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A LOOP GROUP FORMULATION FOR CONSTANT CURVATURE SUBMANIFOLDS OF PSEUDO-EUCLIDEAN SPACE

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Abstract. We give a loop group formulation for the problem of isometric immersions with flat normal bundle of a simply connected pseudo-Riemannian manifold $M^m_{c,r}$, of dimension m, constant sectional curvature $c \neq 0$, and signature r, into the pseudo-Euclidean space $\mathbf{R}^{m+k}_s$, of signature $s \geq r$. In fact these immersions are obtained canonically from the loop group maps corresponding to isometric immersions of the same manifold into a pseudo-Riemannian sphere or hyperbolic space S^{m+k}_s or H^{m+k}_s, which have been known for some time. A simple formula is given for obtaining these immersions from those loop group maps.

1. Introduction

Many special submanifolds can be formulated as maps into loop groups which admit various techniques to produce or analyse solutions (see, for example, [4, 11, 9], and associated references). Concerning the present article, it was shown by Ferus and Pedit [8] that isometric immersions with flat normal bundle, $Q^m_c \to Q^{m+k}_{\tilde{c}}$, between simply connected Riemannian space forms, where $c \neq \tilde{c}$ and $c \neq 0 \neq \tilde{c}$, admit a loop group formulation. They come in natural families parameterised by a spectral parameter λ in either $\mathbf{R}^*$, $i\mathbf{R}^*$ or S^1, and the constant curvature of the immersion varies with λ. They also showed how to produce many local solutions using the AKS theory, when $k \geq m-1$. If $c < \tilde{c}$, then $k = m-1$ is the minimal codimension for even a local isometric immersion, and in this critical codimension the normal bundle is automatically flat.

In [3], it was shown that these loop group maps can also be constructed from flat immersions, using a generalised DPW method. In [2], each map was shown,

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moreover, to correspond to different immersions into different target spaces, depending on whether the map was evaluated for values of the spectral parameter λ in $\mathbf{R}^*$, $i\mathbf{R}^*$ or S^1, showing that various global isometric immersion problems are equivalent.

The immersions obtained for λ in $i\mathbf{R}^*$ and S^1 shrink to a point as λ approaches the degenerate values $\pm i$, and so it is of interest to find another way to interpret the loop group map at these values, to complete the above picture. On the other hand, apart from the case of pseudo-spherical surfaces in $\mathbf{R}^3$, which were studied by M Toda [12, 13] using a different approach, the case $\tilde{c} = 0$ has still not been given a loop group formulation in general. This is therefore also of interest. In this note we will simultaneously solve both of these problems.

We first prove, in Theorem 2.1, that isometric immersions with flat normal bundle from a simply connected pseudo-Riemannian manifold $M^m_{c,r}$, of constant curvature $c \neq 0$ and signature r, to a pseudo-Riemannian sphere or hyperbolic space, S^{m+k}_s or H^{m+k}_s, of signature s, correspond, in a very natural way, to isometric immersions with flat normal bundle of the same manifold, $M^m_{c,r}$, into a pseudo-Euclidean space $\mathbf{R}^{m+k}_s$. We give a simple, coordinate free, proof of this result, which generalises to an arbitrary situation what had, in effect, been shown earlier [1] for the special case that the space being immersed is Riemannian, the codimension is $m-1$, and such that the principle normal curvatures are never equal to $c - \tilde{c}$, which guaranteed the existence of principle coordinates.

In Sections 3 and 4, we look at the loop group formulation mentioned above, which generalises easily to pseudo-Riemannian space forms of arbitrary signatures. Previously, Cieśliński and Aminov [7], had shown locally and for the special case of isometrically immersing the hyperbolic space H^m into the sphere S^{2m-1}, that, by allowing the target space to grow so that the curvature induced on the immersion remains fixed as the parameter λ varies, one obtains, in the limit as the radius of the spherical target space approaches infinity, an isometric immersion of H^m in Euclidean space E^{2n-1}. They also gave a formula of Sym type for the immersion into E^{2n-1}.

We prove, Theorem 4.1, that this result holds globally, and for arbitrary signatures and codimension. The limit as the target space approaches pseudo-Euclidean space corresponds to the above-mentioned degenerate spectral parameter values $\lambda = \pm i$, and the immersion into pseudo-Euclidean space obtained at $\lambda = i$ is the same as the one given in Theorem 2.1. We also prove that, conversely, every isometric immersion with flat normal bundle $M^m_{c,r} \to \mathbf{R}^{m+k}_s$ is associated to one of these loop group maps.

