|} = 0$$

with the derivative operator $\nabla f(\bar{x})$; the latter is always the case when $f \in C^1$ around $\bar{x}$. Furthermore, for mappings $f$ strictly differentiable at $\bar{x}$ we have the representation

$$(2.8) \qquad D_N^* f(\bar{x})(y^*) = D_M^* f(\bar{x})(y^*) = \left\{ \nabla f(\bar{x})^* y^* \right\} \text{ for all } y^* \in Y^*,$$

which shows that the coderivatives under consideration are appropriate extensions of the *adjoint derivative* operator to nonsmooth and set-valued mappings.

We pay the main attention in this paper to evaluating the *mixed coderivative* in case of frontier maps, which plays the most crucial role in infinite-dimensional spaces. The common notation $D^*$ for both mixed and normal is used when there is no difference between them in the cases under consideration.

Finally in this section, recall two "normal compactness" properties that are automatic in finite dimensions while playing a significant role in many aspects of variational analysis and generalized differentiation in infinite dimensional spaces; see [20, 21]. A set $\Omega$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k; \Omega)$ we have the implication

$$\left[ x_k^* \xrightarrow{w^*} 0 \right] \implies \left[ \|x_k^*\| \to 0 \right] \text{ as } k \to \infty,$$

where $\varepsilon_k$ can be omitted as usual if $X$ is Asplund and if $\Omega$ is locally closed around $\bar{x}$. Note that the above SNC property of $\Omega$ is implied by the so-called "compactly epi-Lipschitzian" property of $\Omega$ around $\bar{x}$ in the sense of Borwein and Strójwas, which is formulated in the primal space $X$ with no use of generalized normals; see [20, Subsection 1.1.4] for more details and references. If $\Omega$ is convex, the latter property is equivalent to the finite codimensionality of $\Omega$ having nonempty relative interior with respect to its closed span.

A set-valued mapping $F\colon X \rightrightarrows Y$ is *SNC* at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if its graph enjoys this property at $(\bar{x}, \bar{y})$. A more subtle *partial SNC* (i.e., PSNC) property is defined as follows. A mapping $F$ is *PSNC* at $(\bar{x}, \bar{y})$ if for any sequences $\varepsilon_k \downarrow 0$, $(x_k, y_k) \overset{\operatorname{gph} F}{\to} (\bar{x}, \bar{y})$, and $(x_k^*, y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} F)$ we have the implication

$$\left[ x_k^* \overset{w^*}{\to} 0, \ \|y_k^*\| \to 0 \right] \implies \left[ \|x_k^*\| \to 0 \right] \ \text{ as } \ k \to \infty,$$

where $\varepsilon_k = 0$ in the Asplund space and closed graph setting. The PSNC property always holds when $F$ is *Lipschitz-like* around $(\bar{x}, \bar{y})$, or has the Aubin "pseudo-Lipschitz" property: there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ as well as modulus $\ell \geq 0$ such that

$$(2.9) \qquad F(u) \cap V \subset F(v) + \ell \|u - v\| \mathbb{B} \ \text{ whenever } \ u, v \in U.$$

This reduces to the classical (Hausdorff) *local Lipschitzian* behavior of $F$ around $\bar{z}$ when $V$ is taken as the whole space $Y$ in (2.9).

## 3. Coderivatives of Frontier Maps with Respect to Generalized Order Optimality

In this section we derive verifiable formulas for computing and estimating both normal and mixed coderivatives of *frontier maps* for multiobjective problems with the notion of optimality/efficiency induced by the concept of *generalized order optimality* from [21, Subsection 5.3.1], where the reader can find more details, discussions, and examples of reducing conventional preference relations to generalized order optimality.

**Definition 3.1.** (efficient points with respect to generalized order optimality). Let $\Theta \subset Z$ be an ordering subset of a Banach space $Z$ with $0 \in \Theta$. Then:

(i) Given an nonempty set $\Omega \subset Z$, we say that $\bar{z} \in \Omega$ is a local efficient point of $\Omega$ with respect to $\Theta$ if there is a neighborhood $V$ of $\bar{z}$ and a sequence $\{v_k\} \subset Z$ with $v_k \to 0$ as $k \to \infty$ such that

$$z - \bar{z} \notin \Theta - v_k \ \text{ for all } \ z \in \Omega \cap V \ \text{ and } \ k \in \mathbb{N}.$$

The set of all the local efficient points of $\Omega$ with respect to $\Theta$ is denoted by $\operatorname{Eff}(\Omega | \Theta)$.

(ii) Given a nonempty subset $\Xi \subset X$ of a Banach space $X$ and a mapping $g\colon X \to Z$, we say that $\bar{x} \in \Xi$ is a local efficient/minimal solution to $g$ on $\Xi$ with respect to $\Theta$ if there is a neighborhood $U$ of $\bar{x}$ such that $g(\bar{x}) \in \operatorname{Eff}\big(g(\Xi \cap U) \big| \Theta\big)$.

The notions of generalized order optimality/efficiency from the Definition 3.1 encompass various preference relations; in particular, the so-called *generalized Pareto* ones given by

$$(3.1) \qquad z_1 \prec z_2 \text{ if and only if } z_2 - z_1 \in \Theta \setminus \{0\},$$

where $\Theta \subset Z$ is a convex ordering/positive cone in $Z$. This name for preferences (3.1) comes from the fact that the classical notions of Pareto and weak Pareto optimality as well as their various extensions can be written in form (3.1) by choosing cones $\Theta$ satisfying certain properties; see [20, Section 5.3] and the recent paper [1] for more details and discussions.

To proceed, we fix an ordering set $\Theta \subset Z$ in what follows and introduce the *dual interior set* for the ordering set $\Theta$ defined by

$$(3.2) \qquad \Theta_>^* := \left\{ z^* \in Z^* \,\Big|\, \inf_{\theta \in \Theta \setminus \{0\}} \frac{\langle z^*, \theta \rangle}{\|\theta\|} > 0 \right\}.$$

Observe that for $Z = \mathbb{R}^n$ and $\Theta = \mathbb{R}_+^m$ we have $\Theta_>^* = \operatorname{int} \mathbb{R}_+^m$. The next lemma establishes a relation between the normal and mixed coderivatives of a set-valued mapping from $X$ to $Z$ and those for its $\Theta$-addition in dual interior directions.

**Lemma 3.2.** (coderivatives of set-valued mappings at dual interior directions). *Let* $F \colon X \rightrightarrows Z$ *be a set-valued mapping between Banach spaces, and let* $(\bar{x}, \bar{z}) \in \operatorname{gph} F$. *Then for all* $z^* \in \Theta_>^*$ *we have the inclusion*

$$(3.3) \qquad D_M^* F(\bar{x}, \bar{z})(z^*) \subset D_M^* (F + \Theta)(\bar{x}, \bar{z})(z^*).$$

*Proof.* Fix an arbitrary direction $z^* \in \Theta_>^*$ and pick any $x^* \in D_M^* F(\bar{x}, \bar{z})(z^*)$. By the mixed coderivative description (2.5), find sequences $\varepsilon_k \downarrow 0$, $(x_k, z_k) \to (\bar{x}, \bar{y})$, and $x_k^* \xrightarrow{w^*} x^*$, $z_k^* \to z^*$ as $k \to \infty$ satisfying

$$(3.4) \qquad (x_k, z_k) \in \operatorname{gph} F \text{ and } (x_k^*, -z_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k, z_k); \operatorname{gph} F\big)$$
$$\text{for all } k \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$ and apply the construction of $\varepsilon_k$-normals from (2.1) to $(x_k^*, -z_k^*)$ in (3.4). In this way we get sequences $\gamma_k \downarrow 0$ and $\eta_k \downarrow 0$ for which

$$(3.5) \qquad \langle x_k^*, x - x_k \rangle \le \langle z_k^*, z - z_k \rangle + \varepsilon_k \big(\|x - x_k\| + \|z - z_k\|\big)$$

whenever $x \in x_k + \gamma_k \mathbb{B}$, $z \in z_k + \eta_k \mathbb{B}$, $(x, z) \in \operatorname{gph} F$ and $k \in \mathbb{N}$. It follows from $z_k^* \to z^*$ as $k \to \infty$ that $\langle z_k^* - z^*, \theta \rangle \to 0$ as $k \to \infty$ for any $\theta \in Z$. Furthermore, we claim from the choice of $z^* \in \Theta_>^*$ that there is a number $\varepsilon > 0$ for which

$$(3.6) \qquad \inf_{\theta \in \Theta \setminus \{0\}} \frac{\langle z_k^*, \theta \rangle}{\|\theta\|} \ge \varepsilon$$

whenever $k \in \mathbb{N}$ is sufficiently large. Assuming by the contrary that (3.6) does not hold, for any $n \in \mathbb{N}$ find $z^*_{k_n}$ and $\theta_n \in \Theta \setminus \{0\}$ satisfying

$$\frac{\langle z^*_{k_n}, \theta_n \rangle}{\|\theta_n\|} < \frac{1}{n}, \quad n \in \mathbb{N}.$$

This implies the relations

$$\begin{aligned}
\frac{\langle z^*, \theta_n \rangle}{\|\theta_n\|} &= \left\langle z^*_{k_n}, \frac{\theta_n}{\|\theta_n\|} \right\rangle + \left\langle z^* - z^*_{k_n}, \frac{\theta_n}{\|\theta_n\|} \right\rangle \\
&< \frac{1}{n} + \left\langle z^* - z^*_{k_n}, \frac{\theta_n}{\|\theta_n\|} \right\rangle, \quad n \in \mathbb{N}.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the latter estimate and taking into account the strong convergence $\|z^*_{k_n} - z^*\| \to 0$ as $n \to \infty$ due the *mixed coderivative* construction, we get that

$$\inf_{\theta \in \Theta \setminus \{0\}} \frac{\langle z^*, \theta \rangle}{\|\theta\|} \leq 0,$$

which contradicts (3.2) and hence justifies (3.6).

