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# AN EXTENSION OF JUNGCK AND SESSA RESULT

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**Abstract.** The purpose of this paper is to extend the work of Jungck and Sessa [4] from single-valued maps to multivalued maps.

## 1. Introduction

Brosowski [1], Meinardus [9] and Singh [12] established some results on invariant approximation in normed spaces using fixed point theory. Jungck and Sessa [4] worked over approximation theory for setting of normed spaces. Later on, several generalizations of their results were obtained by Habiniak [2], Hicks and Humphries [3], Khan, Hussain and Thaheem [7], Latif and Bano [8] and Narang [10]. However, another important aspect of their work is still not explored and needs attention and that is the application of their results to multi-valued maps. Therefore, we have decided to extend the result of Jungck and Sessa [4] from single- valued maps.

Throughout the manuscript, the following definitions and results have been used. Let $X$ be a normed space and $M$ be its nonempty subset. We denote the families of all nonempty closed bounded and nonempty compact subsets of $X$ by $CB(X)$ and $K(X)$ respectively. Let $H$ be the Hausdorff metric on $CB(X)$ induced by the norm of $X$ which means

$$H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|\right\}, \sup_{a\in B}\inf_{b\in B}\|a-b\|, \text{for } A,\ B \text{ in } CB(X),$$

$M$ is said to be starshaped with respect to a point $q \in M$ if $(1-h)q + hx \in M$ for all $x \in M$ and all $h \in (0,1)$. Further, each convex set is necessarily starshped, but a starshaped need not be convex.

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Let $P_M(x_0) = \{y \in M : d(y, x_0) = d(x_0, M)\}$ the set of best $M$ approximation to $x_0$. $P_M(x_0)$ is always bounded subset of $X$ and it is closed or convex if $M$ is closed or convex [1]. We adopt the following definitions for convenience. Let $\wp$ be a family of single-valued self map of $M$ and $\wp_f = \{f^n : M \to M, n \geq 0\}(f^0 = I)$. A map $T : M \to CB(M)$ said to be

(i) $\wp$ -nonexpansive [6], if for all $x, y \in M$ there exists $f, g \in \wp$ such that

$$H(Tx, Ty) \leq \|fx - gx\|;$$

(ii) $\wp$ -contraction [6], if for all $x, y \in M$ there exists a real number $h \in (0, 1)$ such that

$$H(Tx, Ty) \leq h\|fx - gy\| \text{ for some } f, g \in \wp;$$

(iii) $\wp_f$ -nonexpansive, if for each $x, y \in M$ there exists $n, m \geq 0$ such that

$$H(Tx, Ty) \leq \|f^n x - f^m y\|;$$

We denote by $F(T)$ and $F(f)$ $(F(\wp))$ the set of all fixed points of $T$ and $f$(the set of all common fixed points of $\wp$ ). Also we say that $\wp$ and $T$ commute, if for each $f \in \wp$ commutes with $T$.

In 1995 Jungck and Sessa [4] obtained the following generalization of Sahab, Khan and Sessa result [11].

**Theorem 1.1.** *Let $M$ be compact subset of a normed linear space, $X$ which is starshaped with respect to $q \in M$. Let $T : M \to M$ be continuous and let $\wp$ be a family of affine maps $f : M \to M$ such that $q \in F(f)$ and $fT = Tf$ If for each pair $(x, y) \in M^2$ there exists $f, g \in \wp$ such that*

$$\|Tx - Ty\| \leq \|fx - gy\|,$$

*then there exists $a \in M$ such that $a = T(a)$ and $a = f(a)$ for all continuous $f \in \wp$.*

We now state a theorem, which is a special case of Theorem 1 in [5].

**Theorem 1.2.** *Let $(X, d)$ be a compact metric space. Let $\wp$ be a family of continuous self mappings of $X$ and $T : X \to K(X)$ be $\wp$-contraction multi-valued map such that $T(X) \subseteq f(X)$ for all $f \in \wp$ and $\wp$ commutes with $T$. Then $F(T) \cap F(\wp) \neq \phi$.*

## 2. Results

In this section Theorem 1.1 has been extended to multi-valued $\wp$ -nonexpansive map.