The Sym-type formula (4.4) allows one to obtain the immersion into pseudo-Euclidean space directly from the loop group map. Thus the loop group methods for producing constant curvature immersions into pseudo-Riemannian spheres and hyperbolic spaces, which have already been developed in [8] and [3], automatically and explicitly produce isometric immersions of the same manifold into pseudo-Euclidean space, with the same codimension.

2. Constant Curvature Immersions with Flat Normal Bundle into Pseudo-riemannian Space Forms

Here, we adopt the approach of moving frames, which entails lifting an immersion into a homogeneous space G/H to a frame in G.

Let $\mathbf{R}^n_s$ denote the pseudo-Euclidean space $\mathbf{R}^n$ with metric of signature s, and S^n_s denote the unit sphere in $\mathbf{R}^{n+1}_s$. The isometry group of S^n_s is $SO_s(n+1,\mathbf{R}) = \{A \in GL(n+1,\mathbf{R}) \mid A^t J A = J\}$, where J is a diagonal matrix, $\mathrm{diag}(\epsilon_1, ..., \epsilon_{n+1})$, whose entries ϵ_i are all ± 1, and where s of these entries are negative. Let $ASO_s(n,\mathbf{R})$ denote the group of pseudo-Euclidean motions of $\mathbf{R}^n_s$, that is the subgroup of $GL(n+1,\mathbf{R})$ consisting of matrices of the form

$$\begin{bmatrix} T & a \\ 0 & 1 \end{bmatrix},$$

where $T \in SO_s(n,\mathbf{R})$ and a is a column vector.

Let M be a manifold of dimension m, and $\hat{f} : M \to \mathbf{R}^{m+k}_s$ an immersion such that the pull-back metric has signature r. An *adapted frame* for $\hat{f}$ is a map, $\hat{F} :\to ASO_s(m+k)$, which has the form

$$\hat{F} = \begin{bmatrix} T & \hat{f} \\ 0 & 1 \end{bmatrix},$$

where $T = [\hat{e}_1, ..., \hat{e}_m, \hat{n}_1, ..., \hat{n}_k]$, and the column vectors $\hat{e}_i$ and $\hat{n}_j$ span the tangent and normal spaces respectively to the image of $\hat{f}$. We fix the matrix J which defines $SO_s(n,\mathbf{R})$ to be of the form

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix},$$

where J_1 is $m \times m$ and of signature r, and J_2 is $k \times k$ of signature $s-r$. J_1 encodes the signature of the induced metric.

The Maurer-Cartan form for $\hat{F}$ is the pull-back of the Maurer-Cartan form on $ASO_s(m+k)$, namely,

$$\hat{A} = \hat{F}^{-1}\mathrm{d}\hat{F} = \begin{bmatrix} \hat{\omega} & \hat{\beta} & \hat{\theta} \\ \hat{\alpha} & \hat{\eta} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.1}$$

where, if $\Omega(M)$ denotes the vector space of real valued 1-forms on M, then $\hat{\omega} = [\hat{\omega}^i_j] \in so_r(m) \otimes \Omega(M)$ and $\hat{\eta} = [\hat{\eta}^i_j] \in so_{s-r}(k) \otimes \Omega(M)$ are the connections for the tangent and normal bundles respectively, $\hat{\beta} = [\hat{\beta}^i_j]$ is the second fundamental form, $\hat{\alpha} = -J_2 \hat{\beta}^t J_1$, and the components of $\hat{\theta} = [\hat{\theta}^1, ..., \hat{\theta}^m]^t$ make up the coframe dual to $\hat{e}_i$. To verify this, one uses the fact that $T^{-1} = JT^tJ$ and checks that the following equations are satisfied:

$$\begin{aligned} \mathrm{d}\hat{f} &= \sum_j \hat{\theta}^j \hat{e}_j, \\ \mathrm{d}\hat{e}_i &= \sum_j \hat{\omega}^j_i \hat{e}_j + \sum_j \hat{\alpha}^j_i \hat{n}_j, \\ \mathrm{d}\hat{n}_i &= \sum_j \hat{\beta}^j_i \hat{e}_j + \sum_j \hat{\eta}^j_i \hat{n}_j. \end{aligned}$$