Assume further with no loss of generality that $\varepsilon_k \leq \varepsilon$ and (3.6) hold for all $k \in \mathbb{N}$. Thus we have the inequalities

$$\begin{aligned}
\langle x^*_k, x - x_k \rangle &\leq \langle z^*_k, z + \theta - z_k \rangle - \varepsilon \|\theta\| + \varepsilon_k \big( \|x - x_k\| + \|z - z_k\| \big) \\
&\leq \langle z^*_k, z + \theta - z_k \rangle + \varepsilon_k \big( \|x - x_k\| + \|z + \theta - z_k\| \big), \quad k \in \mathbb{N},
\end{aligned}$$

for all $x \in x_k + \gamma_k \mathbb{B}$, $z \in z_k + \eta_k \mathbb{B}$, and $\theta \in \Theta$. Since obviously $(x_k, z_k) \in$ gph $(F + \Theta)$ by the first inclusion in (3.4), the latter implies that

$$(x^*_k, -z^*_k) \in \widehat{N}_{\varepsilon_k} \big( (x_k, z_k); \text{gph}\,(F + \Theta) \big) \quad \text{as} \ z^*_k \in \Theta^*_>, \quad k \in \mathbb{N}.$$

Therefore we get $x^* \in D^*_M (F + \Theta)(\bar{x}, \bar{z})(z^*_k)$ with the direction $z^* \in \Theta^*_>$ fixed above. This justifies (3.3) and completes the proof of the lemma. ∎

In what follows, consider the *image map*

$$(3.7) \qquad \Sigma(x) := f\big(x, G(x)\big) = \big\{ f(x, y) \big| \ y \in G(x) \big\}, \quad x \in X,$$

for the multiobjective optimization problem (1.1) with respect to the generalized order optimality induced by the ordering set $\Theta \subset Z$ and the corresponding frontier map $\mathcal{F}(x)$ defined in (1.2). We say that the *domination property* holds for problem (1.1) with respect to the generalized order optimality/efficiency induced by $\Theta$ if

$$(3.8) \qquad \mathcal{F}(x) + \Theta = f\big(x, G(x)\big) + \Theta, \quad x \in X.$$

This property has been studied for some specific preference relations in, e.g., [1, 2, 7, 9, 13, 29, 31] and the references therein.

Our first theorem in this section provides efficient *upper estimates* for both normal and mixed coderivatives of the frontier map (1.2) via the corresponding coderivatives of the initial data in the multiobjective optimization problem (1.1). To proceed, we introduce the *normalized dual interior set* for the ordering set $\Theta$ constructed from the dual interior set (3.2) and the basic normal cone to $\Theta$ by

$$(3.9) \qquad \Theta_N^* := \Theta_>^* \cap \big( - N(0; \Theta) \big).$$

It is easy to observe that the inclusion $\Theta_N^* \subset \Theta_<^*$, which is obviously strict in general, holds as equality $\Theta_N^* = \Theta_>^* = \operatorname{int} \mathbb{R}_+^m$ in the classical setting of $\Theta = \mathbb{R}_+^m$.

**Theorem 3.3.** (upper estimates for coderivatives of frontier maps with respect to generalized order optimality). *Let $\mathcal{F}$ be the frontier map (1.2) for multiobjective problem (1.1) with the ordering set $\Theta$ locally closed around $0 \in \Theta$, and let $(\bar{x}, \bar{y}) \in$ gph $S$ for the optimal solution map $S$ from (1.3). Suppose that all the spaces $X, Y, Z$ are Asplund, that the cost mapping $f$ is locally Lipschitzian around $(\bar{x}, \bar{y})$ and that the graph of the image map $\Sigma$ from (3.7) is locally closed around $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{z} := f(\bar{x}, \bar{y})$. Assume also that the domination property (3.8) is satisfied. Then we have the upper estimate*

$$(3.10) \qquad \begin{aligned} &D_M^* \mathcal{F}(\bar{x}, \bar{z})(z^*) \subset \bigcup_{(x^*, y^*) \in D_M^* f(\bar{x}, \bar{y})(z^*)} \left[ x^* + D_N^* G(\bar{x}, \bar{y})(y^*) \right] \\ &\text{for all } \ z^* \in \Theta_N^* \end{aligned}$$

*along the normalized set (3.9) of dual interior directions $z^* \in \Theta_N^*$. If furthermore $f$ is strictly differentiable at $(\bar{x}, \bar{y})$, then estimate (3.10) can be improved by*

$$(3.11) \qquad \begin{aligned} &D_M^* \mathcal{F}(\bar{x}, \bar{z})(z^*) \subset \nabla_x f(\bar{x}, \bar{y})^* z^* + D_M^* G(\bar{x}, \bar{y}) \big( \nabla_y f(\bar{x}, \bar{y})^* z^* \big) \\ &\text{for all } \ z^* \in \Theta_N^*. \end{aligned}$$

*Proof.* We first justify inclusion (3.10). It follows immediately from the assumed domination property (3.8) that

$$(3.12) \quad D_M^*(\mathcal{F} + \Theta)(\bar{x}, \bar{z})(z^*) = D_M^*(\Sigma + \Theta)(\bar{x}, \bar{z})(z^*) \ \text{ for all } \ z^* \in Z^*.$$

To elaborate the right-hand side of equality (3.12), we apply the *coderivative sum rule* from [20, Theorem 3.10] to the sum $\Sigma(x) + \Theta(x)$, where the second mapping is constant: $\Theta(x) \equiv \Theta$ on $X$. It is easy to check that

$$(3.13) \qquad D_M^* \Theta(\bar{x}, 0)(z^*) = \begin{cases} 0 & \text{for } \ z^* \in -N(0; \Theta), \\ \emptyset & \text{otherwise}. \end{cases}$$

Moreover, it is obvious that $D_M^*\Theta(\bar{x};0)(0) = \{0\}$ and the constant mapping $\Theta(x) = \Theta$ is PSNC at $(\bar{x},0)$. Thus all the assumptions in [20, Theorem 3.10] are satisfied and we get

$$(3.14) \quad D_M^*(\Sigma + \Theta)(\bar{x},\bar{z})(z^*) \subset D_M^*\Sigma(\bar{x},\bar{z})(z^*) \quad \text{whenever} \quad -z^* \in N(0;\Theta).$$

Observing further the composite form of the image map $\Sigma$ in (3.7) and applying to it the *coderivative chain rule* from [20, Theorem 3.18(i)] for the locally *Lipschitzian* cost mapping $f(x,y)$, we arrive at the inclusion

$$(3.15) \quad D_M^*\Sigma(\bar{x},\bar{z})(z^*) \subset \bigcup_{(x^*,y^*)\in D_M^*f(\bar{x},\bar{y})(z^*)} \Big[x^* + D_N^*G(\bar{x},\bar{y})(y^*)\Big], \quad z^* \in Z^*.$$

Combining now the relations in (3.12)–(3.15) with (3.3) from Lemma 3.2, we get (3.10).

