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ON A DIFFERENCE EQUATION MOTIVATED BY A HEAT CONDUCTION PROBLEM

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Abstract. Let $\{\tau_n\}$ be a sequence of numbers recursively defined by

$$f(\tau_n) + f(\tau_n + \tau_{n-1}) + \cdots + f(\tau_n + \tau_{n-1} + \cdots + \tau_1) = 1,$$

where f is a continuous and strictly decreasing function on $(0, \infty)$ with $f(0^+) \geq 1$, and $f(\infty) = 0$. Assume the convexity of $\log f$ or $\log |f'|$. It can be shown that $\{\tau_n\}$ is increasing. Thus $\lim \tau_n$ exists in $(0, \infty]$.

The difference equation above is motivated by a heat conduction problem studied in Myshkis (1997) and Chen, Chow and Hsieh (2006).

1. Introduction

In this note we study the behaviour of a sequence $\{\tau_n\}$ which is recursively defined by

$$\sum_{j=1}^{n} f(\sum_{s=j}^{n} \tau_s) = 1 \ ; \ n = 1, 2, ..., \tag{1.1}$$

where f is a continuous and strictly decreasing function on $(0, \infty)$ with $f(0^+) \geq 1$, and $f(\infty) = 0$. We will characterize the behaviour of the sequence $\{\tau_n\}$ by assuming the convexity of $\log f$ or $\log |f'|$.

Theorem 1.1. *Assume* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies the conditions in* (1.1). *If* $(\log f)''$ *or* $(\log |f'|)''$ *is nonnegative (positive), then the sequence* $\{\tau_n\}$ *defined in* (1.1) *is (strictly) increasing. Moreover,*

$$\lim \tau_n = \beta < \infty \textit{ iff } \sum_{n=1}^{\infty} f(n) < \infty.$$

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In that case, β is uniquely determined by the equation $\sum_{n=1}^{\infty} f(n\beta) = 1$.

We remark that it is not difficult to check that

$$\tau_1 < \tau_2 = \tau_3 = \cdots = \tau_n = \cdots \text{ in case } f(x) = cq^x \text{ with } c > 1 > q > 0.$$

It indicates that if $(\log f)''$ or $(\log |f'|)''$ is nonnegative only, then $\{\tau_n\}$ may not be strictly increasing.

Also note that the condition $(\log f)''$ (or $(\log |f'|)''$) is nonnegative on $(0, \infty)$ can be replaced by $\log f$ (or $\log |f'|$) is convex on $(0, \infty)$, which is slightly weaker. By Proposition 5.17 in Royden (1988) [8], we may use any of the one-sided derivatives of $\log f$ (or of $\log |f'|$) to replace $(\log f)'$ (or $(\log |f'|)'$).

It is easy to verify that $f(x) = cx^{-\delta}$, where c and δ are positive constants, satisfies the condition in Theorem 1.1. Hence we get the following.

Corollary 1.2. ([3-5]). *Let $\{\tau_n\}$ be defined in (1.1) with $f(x) = cx^{-\delta}$ on $(0, \infty)$, then $\{\tau_n\}$ is strictly increasing. Moreover,* $\lim \tau_n = \infty$ *for* $0 < \delta \leq 1$ *and* $\lim \tau_n = (c\zeta(\delta))^{1/\delta}$ *for* $\delta > 1$. *Here,* $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ *is the Riemann-Zeta function.*

This research is motivated by a heat conduction problem studied by Myshkis (1997) [6]. Assume the initial temperature of an infinite homogeneous medium set in $\mathbf{R^d}$ is 0 and at time 0, a heat impulse of size b is applied at the origin. When the temperature at the origin drops to a preset threshold $u_0 > 0$, another heat impulse of the same size is applied at the origin. The same procedure is repeated over and over. Let $t_0 = 0, t_1, t_2, ..., t_{n-1}$ be the heating times obtained in this way. By solving the heat equation

$$\begin{cases} \partial u/\partial t = a \cdot \sum_{i=1}^{d} \partial^2 u/\partial x_i^2, \\ u(\mathbf{x}, t_{n-1}^+) = u(\mathbf{x}, t_{n-1}) + b \cdot \delta(\mathbf{x}), \end{cases} \tag{1.2}$$

where a is the heat conduction coefficient of the medium and $\delta(\mathbf{x})$ the Dirac function at $\mathbf{x} = 0$, it is not difficult to obtain from the superposition principle that for $t_{n-1} < t$,

$$u(\mathbf{x}, t) = b \sum_{j=0}^{n-1} \left(\frac{1}{4\pi a(t - t_j)} \right)^{d/2} \exp \left(-\frac{\sum_{i=1}^{d} x_i^2}{4a(t - t_j)} \right).$$