Given such an immersion $\hat{f}$, an adapted frame always exists locally. If X and Y are matrix-valued 1-forms, their wedge product is defined to have components $(X \wedge Y)^i_j := \sum_k X^i_k \wedge Y^k_j$. The Maurer-Cartan form satisfies the Maurer-Cartan equation

$$\mathrm{d}\hat{A} + \hat{A} \wedge \hat{A} = 0. \tag{2.2}$$

Conversely, given a 1-form of the form (2.1), defined on a simply connected subset $U \subset M$, such that the components $\hat{\theta}^i$ are all linearly independent 1-forms, and which satisfies (2.2), then $\hat{A}$ integrates to an adapted frame $\hat{F}$ for an immersion $\hat{f} : U \to \mathbf{R}^{m+k}$.

Let $Q^n_s(\epsilon)$ denote the pseudo-Riemannian sphere S^n_s and hyperbolic space H^n_s, for $\epsilon = 1$ and $\epsilon = -1$ respectively. The pseudo-Riemannian hyperbolic space is defined as $H^n_s := \{x \in \mathbf{R}^{n+1}_{s+1} \mid x^t J x = -1\}$, where J is the metric on $\mathbf{R}^{n+1}_{s+1}$. Now suppose we have an immersion $f : M \to Q^{m+k}_s(\epsilon)$, with induced metric of signature r. Then one has the analogue of the above description for an adapted frame $F : M \to SO_{s+\delta}(m+k+1)$, where $\delta = \frac{1}{2}(1-\epsilon)$, and this time $J = \mathrm{diag}(J_1, J_2, \epsilon)$,

$$F = [e_1, ..., e_m, n_1, ..., n_k, f],$$

and

$$A = F^{-1}\mathrm{d}F = \begin{bmatrix} \omega & \beta & \theta \\ \alpha & \eta & 0 \\ -\epsilon\theta^t J_1 & 0 & 0 \end{bmatrix}. \tag{2.2}$$

Here ω, β, θ, η and α all have the same form and interpretation as their corresponding objects in the affine case above.

In any of the above cases, constant sectional curvature c and flatness of the normal bundle for the immersion are characterised respectively by the following equations and their analogues, replacing an object x with $\hat{x}$ where appropriate:

$$\mathrm{d}\omega + \omega \wedge \omega = c\theta \wedge \theta^t J_1, \tag{2.4}$$

$$\mathrm{d}\eta + \eta \wedge \eta = 0. \tag{2.5}$$

Theorem 2.1. *Let M be a simply connected manifold of dimension m.*

(1) *Let $f : M \to S_s^{m+k}$ be a smooth immersion with flat normal bundle, such that the induced metric has signature r, and constant curvature $c \in (-\infty, 0) \cup (1, \infty)$. Then there is, uniquely up to an action by the isometry group of $\mathbf{R}_s^{m+k}$, a canonically defined immersion with flat normal bundle $\hat{f} : M \to \mathbf{R}_s^{m+k}$, with the same induced metric. The same statement holds with the roles of S_s^{m+k} and $\mathbf{R}_s^{m+k}$ reversed.*

(2) *Statement (1) is also valid substituting H_s^{m+k} for S_s^{m+k} and $(-\infty, -1) \cup (0, \infty)$ for $(-\infty, 0) \cup (1, \infty)$.*

Proof. Let $f : M \to Q_s^{m+k}(\epsilon)$ be the map from either the first or the second case. Fix a base point p of M. Without loss of generality, we assume that $f(p) = [0, ..., 0, 1]^t$. Choose an adapted frame F for f on a simply connected neighbourhood U of p, normalised to the identity at p. This frame is unique up to right multiplication by a smooth map $G : U \to SO_r(m, \mathbf{R}) \times SO_{s-r}(k, \mathbf{R}) \subset SO_{s+\delta}(m + k + 1, \mathbf{R})$, which has the form

$$G = \mathrm{diag}(G_1, G_2, 1), \qquad G(p) = I. \tag{2.6}$$

This corresponds to a change of orthonormal frames for the tangent and normal bundles, while fixing the last column, f.