It remains to justify inclusion (3.11). Since $f$ is now assumed to be *strictly differentiable* at $(\bar{x},\bar{y})$, we have by (2.8) that

$$D_N^*f(\bar{x},\bar{y})(z^*) = D_M^*f(\bar{x},\bar{y})(z^*) = \big\{(\nabla_x f(\bar{x},\bar{y})^*z^*, \nabla_y f(\bar{x},\bar{y})^*z^*)\big\}, \quad z^* \in Z^*.$$

The latter immediately gives the normal counterpart of inclusion (3.11) but not the claimed one for the mixed coderivatives. However, the above proof allows us to get the conclusion of (3.11) by replacing the application of [20, Theorem 3.18(i)] therein by a more delicate *chain rule* from [20, Theorem 3.18(ii)] held for the mixed coderivative case under the strict differentiability assumption on the outer mapping in the composition. This gives (3.11) and completes the proof of the theorem.    ∎

Next result provides *precise/equality formulas* for computing both normal and mixed coderivatives of the frontier map (1.2) with respect to generalized order optimality. Besides the notions of the $M$-regularity and $N$-regularity of set-valued mappings $F\colon X \rightrightarrows Y$ at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ introduced in Section 2, we recall the following definition of the (local) *upper Lipschitzian property* for single-valued mappings $l\colon \Omega \subset X \to Y$ at $\bar{x} \in \Omega$ given by: there are numbers $\eta > 0$ and $\ell \geq 0$ such that

$$(3.16) \quad \|l(x) - l(\bar{x})\| \leq \ell\|x - \bar{x}\| \quad \text{for all} \quad x \in \Omega \cap (\bar{x} + \eta\mathbb{B}).$$

We say that a set-valued mapping $L\colon \Omega \subset X \rightrightarrows Y$ admits an *upper Lipschitzian selection* around $(\bar{x},\bar{y}) \in \operatorname{gph} L$ if there is a neighborhood $U$ of $\bar{x}$ and a single-valued mapping $l\colon \Omega \cap U \to Y$ such that

$$l(\bar{x}) = \bar{y}, \quad l(x) \in L(x) \quad \text{for all } x \in \Omega \cap U,$$

and $l$ is upper Lipschitzian at $\bar{x}$. In what follows we always assume that $\eta$ in (3.16) is selected so that $\bar{x} + \eta\mathbb{B} \subset U$.

**Theorem 3.4.** (computing coderivatives of frontier maps with respect to generalized order optimality). *In addition to all the assumptions of Theorem 3.3, suppose that the optimal solution map $S \colon \operatorname{dom} G \rightrightarrows Y$ admits an upper Lipschitzian selection around $(\bar{x}, \bar{y})$. Then we have the equality*

$$(3.17) \qquad \begin{aligned} & D_M^* \mathcal{F}(\bar{x}, \bar{z})(z^*) = \nabla_x f(\bar{x}, \bar{y})^* z^* + D_M^* G(\bar{x}, \bar{y})\big(\nabla_y f(\bar{x}, \bar{y})^* z^*\big) \\ & \textit{for all } \ z^* \in \Theta_N^* \end{aligned}$$

*if $G$ is $M$-regular at this point.*

*Proof.* The inclusion "$\subset$" in (3.17) follows from the corresponding results of Theorem 3.3 in the case of strictly differentiable cost mappings. We need to justify the opposite inclusion "$\supset$" in (3.17) under the additional assumptions of the theorem.

To proceed, fix any $x^* \notin D_M^* \mathcal{F}(\bar{x}, \bar{z})(z^*)$ with $z^* \in \Theta_N^*$ and show that

$$(3.18) \qquad x^* - \nabla_x f(\bar{x}, \bar{y})^* z^* \notin D_M^* G(\bar{x}, \bar{y})\big(\nabla_y f(\bar{x}, \bar{y})^* z^*\big).$$

Since $G$ is assumed to be $M$-regular at $(\bar{x}, \bar{y})$, relation (3.18) is equivalent to

$$(3.19) \qquad \big(x^* - \nabla_x f(\bar{x}, \bar{y})^* z^*, -\nabla_y f(\bar{x}, \bar{y})^* z^*\big) \notin \widehat{N}\big((\bar{x}, \bar{y}); \operatorname{gph} G\big).$$

We obviously have from the choice of $x^*$ and the definition of the normal coderivative that

$$(x^*, -z^*) \notin \widehat{N}\big((\bar{x}, \bar{z}); \operatorname{gph} \mathcal{F}\big),$$

which implies by definition (2.1) with $\varepsilon = 0$ that

$$(3.20) \qquad \limsup_{(x,z) \xrightarrow{\operatorname{gph} \mathcal{F}} (\bar{x}, \bar{z})} \frac{\langle x^*, x - \bar{x}\rangle - \langle z^*, z - \bar{z}\rangle}{\|x - \bar{x}\| + \|z - \bar{z}\|} > 0.$$

Using the upper Lipschitzian assumption of the theorem, find $l \colon \operatorname{dom} G \to Y$ such that $l(\bar{x}) = \bar{y}$, that $l$ is upper Lipschitzian at $\bar{x}$, and that $l(x) \in S(x)$ for all $x \in \operatorname{dom} G$ sufficiently close to $\bar{x}$. It follows from (3.20) and from the above properties of $l(\cdot)$ that there exist a number $\gamma > 0$ and a sequence $x_k \to \bar{x}$ as $k \to \infty$ along which

$$(3.21) \qquad \langle z^*, z_k - \bar{z}\rangle - \langle x^*, x_k - \bar{x}\rangle \leq -\gamma\big(\|x_k - x\| + \|z_k - \bar{z}\|\big)$$

with $z_k = f(x_k, y_k)$, $y_k := l(x_k) \in S(x_k)$ and

$$(3.22) \qquad \|x_k - \bar{x}\| \geq \ell^{-1}\|y_k - \bar{y}\|$$

for all $k \in \mathbb{N}$ sufficiently large. By (3.21) and the construction of $z_k$ via the strict differentiable mapping $f$ we have for such $k \in \mathbb{N}$ that

$$
\begin{aligned}
\langle x^*, x_k - \bar{x} \rangle &\geq \langle z^*, f(x_k, y_k) - f(\bar{x}, \bar{y}) \rangle + \gamma \big( \|x_k - \bar{x}\| + \|f(x_k, y_k) - f(\bar{x}, \bar{y})\| \big) \\
&= \langle z^*, \nabla f(\bar{x}, \bar{y})(x_k - \bar{x}, y_k - \bar{y}) \rangle + o\big( \|x_k - \bar{x}\| + \|y_k - \bar{y}\| \big) \\
&\quad + \gamma \big( \|x_k - \bar{x}\| + \|f(x_k, y_k) - f(\bar{x}, \bar{y})\| \big) \\
&= \langle \nabla f(\bar{x}, \bar{y})^* z^*, (x_k - \bar{x}, y_k - \bar{y}) \rangle + o\big( \|x_k - \bar{x}\| + \|y_k - \bar{y}\| \big) \\
&\quad + \gamma \big( \|x_k - \bar{x}\| + \|f(x_k, y_k) - f(\bar{x}, \bar{y})\| \big).
\end{aligned}
$$

The latter implies by (3.22) that

$$
\langle x^* - \nabla_x f(\bar{x}, \bar{y})^* z^*, x_k - \bar{x} \rangle - \langle \nabla_y f(\bar{x}, \bar{y})^* z^*, y_k - \bar{y} \rangle
$$
$$
\geq \frac{\gamma}{2} \|x_k - \bar{x}\| + \frac{\gamma}{2\ell} \|y_k - \bar{y}\| + o\big( \|x_k - \bar{x}\| + \|y_k - \bar{y}\| \big)
$$
$$
\geq \widetilde{\gamma} \big( \|x_k - \bar{x}\| + \|y_k - \bar{y}\| \big) + o\big( \|x_k - \bar{x}\| + \|y_k - \bar{y}\| \big)
$$

with $\widetilde{\gamma} := \min\{\gamma/2, \gamma/(2\ell)\}$. This gives

$$
\limsup_{(x,y) \xrightarrow{\text{gph } G} (\bar{x}, \bar{y})} \frac{\langle x^* - \nabla_x f(\bar{x}, \bar{y})^* z^*, x - \bar{x} \rangle - \langle \nabla_y f(\bar{x}, \bar{y})^* z^*, y - \bar{y} \rangle}{\|x - \bar{x}\| + \|y - \bar{y}\|} \geq \widetilde{\gamma},
$$

which implies (3.19). Thus we get (3.18) and complete the proof of the theorem. ■

## 4. Coderivatives of Frontier Maps in Special Classes of Constrained Multiobjective Problems

In this section we derive efficient specifications of the general results obtained in Section 3 for various classes of multiobjective optimization problems with the constraint mapping $G \colon X \rightrightarrows Y$ given in the form

(4.1) $$ G(x) := \big\{ y \in Y \,\big|\, h(x, y) \in -K \big\}, $$

where $h \colon X \times Y \to W$ is a single-valued mapping between Banach spaces, and where $\emptyset \neq K \subset W$. In what follows we keep the notation of Section 3. Constraints of type (4.1) are known as *operator constraints* (note that the range space $W$ is generally infinite-dimensional). They include geometric, functional, and other types of constraints under appropriate specifications of $h$ and $K$; see [21] for more discussions and examples.

The next theorem provides upper estimating and precise computing formulas to evaluate both normal and mixed coderivatives of the frontier map (1.2) with respect to generalized order optimality for constraints given by (4.1). For simplicity we

focus on the case when $h(\cdot)$ is strictly differentiable at the reference point with the surjective derivative. The coderivative calculus rules from [20] allow us to proceed in more general cases under more involved assumptions on $h$ in smooth and nonsmooth settings.