The heating condition $u(0, t_n) = u_0$ then implies

$$u_0 = u(0, t_n) = b \sum_{j=0}^{n-1} \left(\frac{1}{4\pi a(t_n - t_j)} \right)^{d/2}.$$

For $j \geq 1$, define $\tau_j = 4\pi a(t_j - t_{j-1})(u_0/b)^{2/d}$ as the normalized waiting time between two consecutive heating times t_{j-1} and t_j. A simple computation shows

$$\tau_1 = 1 \quad \text{and} \quad \sum_{j=1}^{n}\{\sum_{s=j}^{n}\tau_s\}^{-d/2} = 1 \quad \text{for} \quad n \geq 2. \tag{1.3}$$

Corollary 1.2 can be applied to (1.3) if we take $c = 1$ and $\delta = d/2$. When the heat problem (1.2) is set in a general domain like a finite or semi-infinite region plus some boundary conditions, its fundamental solution becomes very complicated and can at most be expressed in sum of infinite series. This motivates us to study (1.1) as a generalization to (1.3).

Note that Chen, Chow and Hsieh (2000) [5] showed that for the heat problem (1.2), $\lim \tau_n/n = \pi^2/2$ for $d = 1$ as conjectured in Myshkis (1997). Then Chang, Chow and Wang (2003) [4] showed that $\lim \tau_n/\log n = 1$ for $d = 2$. These results are not covered by Theorem 1.1. It is interesting to see how to get some similar results for (1.1).

2. Proof of Theorem 1.1

First we study the increasing property of the sequence $\{\tau_n\}$. By (1.1), $f(\tau_1) = 1 = f(\tau_2) + f(\tau_2 + \tau_1) > f(\tau_2)$. Since $f(t)$ is strictly decreasing, we have $\tau_2 > \tau_1$. By induction, it suffices to check $\tau_{n+1} > \tau_n$ under the hypothesis that

$$\tau_n > \tau_{n-1} > \cdots > \tau_1. \tag{2.1}$$

Define $\widetilde{T}_j^k = \sum_{s=j}^{k}\tau_s$ and $\Phi(t) = f(t) + \sum_{j=1}^{n} f(t + \widetilde{T}_j^n)$, which is strictly decreasing. Assume temporarily that

$$\sum_{j=1}^{n} f(\tau_n + \widetilde{T}_j^n) > \sum_{j=1}^{n-1} f(\widetilde{T}_j^n). \tag{2.2}$$

Adding $f(\tau_n)$ to both sides above and using (1.1),

$$\Phi(\tau_n) > f(\tau_n) + \sum_{j=1}^{n-1} f(\widetilde{T}_j^n) = 1 = \sum_{j=1}^{n+1} f(\widetilde{T}_j^{n+1}) = \Phi(\tau_{n+1}),$$

from which the desired inequality $\tau_{n+1} > \tau_n$ follows.

It remains to verify (2.2). First we show it under the assumption that $(\log f)'' > 0$, which implies f'/f is strictly increasing.

Since both $\{\widetilde{T}_j^n\}, \{\widetilde{T}_j^{n-1}\}$ are strictly decreasing in j and $\widetilde{T}_{j+1}^n > \widetilde{T}_j^{n-1}$ for $1 \leq j \leq n-1$ under the induction hypothesis (2.1), we have

$$f(\widetilde{T}_j^{n-1}) > f(\widetilde{T}_{j+1}^n) > f(\widetilde{T}_1^n) \text{ and } f(\tau_{n-1}) > f(\widetilde{T}_{j-1}^{n-1}). \tag{2.3}$$

Denote $q(t) = f(\tau_n + t)/f(t)$. Because f'/f is strictly increasing, $q'(t) = \{f(\tau_n+t)/f(t)\} \cdot (f'(\tau_n+t)/f(\tau_n+t) - f'(t)/f(t)) > 0$. So q is strictly increasing. Therefore for $2 \le n$ and $1 \le j \le n-1$,

$$q(\widetilde{T}_1^n) > q(\widetilde{T}_{j+1}^n) > q(\widetilde{T}_j^{n-1}) \text{ and } q(\widetilde{T}_{j-1}^{n-1}) > q(\tau_{n-1}) > 0. \tag{2.4}$$