Let A be the Maurer-Cartan form for F, with components labeled as in the equation (2.3). Now set

$$\hat{A} = \begin{bmatrix} \omega & \frac{i\sqrt{\epsilon c}}{\sqrt{1-\epsilon c}}\beta & \theta \\ \frac{i\sqrt{\epsilon c}}{\sqrt{1-\epsilon c}}\alpha & \eta & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.7}$$

The allowed ranges for c ensure that the factor $\frac{i\sqrt{\epsilon c}}{\sqrt{1-\epsilon c}}$ is real, so $\hat{A}$ is a 1-form with values in the Lie algebra of $ASO_s(m + k, \mathbf{R})$. It is a straightforward computation to verify that A satisfies the integrability condition (2.2) together with the equations (2.4) and (2.5) if and only $\hat{A}$ does also. In the computation, one uses (2.4) to obtain the equivalence of the first diagonal components of the Maurer-Cartan equations, which for A and $\hat{A}$ are, respectively,

$$\mathrm{d}\omega + \omega \wedge \omega + \beta \wedge \alpha - \theta \wedge \epsilon\theta^t J_1 = 0,$$
$$\mathrm{d}\omega + \omega \wedge \omega - \frac{\epsilon c}{1 - \epsilon c}\beta \wedge \alpha = 0,$$

and the equation (2.5) is needed for the second diagonal components. Thus constant curvature and flatness of the normal bundle are essential here.

Now we can integrate $\hat{A}$ on U to get a unique adapted frame $\hat{F}$ for the desired immersion $\hat{f}$, with the initial condition $\hat{F}(p) = I$. The freedom for the choice of adapted frame for $\hat{F}$ is also post-multiplication by a smooth map $\hat{G} : U \to SO_r(m, \mathbf{R}) \times SO_{s-r}(k, \mathbf{R}) \subset ASO_s(m + k, \mathbf{R})$, which has the same form as (2.6), and has exactly the same effect on the Maurer-Cartan form $\hat{A}$, whether it is applied *first* to F, and then constructing $\hat{A}$ as prescribed above, or whether it is applied to $\hat{F}$ *after* the construction from F and then differentiating $\hat{F}$. Hence the map $\hat{f}$ is uniquely determined by our choice of normalisation point p, which corresponds to an action of the isometry group $ASO_s(m + k, \mathbf{R})$.

For the global picture, one observes that for any point q in M, there is a simply connected neighbourhood U_q of q, which contains p, and an adapted frame F_q on U_q, normalised at p. Thus the same procedure can be carried out on U_q. On the overlap, $U_q \cap U$, F and F_q differ only by right multiplication by a matrix of the form (2.6), which has already been taken into account in our construction of $\hat{f}$ described above.

The induced metric for both f and $\hat{f}$ is given in terms of the local coframe, $J_1\theta =: [\theta_1, ..., \theta_m]^t$, by the formula $\theta_1^2 + ... + \theta_m^2$, and thus is identical. Clearly the same argument holds with the roles of the target spaces reversed. ∎

3. The Loop Group Formulation

The loop group formulation for isometric immersions of space forms given in [8] works also for the pseudo-Riemannian case. Here is a brief outline of the formulation of Ferus and Pedit, the only difference here being that we allow non-Riemannian signatures. The computations are easily verified to go through exactly as in [8].

Let M be a simply connected pseudo-Riemannian space form, with constant curvature $c \neq 0$ and of signature r, and fix a base point p of M. Given an isometric immersion with flat normal bundle, f, of M into the pseudo-Riemannian sphere or hyperbolic space, $Q_s^{m+k}(\epsilon)$, and an adapted frame F, one inserts a complex parameter λ into the Maurer-Cartan form (2.2), to obtain a family of 1-forms,

$$(3.1) \quad A_\lambda = \begin{bmatrix} \omega & \frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda - \lambda^{-1})\beta & \frac{\sqrt{\epsilon c}}{2}(\lambda + \lambda^{-1})\theta \\ \frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda - \lambda^{-1})\alpha & \eta & 0 \\ -\epsilon\frac{\sqrt{\epsilon c}}{2}(\lambda + \lambda^{-1})\theta^t J_1 & 0 & 0 \end{bmatrix}.$$