**Theorem 4.1.** (coderivatives of frontier maps for multiobjective problems with operator constraints). *Let* $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ *for problem* (1.1) *with the constraints given by* (4.1)*, let the assumptions of Theorem 3.3 be satisfied in the case of the cost mapping $f$ locally Lipschitzian around $(\bar{x}, \bar{y})$, and let $\bar{w} := h(\bar{x}, \bar{y})$. The following assertions hold:*

(i) *Suppose that $h$ in* (4.1) *is strictly differentiable at $(\bar{x}, \bar{y})$ with the surjective derivative operator $\nabla h(\bar{x}, \bar{y})$, where the range space $W$ is Banach. Then we have the estimate*

$$
\begin{aligned}
(4.2) \quad D_M^* \mathcal{F}(\bar{x}, \bar{z})(z^*) \subset \bigcup_{(x^*, y^*) \in D_M^* f(\bar{x}, \bar{y})(z^*)} & \Big\{ u^* + x^* \Big| (u^*, -y^*) \\
& \in \nabla h(\bar{x}, \bar{y})^* N(\bar{w}; -K) \Big\}
\end{aligned}
$$

*satisfied whenever $z^* \in \Theta_N^*$.*

(ii) *Suppose in addition to* (i) *that $f$ is strictly differentiable at $(\bar{x}, \bar{y})$, and that the solution map $S \colon \operatorname{dom} G \rightrightarrows Y$ admits a local upper Lipschitzian selection around $(\bar{x}, \bar{y})$, and that the set $K$ is normally regular at $-\bar{w}$. Then we have the equality*

$$
\begin{aligned}
(4.3) \quad D^* \mathcal{F}(\bar{x}, \bar{z})(z^*) = & \Big\{ u^* + \nabla_x f(\bar{x}, \bar{y})^* z^* \Big| (u^*, -\nabla_y f(\bar{x}, \bar{y})^* z^*) \\
& \in \nabla h(\bar{x}, \bar{y})^* N(\bar{w}; -K) \Big\}
\end{aligned}
$$

*satisfied whenever $z^* \in \Theta_N^*$ for both coderivatives $D^* = D_N^*, D_M^*$.*

*Proof.* Observe that the graph of the constraint mapping $G$ in case (4.1) admits the *inverse image* representation

$$
(4.4) \qquad \operatorname{gph} G = h^{-1}(-K) := \big\{ (x, y) \in X \times Y \,\big|\, h(x, y) \in -K \big\}.
$$

By the *surjectivity* assumption on the derivative operator $\nabla h(\bar{x}, \bar{y})$ we have from [20, Theorem 1.17] that, in arbitrary Banach spaces,

$$
N\big((\bar{x}, \bar{y}); h^{-1}(-K)\big) = \nabla h(\bar{x}, \bar{y})^* N(\bar{w}; -K).
$$

This allows us to conclude that the normal coderivative (2.3) of the constraint mapping (4.1) is computed by

$$
(4.5) \quad D_N^* G(\bar{x}, \bar{y})(y^*) = \big\{ u^* \in X^* \,\big|\, (u^*, -y^*) \in \nabla h(\bar{x}, \bar{y})^* N(\bar{w}; -K) \big\}.
$$

Substituting (4.5) into the coderivative estimate (3.10) of Theorem 3.3, we arrive at inclusion (4.2) for mixed coderivative $D_M^*$ of the frontier map $\mathcal{F}$ in the case of operator constraints (4.1) and thus get (i).

To justify assertion (ii), we employ Theorem 3.4 and the coderivative representation (4.5) to arrive at equality (4.3) for the normal coderivative case $D^* = D_N^*$ *provided* that the constraint mapping (4.1) is $N$-regular at $(\bar{x}, \bar{y})$. It further follows from [20, Theorem 1.19] that the $N$-regularity of the above mapping $G$ at $(\bar{x}, \bar{y})$ is *equivalent*, by the inverse image representation and the surjectivity assumption on $\nabla h(\bar{x}, \bar{y})$, to the normal regularity of the set $K$ at $-\bar{w}$. This gives (4.3) for the normal coderivative $D_N^* \mathcal{F}(\bar{x}, \bar{z})$. The mixed coderivative equality in (4.3) follows by the above arguments from Theorem 3.4 in the mixed coderivative case and the obvious fact that the $N$-regularity of any mapping $G$ implies its $M$-regularity at the reference point. ∎

Next we consider the multiobjective problems (1.1) with respect to generalized order optimality subject to *functional constraints* of the equality and inequality type given by

$$
\text{(4.6)} \quad
\begin{aligned}
G(x) := \big\{ y \in Y \big| \ & \varphi_i(x, y) \le 0, \ i = 1, \ldots, m, \\
& \varphi_i(x, y) = 0, \ i = m+1, \ldots, m+r \big\},
\end{aligned}
$$

where $\varphi_i$, $i = 1, \ldots, m+r$, are real-valued functions on the Asplund space $X \times Y$. Constraints of this type can be treated as a particular case of the operator constraints (4.1) with $h(x, y) := (\varphi_1(x, y), \ldots, \varphi_{m+r}(x, y))$ and $K = \mathbb{R}_+^{m+r}$. Similarly to Theorem 4.1 we concentrate in the next theorem on the case when the constraint functions $\varphi_i$ are strictly differentiable at the reference point $(\bar{x}, \bar{y})$. However, we *do not* impose assumptions on their gradients ensuring the surjectivity of $\nabla h(\bar{x}, \bar{y})$ for the corresponding mapping $h(\cdot)$; such assumptions reduce in setting (4.6) to the linear independence of the gradients $\nabla \varphi_1(\bar{x}, \bar{y}), \ldots, \nabla \varphi_{m+r}(\bar{x}, \bar{y})$. Instead we require the more relaxed *Mangasarian-Fromovitz constraint qualification* formulated as follows:

$$
\text{(4.7)} \quad
\begin{aligned}
& \text{the gradients } \nabla \varphi_{m+1}(\bar{x}, \bar{y}), \ldots, \nabla \varphi_{m+r}(\bar{x}, \bar{y}) \text{ are linearly independent, and} \\
& \text{there is } v \in X \times Y \text{ such that } \langle \nabla \varphi_i(\bar{x}, \bar{y}), v \rangle = 0 \text{ for } i = m+1, \ldots, m+r \\
& \text{and that } \langle \nabla \varphi_i(\bar{x}, \bar{y}), v \rangle < 0 \text{ whenever } i = 1, \ldots, m \text{ with } \varphi_i(\bar{x}, \bar{y}) = 0.
\end{aligned}
$$

For each $y^* \in Y^*$ we consider the set of *Lagrange multipliers* satisfying the *sign* and *complementary slackness conditions*:

$$
\text{(4.8)} \quad
\begin{aligned}
\Lambda(\bar{x}, \bar{y}; y^*) = \Big\{ & \lambda = (\lambda_1, \ldots, \lambda_{m+r}) \in \mathbb{R}^{m+r} \Big| y^* + \sum_{i=1}^{m+r} \lambda_i \nabla_y \varphi_i(\bar{x}, \bar{y}) = 0, \\
& \lambda_i \ge 0, \ \lambda_i \varphi_i(\bar{x}, \bar{y}) = 0 \text{ for } i = 1, \ldots, m \Big\}.
\end{aligned}
$$

**Theorem 4.2.** (coderivatives of frontier maps for multiobjective problems with functional constraints). *Let $\mathcal{F}$ be the frontier map for the multiobjective problem (1.1) in the sense of generalized order optimality/efficiency from Definition 3.1 with the constraint mapping $G$ given by (4.6), let $\Theta$, $X$, $Y$, and $Z$ be the same as in Theorem 3.3, let the domination property (3.8) hold in (1.1), and let $(\bar{x}, \bar{y}) \in \mathrm{gph}\, S$ for the efficient solution map (1.3). Assume in addition that all the functions $\varphi_i$, $i = 1, \ldots, m + r$, are strictly differentiable at $(\bar{x}, \bar{y})$ and that the Mangasarian-Fromovitz constraint qualification (4.7) is satisfied. Then we have the following assertions:*

(*i*) *If the cost $f$ in (1.1) is locally Lipschitzian around $(\bar{x}, \bar{y})$, then the upper estimate*

$$(4.9) \quad D_M^* \mathcal{F}(\bar{x}, \bar{z})(z^*) \subset \bigcup_{(x^*, y^*) \in D_M^* f(\bar{x}, \bar{y})(z^*)} \bigcup_{\lambda \in \Lambda(\bar{x}, \bar{y}, y^*)} \left[ x^* + \sum_{i=1}^{m+r} \lambda_i \nabla_x \varphi_i(\bar{x}, \bar{y}) \right]$$