**Theorem 2.1.** *Let $M$ be a compact subset of a normed linear space, $X$ which is starshaped with respect to $q \in M$. Let $\wp$ be family of continuous affine maps $f : M \to M$ such that $q \in F(f)$ for all $f \in \wp$. If $T : M \to K(M)$ is multi-valued $\wp$-nonexpansive map which commutes with $f$ for all $f \in \wp$ then there exists a point $z \in M$ such that $z \in F(T)$ and also $z \in F(f)$ for all $f \in \wp$ then $F(T) \cap F(f) \neq \phi$*

*Proof.* Consider a sequence $\{h_n\}$ of real numbers for which $0 < h_n < 1$ and $h_n \to 1$ as $n \to \infty$ . For each $n$, a multi-valued map $T_n$ is defined by setting

$$T_n(x) = h_n Tx + (1 - h_n)q, \quad (x \in M).$$

Now we prove that for each $n \geq 1$, $T_n$ maps $M$ into $K(M)$. Indeed we show that $T_n(x)$ is compact for all $x \in M$. Let $x_k \in T_n(x), k = 1, 2, ....,$ be a sequence of $T_n(x)$. We get $x_k = h_n u_k + (1 - h_n)q$ for some $u_k \in T(x)$ and, by using the compactness of $T(x)$ there exists a suitable subsequence $\{u_{k(p)}, p = 1, 2, ..., \}$ such that $u_{k(p)} \to u \in T(x)$ By setting $x_0 = h_n u + (1 - h_n)q$, we have that there is a subsequence $\{x_{k(p)}, k = 1, 2, ..., \}$ of $x_k$, such that $x_{k(p)} = h_n u_{k(p)} + (1 - h_n)q \to x_0$. This proves that $T_n(x)$ is compact. ■

Next we show that $T_n$ is $\wp$ -contraction and commutes with $\wp$. Let $x, y \in M$ and $u_x \in T_n(x)$ then $u_x = h_n v_x + (1 - h_n)q$ for some $v_x \in T(x)$. Since for all $f, g \in \wp$, $T$ is multivalued $\wp$-nonexpansive so there exists $v_y \in T(y)$ for all $y \in M$ such that

$$d(v_x, v_y) \leq H(Tx, Ty).$$

Put $u_y = h_n v_y + (1 - h_n)q$ then $v_y \in T(y)$ and

$$d(u_x, u_y) = h_n d(v_x, v_y) \leq h_n H(Tx, Ty).$$

so by the $\wp$ -nonexpansiveness of $T$ we have

$$d(u_x, u_y) \leq h_n d(f(x), g(y)).$$

It follows that

$$\sup_{u_x \in T_n(x)} d(u_x, T_n(y)) \leq d(u_x, u_y) \leq h_n d(f(x), g(y)).$$

The same argument concludes that

$$\sup_{u_y \in T_n(y)} d(u_y, T_n(x)) \leq h_n d(f(x), g(y)).$$

Hence

$$H(T_nx, T_ny) \leq h_n\|fx - gy\|, \forall f, g \in \wp$$

Which proves that each $T_n$ is $\wp$ -contraction for each $x \in M$. Moreover, since $T$ commutes with $\wp$ and each $f \in \wp$ is affine so for each $x \in M$,

$$\begin{aligned} T_nfx &= h_nTfx + (1 - h_n)fq \\ &= h_nfTx + (1 - h_n)fq \\ &= f(h_nTx + (1 - h_n)q) \\ &= fT_nx \end{aligned}$$