For brevity, introduce $a_j = f(\widetilde{T}_{j+1}^n)$, $\alpha_j = f(\widetilde{T}_j^{n-1})$, $b_j = q(\widetilde{T}_{j+1}^n)$ and $\beta_j = q(\widetilde{T}_j^{n-1})$. Note that $f(\tau_n + \widetilde{T}_j^n) = f(\widetilde{T}_j^n) \cdot q(\widetilde{T}_j^n) = a_{j-1}b_{j-1}$ and $f(\widetilde{T}_j^n) = f(\widetilde{T}_j^{n-1} + \tau_n) = f(\widetilde{T}_j^{n-1}) \cdot q(\widetilde{T}_j^{n-1}) = \alpha_j\beta_j$. By (2.3) and (2.4), we have $b_0 > b_j > \beta_j > 0$ and $\alpha_j > a_j > 0$ for $1 \le j \le n-1$ and $2 \le n$. Using (1.1), the difference on both sides of (2.2) is

$$\begin{aligned}\sum_{j=0}^{n-1} a_j b_j - \sum_{j=1}^{n-1} \alpha_j\beta_j &= a_0b_0 + \sum_{j=1}^{n-1} (a_j(b_j - \beta_j) + \beta_j(a_j - \alpha_j)) \\ &> a_0b_0 + b_0\sum_{j=1}^{n-1}(a_j - \alpha_j) = b_0(\sum_{j=0}^{n-1} a_j - \sum_{j=1}^{n-1} \alpha_j) = 0.\end{aligned} \tag{2.5}$$

This verifies (2.2) and thus $\{\tau_n\}$ is strictly increasing under the assumption that $(\log f)'' > 0$. Under the weaker assumption that $(\log f)'' \ge 0$, we have f'/f is increasing. It is easy to see that all the strictly inequality from (2.1) to (2.5) can be replaced by $\ge$. Hence $\{\tau_n\}$ is increasing.

Now we check (2.2) under the assumption that $(\log|f'|)'' > 0$, which implies f''/f' is strictly increasing. For brevity, denote $\delta_i = \sum_{j=n-i}^{n-1} \tau_j$ and $\gamma_i = \sum_{j=n-i+1}^{n} \tau_j$. By the induction hypothesis (2.1), $\delta_i < \gamma_i$ for $1 \le i \le n-1$.

Denote $g(t) = -f'(t)$. Since f is strictly decreasing with $f(\infty) = 0$, it is easy to verify that $g(t) > 0$ and $g(\infty) = 0$. In terms of g, (1.1) becomes

$$\int_{\delta_1}^{\gamma_1} g(t)dt + \int_{\delta_2}^{\gamma_2} g(t)dt + \cdots + \int_{\delta_{n-1}}^{\gamma_{n-1}} g(t)dt = \int_{\gamma_n}^{\infty} g(t)dt. \tag{2.6}$$

Define $\phi_k(u)$ on $[\gamma_n, \infty)$ for $1 \le k \le n-1$ in the following way :

$$\int_{\delta_k}^{\phi_k(u)} g(t)dt = \left(\frac{\int_{\delta_k}^{\gamma_k} g(t)dt}{\int_{\gamma_n}^{\infty} g(t)dt}\right) \int_u^{\infty} g(t)dt.$$

Note that $\phi_k(\gamma_n) = \gamma_k$ and $\phi_k(\infty) = \delta_k$. Moreover, $\phi_k'(u) < 0$ and thus $\phi_k(u)$ is strictly decreasing on $[\gamma_n, \infty)$. Rewrite (2.6) as

$$\int_{\delta_1}^{\phi_1(u)} g(t)dt + \int_{\delta_2}^{\phi_2(u)} g(t)dt + \cdots + \int_{\delta_{n-1}}^{\phi_{n-1}(u)} g(t)dt = \int_u^{\infty} g(t)dt.$$

Differentiating both sides of this equation, we get

$$g(u) + \sum_{j=1}^{n-1} g(\phi_j(u))\phi_j'(u) = 0. \tag{2.7}$$

Let $H(u) = -g(u) + \sum_{j=1}^{n-1} g(\delta_j) - g(\phi_j(u))$ on $[\gamma_n, \infty)$. Using $g = -f'$ and thus $g' = g\frac{f''}{f'}$,

$$H'(u) = -(g \cdot \frac{f''}{f'})(u) - \sum_{j=1}^{n-1}(g \cdot \frac{f''}{f'})(\phi_j(u)) \cdot \phi_j'(u). \tag{2.8}$$

Because f''/f' is strictly increasing, $g > 0$ and $\phi_j(u) \leq \gamma_j < \gamma_n$, $\phi_j'(u) < 0$ for $u \geq \gamma_n$ and $1 \leq j \leq n-1$, we have

$$-(g \cdot \frac{f''}{f'})(\phi_j(u)) \cdot \phi_j'(u) \leq -\frac{f''}{f'}(u) \cdot g(\phi_j(u)) \cdot \phi_j'(u).$$