*holds whenever $z^* \in \Theta_N^*$, where $\Lambda(\bar{x}, \bar{y}, y^*)$ in the second union of (4.9) is the corresponding set of Lagrange multipliers defined in (4.8).*

(*ii*) *Suppose in addition to the assumptions of* (i) *that the cost $f$ is strictly differentiable at $(\bar{x}, \bar{y})$ and that the solution map $S \colon \mathrm{dom}\, G \rightrightarrows Y$ admits a local upper Lipschitzian selection around $(\bar{x}, \bar{y})$. Then we have the equality*

$$(4.10) \quad D^* \mathcal{F}(\bar{x}, \bar{z})(z^*) = \bigcup_{\lambda \in \Lambda\left(\bar{x}, \bar{y}, \nabla_y f(\bar{x}, \bar{y})^* z^*\right)} \left[ \nabla_x f(\bar{x}, \bar{y})^* z^* + \sum_{i=1}^{m+r} \lambda_i \nabla_x \varphi_i(\bar{x}, \bar{y}) \right]$$

*whenever $z^* \in \Theta_N^*$ for both coderivatives $D^* = D_N^*$, $D_M^*$ of the frontier map $\mathcal{F}$.*

*Proof.* To justify (i), we employ assertion (i) of Theorem 3.3 with a subsequent upper approximation of the normal coderivative $D_N^* G(\bar{x}, \bar{y})$ of the mapping $G$ given by the classical functional constraint system (4.6) in nonlinear differentiable programming. In fact, by [20, Corollary 4.35] we have the *precise formula* to compute this coderivative under the assumptions made, which ensure furthermore the *N-regularity* of the mapping $G$ at $(\bar{x}, \bar{y})$ in the Asplund space setting:

$$(4.11) \quad D_N^* G(\bar{x}, \bar{y})(y^*) = \Big\{ x^* \in X^* \Big| (x^*, -y^*) \in \sum_{i=1}^{m+r} \lambda_i \nabla \varphi_i(\bar{x}, \bar{y}),$$
$$\lambda_i \geq 0, \quad \lambda_i \varphi_i(\bar{x}, \bar{y}) = 0 \ \text{ for } \ i = 1, \ldots, m \Big\}.$$

Substituting (4.11) into (3.10) and taking into account construction (4.8) of the Lagrange multiplier set $\Lambda(\bar{x}, \bar{y}, y^*)$, we arrive at the upper estimate (4.9) for mixed coderivatives $D_M^*$ of the frontier map under consideration and thus get (i). Furthermore, equality (4.11) and the afore-mentioned $N$-regularity of the constraint mapping (4.6) allow us to derive the coderivative formula (4.10) directly from Theorem 3.4 under the assumptions made in (ii). This completes the proof of the theorem. ∎

Finally in this section, we consider the multiobjective optimization problem (1.1) with the notion of efficiency from Definition 3.1 and with the constraint mapping $G$ given by the parameterized *generalized equation*

(4.12) $$G(x) := \big\{ y \in Y \,\big|\, 0 \in q(x, y) + Q(x, y) \big\}$$

in the sense of Robinson [27], where $q \colon X \times Y \to W$ is a single-valued *base* mapping and where $Q \colon X \times Y \rightrightarrows W$ is a set-valued *field* mapping between Banach spaces. It has been well recognized that generalized equations of type (4.12) provide a convenient framework for modeling various parameterized equilibrium systems including variational inequalities, complementarity problems, etc. Multiobjective problems involving such constraints encompass and/or are closely related to some classes of the so-called *equilibrium problems with equilibrium constraints* (EPECs). We refer the reader to [21, 22, 24, 26, 32] for more details, examples, and discussions.

The following two theorems provide, respectively, upper estimating and precise computing the normal and mixed coderivatives of the frontier map (1.2) to the multiobjective optimization problem (1.1) with the equilibrium constraints (4.12) based on the general results of Theorem 3.3 and Theorem 3.4 combined with the corresponding coderivative calculus formulas for equilibrium constraint mappings of type (4.12). To proceed in this way, we employ some of the calculus results from [20, Subsection 4.4.1], where the reader can find more calculus results, particular cases, and discussions.

**Theorem 4.3.** (coderivatives of frontier maps for multiobjective problems with equilibrium constraints). *Consider the frontier map* (1.2) *in* (1.1) *with the constraint system given by* (4.12) *and assume that the spaces* $X, Y, Z$ *and the ordering set* $\Theta$ *are the same as in Theorem 3.3, that the cost mapping* $f$ *is locally Lipschitzian around* $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ *with* $\bar{z} := f(\bar{x}, \bar{y})$, *and that the domination property* (3.8) *is satisfied. Suppose in addition that* $W$ *is Asplund, that* $q \colon X \times Y \to W$ *is locally Lipschitzian around* $(\bar{x}, \bar{y})$, *that* $Q \colon X \times Y \rightrightarrows W$ *is closed-graph around* $(\bar{x}, \bar{y}, \bar{w})$ *with* $\bar{w} := -q(\bar{x}, \bar{y})$, *and that the Fredholm qualification condition*

(4.13) $$\big[ (x^*, y^*) \in D_N^* q(\bar{x}, \bar{y})(v^*) \cap \big( -D_N^* Q(\bar{x}, \bar{y}, \bar{w})(v^*) \big) \big] \implies (x^*, y^*, v^*)$$
$$= (0, 0, 0)$$

*holds. Then we have the upper estimate*

$$
\text{(4.14)} \quad D_M^* \mathcal{F}(\bar{x}, \bar{z})(z^*) \subset \bigcup_{(x^*, y^*) \in D_M^* f(\bar{x}, \bar{y})(z^*)} \left\{ u^* + x^* \,\middle|\, \exists\, v^* \in W^* \text{ with} \right.
$$
$$
\left. (u^*, -y^*) \in D_N^* q(\bar{x}, \bar{y})(v^*) + D_N^* Q(\bar{x}, \bar{y}, \bar{w})(v^*) \right\}
$$

*whenever $z^* \in \Theta_N^*$ provided that either $Q$ is SNC at $(\bar{x}, \bar{y}, \bar{w})$ or $\dim W < \infty$.*

*Proof.* It is easy to check that under the assumptions made we are in the setting of Theorem 4.3 and hence have the mixed coderivative inclusion (3.10) with the constraint mapping $G$ given by (4.12). Applying now [20, Theorem 4.46], we get the inclusion

$$
\text{(4.15)} \quad D_N^* G(\bar{x}, \bar{y})(v^*) \subset \left\{ u^* \in X^* \,\middle|\, \exists\, v^* \in W^* \text{ with} \right.
$$
$$
\left. (u^*, -y^*) \in D_N^* q(\bar{x}, \bar{y})(v^*) + D_N^* Q(\bar{x}, \bar{y}, \bar{w})(v^*) \right\}
$$

provided that the qualification condition (4.13) is satisfied and that either $Q$ is SNC at $(\bar{x}, \bar{y}, \bar{w})$ or $\dim W < \infty$. Substituting (4.15) into (3.10) gives us inclusion (4.14) and thus completes the proof of the theorem. ∎

The next result ensures the *equality* in (4.14) for both coderivatives $D^* = D_N^*, D_M^*$ under some additional assumptions on the initial data in (1.1) and (4.12).

**Theorem 4.4.** (computing coderivatives of frontier maps in multiobjective problems with equilibrium constraints). *In addition to the assumptions of Theorem 4.3, suppose that the solution map $S \colon \operatorname{dom} G \rightrightarrows Y$ admits an upper Lipschitzian selection around $(\bar{x}, \bar{y})$, that $Q$ is $N$-regular at $(\bar{x}, \bar{y}, \bar{w})$, and that both mappings $f$ and $q$ are strictly differentiable at $(\bar{x}, \bar{y})$; the latter ensures that the Fredholm qualification condition (4.13) is equivalent to the fact that the adjoint generalized equation*

$$
\text{(4.16)} \quad 0 \in \nabla q(\bar{x}, \bar{y})^* v^* + D_N^* Q(\bar{x}, \bar{y}, \bar{w})(v^*)
$$

*admits only the trivial solution $v^* = 0$. Then we have the equality*

$$
\text{(4.17)} \quad
\begin{aligned}
&D^* \mathcal{F}(\bar{x}, \bar{z})(z^*) \\
&= \left\{ u^* + \nabla_x f(\bar{x}, \bar{y})^* z^* \,\middle|\, \exists\, v^* \in W^* \text{ with } \left( u^* - \nabla_x q(\bar{x}, \bar{y})^* v^*, \right.\right. \\
&\qquad \left.\left. -\nabla_y f(\bar{x}, \bar{y})^* z^* - \nabla_y q(\bar{x}, \bar{y})^* v^* \right) \in D^* Q(\bar{x}, \bar{y}, \bar{w})(v^*) \right\}
\end{aligned}
$$