Thus each $T_n$ commutes with $f$. As all the conditions of Theorem 1.2 are satisfied hence $x_n \in M$ such that $x_n \in T_nx_n$ and $x_n = fx_n$ for all $f \in \wp$. So by the definition of $T_nx_n$ there is some $w_n \in Tx_n$ such that

$$x_n = h_nw_n + (1 - h_n)q$$

Since $\{x_n\}$ is a sequence in a compact set $M$, there exists a subsequence $\{x_{n_i}\}$. with $x_{n_i} \to z \in M$ As $z = \lim_i x_{n_i} = \lim_i h_{n_i}w_{n_i} + \lim_i(1 - h_{n_i})q$ and that $h_{n_i} \to 1$ then it follows that $z \in Tz$, which implies that $z \in F(T)$. Further, if $f \in \wp$ is continuous, then it follows that

$$z = \lim_i x_{n_i} = f(\lim_i x_{n_i}) = f(z)$$

which implies that $z = f(z)$

**Corollary 2.2.** *Let $M$ be a compact subset of a normed space $X$ which is starshaped with respect to $q \in M$. Let $f$ be self affine map of $M$ with $q \in F(f)$. If $T : M \to K(M)$ is multi-valued $\wp_f$ -nonexpansive map and commute with $T$ then $F(T) \neq \phi$. Moreover, if $f$ is continuous, then $F(T) \cap F(\wp_f) \neq \phi$*

*Proof.* Let $\wp_f = \{f^n : n \geq 0\}\{f^0 = I\}$ For each $n$, $f^n$ is affine, $Tf^n = f^nT$ and $f^nq = q$ and $f^n : M \to M$ since $f$ has these properties. Now the proof of the corollary follows from Theorem 2.1. ■

## 3. A Result in Best Approximation Theory

We shall require the following Lemma, proved by Hicks and Humphries ([3], p. 221) for normed spaces.

**Lemma 3.1.** *Let $M$ be a subset of a normed space $X$. Then, for any $x_0 \in X, P_M(x_0) \subseteq \partial M$(the boundary of M).*

**Theorem 3.2.** *Let $X$ be a normed space and $T : X \to K(X)$ be multivalued map such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$ and $f : X \to X$ be single-valued map such that $x_0 \in F(f)$. Let $M$ be a subset of $X$ such that $T(\partial M) \subseteq M$. Suppose $T$ is $\wp_f$ -nonexpansive on $P_M(x_0) \cup \{x_0\}$, $f$ is affine and commutes with $T$ on $P_M(x_0)$. If $P_M(x_0)$ is nonempty, compact and starshaped with respect to $q \in F(f)$ and if, $f(P_M(x_0)) \subseteq P_M(x_0)$, then $P_M(x_0) \cap F(T) \neq \phi$. Further if $f$ is continuous on $P_M(x_0)$, then $P_M(x_0) \cap F(T) \cap F(\wp_f) \neq \phi$.*

*Proof.* Let $D = P_M(x_0)$ and $u \in D$. Then $u \in M$ and

$$\|x_0 - u\| = d(x_0, M).$$

Let $v \in T(u) \subset M$. Then we have

$$\|v - x_0\| \leq H(T(u), T(x_0))$$

Using the $\wp$ -nonexpansiveness of $T$, one gets

$$H(T(u), T(x_0)) \leq \|f^n(u) - f^m(x_0)\|$$

for some $m, n \geq 0$. As $f^m(x_0) = x_0$ for $m \geq 0$, and $f(P_M(x_0)) \subseteq P_M(x_0)$, so for all $n \geq 0$, $f^n(u) \in P_M(x_0)$, and

$$\|v - x_0\| = d(x_0, M).$$

implies that $v \in D$ and thus $T(u) \subset D$. Therefore $T$ carries $D$ into $K(D)$.

Thus, by Corollary 2.2 the conclusion holds.

We observe that if $f = I$ the identity map on $P_M(x_0)$, then $\wp_f$-nonexpansive map is the usual nonexpansive map. ■

In this case we have the following result.

**Corollary 3.3.** *Let $X$ be a normed space and $T : X \to K(X)$ such that $T(x_0) = x_0$ for some $x_0 \in X$. $M$ be a subset of $X$ such that $T(\partial M) \subseteq M$ Let $T$ be nonexpansive map on $P_M(x_0) \cup \{x_0\}$. If $P_M(x_0)$ is nonempty compact, and starshaped, then $P_M(x_0) \cap F(T) \neq \phi$*

**Remark 3.4.**

(1) Theorem 3.2 is an extension of Theorem 4 of Jungck and Sessa [4] which in turn includes the main result of Sahab, Khan and Sessa [11].

(2) Corollary 3.3 extends the main result of Singh [12].

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