By (2.7),

$$H'(u) < -\frac{f''}{f'}(u) \left(g(u) + \sum_{j=1}^{n-1} g(\phi_j(u))\phi_j'(u) \right) = 0 \text{ for } u \geq \gamma_n, \tag{2.9}$$

which means $H(u)$ is strictly decreasing for $u \geq \gamma_n$. Since $H(\infty) = 0$, we assert that $H(u) > 0$ on $[\gamma_n, \infty)$. In particular,

$$H(\gamma_n) = \sum_{j=1}^{n} f'(\gamma_j) - \sum_{j=1}^{n-1} f'(\delta_j) > 0. \tag{2.10}$$

By continuity and (1.1),

$$\sum_{j=1}^{n} f(\gamma_j + x) \gtrless \sum_{j=1}^{n-1} f(\delta_j + x) \text{ for } x \gtrless 0 \text{ and small.} \tag{2.11}$$

We claim that $\sum_{j=1}^{n} f(\gamma_j + x) > \sum_{j=1}^{n-1} f(\delta_j + x)$ for all $x > 0$. Then letting $x = \tau_n$, we obtain (2.2) and thus $\{\tau_n\}$ is strictly increasing as desired. Suppose the contrary. By (2.11), there exists a number $v > 0$ such that

$$\sum_{j=1}^{n} f(\gamma_j + x) > \sum_{j=1}^{n-1} f(\delta_j + x) \text{ for } 0 < x < v, \tag{2.12}$$

but $\sum_{j=1}^{n} f(\gamma_j + v) = \sum_{j=1}^{n-1} f(\delta_j + v)$. Repeating the same reasoning from (2.6) on, but with δ_j and γ_j replaced by $\delta_j + v$ and $\gamma_j + v$ respectively, we would obtain

$$\sum_{j=1}^{n} f(\gamma_j + v + x) \gtrless \sum_{j=1}^{n-1} f(\delta_j + v + x) \text{ for } x \gtrless 0,$$

which contradicts to (2.12). This verifies the claim and thus the conclusion.

If we assume the weaker condition that $(\log |f'|)'' \geq 0$, then f''/f' is increasing only. Equation (2.9) becomes $H'(u) \leq 0$ in (γ_n, ∞) and then it is possible that $H(\gamma_n) = 0$ in (2.10). In that case, $H(u) = 0$ for $u \geq \gamma_n$ due to $H'(u) \leq 0$ and $H(\infty) = 0$. Consequently, $H'(u) = 0$ on (γ_n, ∞). In view of (2.8) and (2.9),

$$-(g \cdot \frac{f''}{f'})(\phi_j(u)) \cdot \phi_j'(u) = -\frac{f''}{f'}(u) \cdot g(\phi_j(u)) \cdot \phi_j'(u)$$

for $1 \leq j \leq n-1$ and $u \in (\gamma_n, \infty)$. Since $g > 0$ and $\phi_j' < 0$, we get

$$\frac{f''}{f'}(\phi_j(u)) = \frac{f''}{f'}(u)$$

for $1 \leq j \leq n-1$ and $u \in (\gamma_n, \infty)$. Taking $j = 1$ and $u = \infty$, we have $(f''/f')(\tau_{n-1}) = (f''/f')(\infty)$. This implies that f''/f' is a constant function on $[\tau_{n-1}, \infty)$ due to the monotone assumption of f''/f'. Note that $f(\infty) = 0$. A simple integration shows $f(x) = cq^x$ on $[\tau_{n-1}, \infty)$, where $c > 0$ and $1 > q > 0$ are constants. It is then easy to check that

$$\tau_1 < \tau_2 < \cdots < \tau_n = \tau_{n+1} = \tau_{n+2} = \cdots .$$

So we can only claim that $\{\tau_k; k \geq 1\}$ is increasing as remarked after Theorem 1.1.

Once we know $\{\tau_k; k \geq 1\}$ is increasing. What remains is easy. Let $\lim \tau_n = \beta \leq \infty$. If $\beta < \infty$, we get from applying Monotone Convergence Theorem to (1.1) that

$$\sum_{n=1}^{\infty} f(n\beta) = 1. \tag{2.13}$$

Since f is strictly decreasing, $\sum_{n=1}^{\infty} f(n) = \infty$ iff $\sum_{n=1}^{\infty} f(nx) = \infty$ for any $x > 0$. In that case, it is not difficult to see that $\beta = \infty$ by contradiction. Otherwise, $\beta < \infty$ and is uniquely determined by (2.13).

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