*held for both coderivatives $D^* = D_N^*, D_M^*$ whenever $z^* \in \Theta_N^*$.*

*Proof.* Observe first that the equivalence between the Fredholm qualification condition (4.13) and the triviality of solutions to the adjoint generalized equation (4.16) follows from the coderivative representation (2.8) in the case of strictly differentiable mappings. By [20, Theorem 4.44(ii)] we have the formula

$$
\begin{aligned}
(4.18) \quad & D_N^* G(\bar{x}, \bar{y})(y^*) \\
& = \Big\{ u^* \in X^* \Big| \, \exists v^* \in W^* \text{ with} \\
& \qquad \big( u^* - \nabla_x q(\bar{x}, \bar{y})^* v^*, -y^* - \nabla_y q(\bar{x}, \bar{y})^* v^* \big) \in D_N^* Q(\bar{x}, \bar{y}, \bar{w})(v^*) \Big\}
\end{aligned}
$$

for computing the normal coderivative of the equilibrium constraint mapping $G$ from (4.12) under the validity of the assumptions of this theorem concerning $G$. Furthermore, these assumptions ensure that the constraint mapping $G$ is $N$-regular at $(\bar{x}, \bar{y})$. Substituting (4.18) into equality (3.17) of Theorem 3.4 in the normal coderivative case, we arrive at (4.17) for $D^* = D_N^*$. The fulfillment of (4.17) in the mixed coderivative case follows by Theorem 3.4 from the assumed $N$-regularity (and hence $M$-regularity) of the field mapping $Q$ and the established $N$-regularity of the equilibrium constraint mapping $G$ at the corresponding points. This completes the proof of the theorem. ∎

## 5. Coderivatives of Efficient Solution Maps

The primary goal of this section is to establish verifiable formulas for upper estimating and/or precise computing the normal and mixed coderivatives of the *optimal/efficient solution map* $S$ in the parametric multiobjective problem (1.1) under consideration via the corresponding coderivatives of the frontier map $\mathcal{F}$. To accomplish this, we employ the *coderivative calculus* developed in [20] and also derive new calculus results needed in what follows, which are certainly of independent interest.

Observe first that the optimal solution map (1.3) can be written in the constrained *generalized equation form*

$$
(5.1) \qquad S(x) = \big\{ y \in G(x) \big| \, 0 \in -f(x, y) + \mathcal{F}(x) \big\}
$$

via the initial cost and constraint data of (1.1) and the frontier map in this problem. Comparing (5.1) with the generalized equation form (4.12) used in Section 4 for modeling equilibrium constraints, we emphasize the *two main differences*:

(1) There are *constraints* on the *decision* variable $y \in G(x)$ in (5.1) in contrast to (4.12).

(2) The set-valued *field* mapping $\mathcal{F}$ of the generalized equation in (5.1)–which is now the frontier map in the multiobjective optimization problem–depends

*only* on the *parameter* $x \in X$. Note that, although form (4.12) involves field mappings $Q$ depending on both decision and parameter variables, the main interest in theory and applications of optimization problems with equilibrium constraints relate to the case of *parameter-independent* fields $Q = Q(y)$ that describe major models in parametric variational inequalities, complementarity problems, KKT systems in optimality conditions, etc., where perturbation parameters enter only the single-valued *base* mappings in the corresponding generalized equations; see, e.g., [21, 25, 27, 28, 32] and the references therein.

Having in mind applications to the optimal solution map $S$ for (1.1) written as (5.1), we pay the main attention in this section to the coderivative analysis of solutions to the parametric *constrained generalized equations* with *parameter-independent fields* given by

$$(5.2) \qquad H(x) := \big\{ y \in P(x) \big| \ 0 \in q(x, y) + Q(x) \big\},$$

where $q \colon X \times Y \to W$, $Q \colon X \rightrightarrows W$, and $P \colon X \rightrightarrows Y$ are, respectively, single-valued and set-valued mappings between Banach spaces. As we can see below, the generalized equation model (5.2) with parameter-independent fields possesses certain specific features, which significantly distinguish it from its formally more general counterpart with $Q = Q(x, y)$ and allow us to establish in this way coderivative results that do not hold for general parameter-dependent field models.

The results derived in the next two theorems–on, respectfully, upper estimating and precise calculating the coderivatives of $H$ in (5.2)–extend to the constrained case the previous developments in [20, Chapter 4] for general fields $Q = Q(x, y)$ and also, on the other hand, exploit specific features of parameter-independent fields in (5.2). The first theorem gives *upper estimates* for both normal and mixed coderivatives of solution maps to the constrained generalized equations (5.2).

**Theorem 5.1.** (upper estimates for coderivatives of solution maps to constrained generalized equations with parameter-independent fields). *Let* $q \colon X \times Y \to W$, $Q \colon X \rightrightarrows W$, *and* $P \colon X \rightrightarrows Y$ *be mappings between Asplund spaces, and let* $(\bar{x}, \bar{y}) \in \operatorname{gph} H$ *with* $\bar{w} := -q(\bar{x}, \bar{y}) \in Q(\bar{x})$. *Assume that* $q$ *is locally Lipschitz continuous around* $(\bar{x}, \bar{y})$ *and that* $P$ *and* $Q$ *are locally closed-graph around* $(\bar{x}, \bar{y})$ *and* $(\bar{x}, \bar{w})$, *respectively. Suppose also that the constraint qualification conditions*

$$(5.3) \quad \begin{aligned} &\big[ (x^*, y^*) \in D_N^* q(\bar{x}, \bar{y})(v^*) + \big( D_N^* Q(\bar{x}, \bar{w})(v^*), 0 \big), \ -x^* \in D_N^* P(\bar{x}, \bar{y})(y^*) \big] \\ &\qquad \Longrightarrow x^* = 0, \ y^* = 0 \quad \text{and} \end{aligned}$$

$$(5.4) \quad (x^*, 0) \in D_N^* q(\bar{x}, \bar{y})(v^*) \cap \big( - D_N^* Q(\bar{x}, \bar{w})(v^*), 0 \big) \Longrightarrow x^* = 0, \ v^* = 0$$

*hold and that one of the following requirements* (a)*,* (b) *is satisfied:*

(a) *P is SNC at $(\bar{x}, \bar{y})$, and either Q is SNC at $(\bar{x}, \bar{w})$ or $\dim W < \infty$;*

(b) *Either Q is SNC at $(\bar{x}, \bar{w})$, or the spaces X and W are finite-dimensional.*

  *Then for both coderivatives $D^* = D_N^*, D_M^*$ of the mapping H in (5.2) we have*

$$D^* H(\bar{x}, \bar{y})(y^*)$$

(5.5) $\subset \left\{ x^* \in X^* \middle| \exists v^* \in W^* \text{ with } (x^*, -y^*) \in D_N^* q(\bar{x}, \bar{y})(v^*) \right.$

$$\left. + \big(D_N^* Q(\bar{x}, \bar{w})(v^*), 0\big) + N\big((\bar{x}, \bar{y}); \operatorname{gph} P\big) \right\}, \quad y^* \in Y^*.$$

*Proof.* Since $D_M^* H(\bar{x}, \bar{y})(y^*) \subset D_N^* H(\bar{x}, \bar{y})(y^*)$, it is sufficient to justify the upper estimate (5.5) for the *normal coderivative*. To proceed, we observe the graph *intersection* relation

(5.6) $$\operatorname{gph} H = \operatorname{gph} P \cap \operatorname{gph} T$$

for the mappings in (5.2), where the graph of $T \colon X \rightrightarrows Y$ is given by the *inverse image*

(5.7)  $\operatorname{gph} T := g^{-1}(\Lambda) = \big\{ (x, y) \in X \times Y \,\big|\, g(x, y) \in \Lambda \text{ with } \Lambda := \operatorname{gph} Q \big\}$

of the graphical set $\Lambda \subset X \times W$ under the mapping $g \colon X \times Y \to X \times W$ defined by

(5.8) $$g(x, y) := \big(x, -q(x, y)\big), \quad x \in X, \ y \in Y.$$

To obtain the claimed estimate for the normal coderivative of the mapping $H$ generated by the normal cone to the graph of $H$, we start with applying the fundamental *normal intersection rule* from [20, Theorem 3.4] to the sets in (5.6) that belong to the Asplund space $X \times Y$; in fact, we employ just [20, Corollary 3.5] for simplicity. Observe that the assumptions made in the theorem ensures the local closedness of the sets in (5.6) around $(\bar{x}, \bar{y})$. Employing the afore-mentioned intersection rule, we get the inclusion

(5.9)  $N\big((\bar{x}, \bar{y}); \operatorname{gph} H\big) \subset N\big((\bar{x}, \bar{y}); \operatorname{gph} P\big) + N\big((\bar{x}, \bar{y}); \operatorname{gph} T\big)$

provided that either $P$ or $T$ is SNC at $(\bar{x}, \bar{y})$ and that the normal qualification condition

(5.10)   $N\big((\bar{x}, \bar{y}); \operatorname{gph} P\big) \cap \big[ -N\big((\bar{x}, \bar{y}); \operatorname{gph} T\big) \big] = \{0\}$

is satisfied. Assume first that the mapping $P$ is SNC at $(\bar{x}, \bar{y})$, which corresponds to case (a) in the assumptions of the theorem, and proceed with the representation of

the normal cone to gph $T$ in (5.9) and (5.10). By the inverse image description (5.7) we employ the calculus rule from [20, Theorem 3.8] on *basic normals to inverse images* that gives the inclusion

$$(5.11) \quad N\big((\bar{x}, \bar{y}); \operatorname{gph} T\big) \subset \bigcup_{(x^*, v^*) \in N(g(\bar{x}, \bar{y}); \operatorname{gph} Q)} D_N^* g(\bar{x}, \bar{y})(x^*, v^*)$$

provided that the qualification condition

$$(5.12) \quad N\big(g(\bar{x}, \bar{y}); \operatorname{gph} Q\big) \cap \ker D_N^* g(\bar{x}, \bar{y}) = \{0\}$$

is satisfied and that either $Q$ is SNC at $(\bar{x}, \bar{w})$ or $g^{-1}$ is PSNC at $(g(\bar{x}, \bar{y}), \bar{x}, \bar{y})$. Due to the obvious sum representation $g(x, y) = (x, 0) + (0, -q(x, y))$ of the mapping $g$ in (5.8) we have by the *sum rule* from [20, Theorem 1.62(ii)] that

$$(5.13) \quad \begin{aligned} D_N^* g(\bar{x}, \bar{y})(x^*, v^*) &= (x^*, 0) + D_N^* q(\bar{x}, \bar{y})(-v^*) \\ &\quad \text{for all} \ \ x^* \in X^* \ \ \text{and} \ \ v^* \in W^*. \end{aligned}$$

It follows from the proof of [20, Theorem 4.46] that the mapping $g^{-1}$ is PSNC at $(g(\bar{x}, \bar{y}), \bar{x}, \bar{y})$ if (actually *if and only if*) dim $W < \infty$. It is a simple matter to check that (5.3) and (5.4) imply (5.10) and (5.12). Substituting now the coderivative representation (5.13) into (5.11) and (5.4) and then into (5.9) and (5.10), we arrive at the coderivative upper estimate (5.5) under the qualification condition (5.3) and (5.4) in case (a) of the theorem. To finish the proof in this case, observe that the qualification condition (5.4) automatically holds when dim $W < \infty$ due to the local Lipschitz continuity of $q$ around $(\bar{x}, \bar{y})$, since the latter property implies that $D_M^* q(\bar{x}, \bar{y})(0) = \{0\}$ by [20, Theorem 1.44].

Consider further the remaining case when $P$ is *not* assumed to be SNC at $(\bar{x}, \bar{y})$. Thus to get (5.9) and proceed further by the above arguments, we need to derive verifiable conditions ensuring the *SNC property* of the mapping $T$ from (5.7) at the point $(\bar{x}, \bar{y})$. The *SNC calculus/preservation* results from [20, Theorem 3.84] allow us to conclude that the inverse image (5.7) is SNC at $(\bar{x}, \bar{y})$ if the qualification condition (5.12) holds and if either $Q$ is SNC at $(\bar{x}, \bar{w})$ or $g$ is SNC at $(\bar{x}, \bar{y})$; note that $g$ is surely PSNC at this point. Taking into account the structure of $g$ in (5.8) and using [20, Corollary 3.30], we get that the SNC property of $g$ at $(\bar{x}, \bar{y})$ is *equivalent* to the finite dimensionality of both spaces $X$ and $W$. It allows us to justify by the above arguments the validity of the coderivative upper estimate (5.5) in case (b) under the corresponding assumptions made therein and therefore to complete the proof of the theorem. ∎

The next result presents verifiable requirements on the initial data of (5.2), generally *different* from those in Theorem 5.1, ensuring *precise formulas* for computing

the normal and mixed coderivatives of the solution map $H$, which specify and simplify both the qualification conditions and the coderivative expression–held as *equality*–of Theorem 5.1.

**Theorem 5.2.** (computing coderivatives of solution maps to constrained generalized equations with parameter-independent fields). *Let* $(\bar{x}, \bar{y}) \in \operatorname{gph} H$ *in the notation of Theorem 5.1, where the mapping* $q\colon X \times Y \to W$ *is strictly differentiable at* $(\bar{x}, \bar{y})$. *The following assertions hold:*

(i) *Assume that* $P(x) \equiv Y$, *that all the spaces are Banach, and that the partial derivative operator* $\nabla_y q(\bar{x}, \bar{w})\colon X \times Y \to W$ *is surjective. Then we have the equality*

(5.14)
$$D_N^* H(\bar{x}, \bar{y})(y^*) = \left\{ \nabla_x q(\bar{x}, \bar{y})^* v^* + D_N^* Q(\bar{x}, \bar{w})(v^*) \middle| \; - y^* \right.$$
$$= \nabla_y q(\bar{x}, \bar{y})^* v^* \Big\}$$

*for the normal coderivative of* $H$ *at* $(\bar{x}, \bar{y})$ *whenever* $y^* \in Y^*$.

(ii) *Let the assumptions of Theorem 5.1 be satisfied, where the qualification conditions* (5.3) *and* (5.4) *reduce to*

(5.15)
$$\left[ 0 \in \nabla_x q(\bar{x}, \bar{y})^* v^* + D_N^* Q(\bar{x}, \bar{w})(v^*) + D_N^* P(\bar{x}, \bar{y})(y^*), \right.$$
$$\left. y^* = \nabla_y q(\bar{x}, \bar{y})^* v^* \right] \implies v^* = 0.$$

*Suppose in addition that* $P$ *and* $Q$ *are* $N$-*regular at the points* $(\bar{x}, \bar{y})$ *and* $(\bar{x}, \bar{w})$, *respectively. Then for both coderivatives* $D^* = D_N^*, D_M^*$ *we get the equality*

(5.16)
$$D^* H(\bar{x}, \bar{y})(y^*) = \left\{ \nabla_x q(\bar{x}, \bar{y})^* v^* + D^* Q(\bar{x}, \bar{w})(v^*) + u^* \middle| \right.$$
$$\left. u^* \in D^* P(\bar{x}, \bar{y}) \big( y^* + \nabla_y q(\bar{x}, \bar{y})^* v^* \big) \right\}.$$

*Proof.* To justify assertion (i), observe that in this case we have

(5.17)
$$\operatorname{gph} H = g^{-1}(\operatorname{gph} Q),$$

where $g\colon X \times Y \to W$ is defined by (5.8). The *inverse image calculus rule* from [20, Theorem 1.17] in arbitrary Banach spaces yields the equality

(5.18)
$$N\big((\bar{x}, \bar{y}); \operatorname{gph} H\big) = \nabla g(\bar{x}, \bar{y})^* N\big(g(\bar{x}, \bar{y}); \operatorname{gph} Q\big)$$

provided that the derivative operator $\nabla g(\bar{x}, \bar{y})\colon X \times Y \to X \times W$ is *surjective*. It is easy to conclude from the structure of $g$ in (5.8) that the surjectivity of $\nabla g(\bar{x}, \bar{y})$ is

*equivalent* to the surjective of the partial derivative $\nabla_y q(\bar{x}, \bar{y})$. Thus, by elementary calculation, we get formula (5.14) from (5.18), (5.8), and definition (2.3) of the normal coderivative.

Let us now justify assertion (ii) for both coderivatives $D^* = D_N^*, D_M^*$. Considering the normal coderivative case in (5.16), we will see from the proof below that the assumptions made in (ii) ensure that the normal and mixed coderivatives of $H$ agree, and thus we simultaneously justify equality (5.16) in the mixed coderivative case as well.

Since $g$ is assumed to be strictly differentiable at $(\bar{x}, \bar{y})$, we get from the coderivative representation (2.8) that the qualification conditions (5.3) and (5.4) reduce to (5.15), and the right-hand side of (5.5) reduces to that of (5.16). Proceeding as in the proof of Theorem 5.1, observe that the additional $N$-regularity assumption on $P$ in (ii) and the $N$-regularity property of $T$ ensure the *equality* in the intersection rule (5.9) and the $N$-*regularity* of $H$ by the equality and regularity conclusions of [20, Theorem 3.4] in the Asplund space setting. Furthermore, the strict differentiability of $g$ and the $N$-regularity assumption on $Q$ in (ii) imply the *equality* in (5.11) and the $N$-*regularity* of $T$ by the equality and regularity conclusions of [20, Corollary 3.16] applied to the *set indicator* outer function $F(\cdot) = \delta(\cdot; \text{gph}\, Q)$ therein. Taking all this into account and following the proof of Theorem 5.1, we justify assertion (ii) under the assumptions made and thus complete the proof of the theorem. ∎

Now it is easy to derive from Theorems 5.1 and 5.2 the corresponding results on upper estimating and precise computing coderivatives of the efficient solution map for the initial multiobjective problem (1.1) under consideration.

**Theorem 5.3.** (upper estimates for coderivatives of efficient solution maps in multiobjective optimization). *Let $S$ in* (1.3) *be the efficient solution map for the multiobjective optimization problem* (1.1)*, where the spaces $X$, $Y$, and $Z$ are Asplund. Given $(\bar{x}, \bar{y}) \in \text{gph}\, S$ with $\bar{z} := f(\bar{x}, \bar{y})$, assume that $f$ is strictly Lipschitz continuous around $(\bar{x}, \bar{y})$ and that $G$ and $\mathcal{F}$ are locally closed-graph around $(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{z})$, respectively. Suppose also that the constraint qualification conditions*

$$\left[(x^*, y^*) \in D_N^* f(\bar{x}, \bar{y})(-z^*) + \left(D_N^* \mathcal{F}(\bar{x}, \bar{z})(z^*), 0\right), \quad -x^* \in D_N^* G(\bar{x}, \bar{y})(y^*)\right]$$
$$\implies x^* = 0, \ y^* = 0 \quad and$$

$$(x^*, 0) \in D_N^* f(\bar{x}, \bar{y})(-z^*) \cap \left(-D_N^* \mathcal{F}(\bar{x}, \bar{z})(z^*), 0\right) \implies x^* = 0, \ z^* = 0$$

*hold and that one of the following requirements* (a)*,* (b) *is satisfied:*

(a) *$G$ is SNC at $(\bar{x}, \bar{y})$, and either $\mathcal{F}$ is SNC at $(\bar{x}, \bar{z})$ or $\dim W < \infty$;*

(b) *Either $\mathcal{F}$ is SNC at $(\bar{x}, \bar{z})$, or the spaces $X$ and $Z$ are finite-dimensional.*

*Then for both coderivatives $D^* = D_N^*, D_M^*$ of the solution map $S$ we have*

$$D^*S(\bar{x}, \bar{y})(y^*) \subset \left\{ x^* \in X^* \middle| \exists z^* \in Z^* \text{ with } (x^*, -y^*) \in D_N^* f(\bar{x}, \bar{y})(-z^*) \right.$$
$$\left. + \left( D_N^* \mathcal{F}(\bar{x}, \bar{z})(z^*), 0 \right) + N\left( (\bar{x}, \bar{y}); \operatorname{gph} G \right) \right\}, \quad y^* \in Y^*.$$

*Proof.* Since $f$ is strictly Lipschitz continuous around $(\bar{x}, \bar{y})$, it follows from [20, Theorem 3.28] that

$$(5.19) \qquad D^*(-f)(\bar{x}, \bar{y})(z^*) = D^* f(\bar{x}, \bar{y})(-z^*), \quad z^* \in Z^*.$$

Due to representation (5.1) of the efficient solution map we deduce these results from Theorem 5.1 with $q = -f$, $Q = \mathcal{F}$, and $P = G$. The proof is complete. ∎

Similarly we deduce from Theorem 5.2 precise formulas for computing coderivatives of the efficient solution map $S$ for the multiobjective problem (1.1).

**Theorem 5.4.** (computing coderivatives of efficient solution maps in multiobjective optimization). *Let $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ in the notation of Theorem 5.3, where the mapping $f \colon X \times Y \to Z$ is strictly differentiable at $(\bar{x}, \bar{y})$. The following assertions hold:*

(i) *Assume that $G(x) \equiv Y$, that all the spaces are Banach, and that the partial derivative operator $\nabla_y f(\bar{x}, \bar{w}) \colon X \times Y \to Z$ is surjective. Then we have the equality*

$$D_N^* S(\bar{x}, \bar{y})(y^*) = \left\{ D_N^* \mathcal{F}(\bar{x}, \bar{z})(z^*) - \nabla_x f(\bar{x}, \bar{y})^* z^* \middle| y^* = \nabla_y f(\bar{x}, \bar{y})^* z^* \right\}$$

*for the normal coderivatives of the efficient solution map $S$ at $(\bar{x}, \bar{y})$ whenever $y^* \in Y^*$.*

(ii) *Let the assumptions of Theorem 5.3 be satisfied, where the qualification conditions reduce to*

$$\left[ \begin{array}{l} 0 \in D_N^* \mathcal{F}(\bar{x}, \bar{z})(z^*) - \nabla_x f(\bar{x}, \bar{y})^* z^* + D_N^* G(\bar{x}, \bar{y})(y^*), \\ -y^* = \nabla_y f(\bar{x}, \bar{y})^* z^* \end{array} \right] \Longrightarrow z^* = 0.$$

*Suppose in addition that $G$ and $\mathcal{F}$ are $N$-regular at the points $(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{z})$, respectively. Then for both coderivatives $D^* = D_N^*, D_M^*$ we have the equality*

$$D^* S(\bar{x}, \bar{y})(y^*) = \left\{ D^* \mathcal{F}(\bar{x}, \bar{z})(z^*) - \nabla_x f(\bar{x}, \bar{y})^* z^* + u^* \middle| \right.$$
$$\left. u^* \in D^* G(\bar{x}, \bar{y})\left( y^* - \nabla_y f(\bar{x}, \bar{y})^* z^* \right) \right\}.$$

*Proof.* Follows from Theorem 5.2 with $q = -f$, $Q = \mathcal{F}$, and $P = G$ due to representation (5.1) and the coderivative relation (5.19). ∎

We conclude the paper by the following remark on *robust Lipschitzian stability* of the frontier and efficient solution maps for the multiobjective optimization problem (1.1).

**Remark 5.5.** (Lipschitzian stability of frontier and efficient solution maps in multiobjective optimization). As mentioned in Section 1, one of the most important applications of the coderivatives under consideration is the possibility to establish *complete characterizations* of *robust Lipschitzian stability* for general set-valued mappings between finite-dimensional and infinite-dimensional spaces. By this we mean deriving verifiable *necessary and sufficient coderivative conditions* for the *Lipschitz-like* property (2.9), which encompasses classical local Lipschitzian behavior of single-valued and set-valued mappings being *robust/stable* with respect of perturbations of the initial data and being *equivalent* to the *linear openness/covering property* of the mapping in question and to the *metric regularity* property of its inverse. Moreover, we get *precise coderivative formulas* for computing *exact bounds* of Lipschitzian (linear openness, metric regularity) moduli. The coderivative results derived in this vein allow us to obtain numerous applications to broad classes of structural problems in variational analysis, optimization, and related areas due to well-developed *coderivative calculus*. We refer the reader to [18, 20, 28] for more details, discussions, and applications.

For closed-graph mappings $F: X \rightrightarrows Y$ between *finite-dimensional* spaces the *coderivative/Mordukhovich criterion* for the Lipschitz-like/Aubin property of $F$ around $(\bar{x}, \bar{y})$ is given by the simple formula

$$(5.20) \qquad\qquad D^*F(\bar{x}, \bar{y})(0) = \{0\},$$

and the *exact bound* (infimum) of all the Lipschitzian moduli $\ell$ in (2.9) denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$ is computed by the coderivative norm

$$(5.21) \quad \operatorname{lip} F(\bar{x}, \bar{y}) = \|D^*F(\bar{x}, \bar{y})\| := \sup \left\{ \|x^*\| \,\big|\, x^* \in D^*F(\bar{x}, \bar{y})(y^*), \ \|y^*\| \leq 1 \right\};$$

see [18, Theorem 5.7] and [28, Theorem 9.40]. Thus the coderivative results (upper estimates and precise formulas) obtained in this paper allow us to derive from (5.20) and (5.21) verifiable *sufficient* as well as *necessary and sufficient conditions* for robust Lipschitzian properties of frontier and efficient solution maps in the multiobjective problems under consideration and also estimate/compute exact bounds of Lipschitzian moduli.

The case of set-valued mappings $F: X \rightrightarrows Y$ between *infinite-dimensional* spaces is significantly more involved in comparison with its finite-dimensional

counterpart. Pointbased necessary and sufficient coderivative conditions for the Lipschitz-like property of mappings between Asplund spaces (the necessity part holds in arbitrary Banach spaces) are derived in [19, Theorem 3.3] and [20, Theorem 4.10] via the *mixed* coderivative of $F$ and the *PSNC* property of this mapping, while upper and lower estimates of the exact Lipschitzian bound employ both normal and mixed coderivatives. In our further research we intend to implement the afore-mentioned results as well as those developed in this paper to deriving verifiable conditions for robust Lipschitzian stability of frontier and efficient solution maps in various problems of constrained multiobjective optimization and equilibria.

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