The original Maurer-Cartan form is obtained at $\lambda_0 = \frac{1}{\sqrt{\epsilon c}}(1 + \sqrt{1 - \epsilon c})$. The assumptions that f has constant curvature and flat normal bundle are equivalent to the assumption that A_λ satisfies the Maurer-Cartan equation (2.2) for all λ in the punctured plane $\mathbf{C}^*$. Depending on the original curvature value c, A_λ is real for λ in one of $i\mathbf{R}^*$, $\mathbf{R}^*$, or S^1, and integrates to an adapted frame $F_\lambda = [e_1^\lambda, ..., e_m^\lambda, n_1^\lambda, ..., n_k^\lambda, f_\lambda]$ for a family of immersions with flat normal bundle, with constant curvature in one of the corresponding ranges $(-\infty, 0)$, $(0, 1)$, or $(1, \infty)$, for the case $\epsilon = 1$, and their reflections about 0 for the case $\epsilon = -1$. The family F_λ is unique with the normalisation $F_\lambda(p) = I$. As mentioned in the introduction, there are several methods for producing the loop group maps F_λ, for $k \geq m - 1$, using techniques from integrable systems.

Remark 3.1. An important point for our discussion in the next section is that, even though a single global adapted frame may not exist for f, one can nevertheless show, [2], that the family f_λ is well defined globally on M.

The coframe of the immersion f_λ, is given, from (3.1), by

$$(3.2) \qquad \theta_\lambda = \frac{\sqrt{\epsilon c}}{2}(\lambda + \lambda^{-1})\theta,$$

and the curvature tensor turns out to be given by the expression

$$\begin{aligned} d\omega + \omega \wedge \omega &= c\theta \wedge \theta^t J_1 \\ &= \frac{4\epsilon}{(\lambda + \lambda^{-1})^2}\theta_\lambda \wedge \theta_\lambda^t J_1. \end{aligned}$$

Thus, the constant curvature, which varies with λ, is given by the formula

$$c_\lambda = \frac{4\epsilon}{(\lambda + \lambda^{-1})^2}.$$

Because the original map, $f = f_{\lambda_0}$, is an immersion, it follows from (3.2) that f_λ is an immersion for all $\lambda \neq \pm i$. However, $f_{\pm i}$ maps M to a single point, because its coframe is zero. If we take the normalisation $F_\lambda(p) = I$, for some $p \in M$, then

$$f_{\pm i} = [0, ..., 0, 1]^t.$$

4. Interpretation of the Loop Group Map at $\lambda = \pm i$

We seek an interpretation of the map F_λ at $\lambda = \pm i$. Our argument is similar to that in [7]; the loop group setting is what makes our approach work globally. The idea of "blowing up" the ambient space has been used in various places before, for example in [5, 6, 10].

If we scale the space that we are immersing into, so that the curvature of the immersion is unchanged, instead of varying with λ, then, as λ approaches $\pm i$, we must have the target space approaching flat space, and so, in the limit, we hope to have a constant curvature immersion $\tilde{f} : M \to \mathbf{R}_s^{m+k}$.

To carry this out, set

$$\tilde{f}_\lambda := \frac{2}{\sqrt{\epsilon c}(\lambda + \lambda^{-1})} f_\lambda. \tag{4.1}$$

As with the expression (3.1), it is easy to verify that $\tilde{f}_\lambda$ is real for values of λ in the appropriate ranges corresponding to c and ϵ, described above. Thus $\tilde{f}_\lambda$ is a map from M into $Q_s^{m+k}(\epsilon|R|) := \{x \in \mathbf{R}_{s+\delta}^{m+k+1} \mid x^t J x = \epsilon R^2\}$, where $R := \frac{2}{\sqrt{\epsilon c}(\lambda+\lambda^{-1})}$. Now the image of $\tilde{f}$ can be identified with the image of f if we scale the ambient space $\mathbf{R}_{s+\delta}^{m+k+1}$ by a factor of R. This has the effect of scaling the metric by a constant conformal factor of R^2, and the curvature by $\frac{1}{R^2}$. Thus $\tilde{f}_\lambda$ has constant curvature

$$c_\lambda \frac{\epsilon c(\lambda + \lambda^{-1})^2}{4} = c.$$

Because all we have done is scale the target space, an adapted frame, $\tilde{F}_\lambda \in ASO_{s+\delta}(m+k+1, \mathbf{R})$, for $\tilde{f}_\lambda : M \to \mathbf{R}_{s+\delta}^{m+k+1}$, is

$$\tilde{F}_\lambda = \begin{bmatrix} e_1^\lambda & \dots & e_m^\lambda & n_1^\lambda & \dots & n_k^\lambda & f_\lambda & \tilde{f}_\lambda \\ & 0 & & & 0 & & 0 & 1 \end{bmatrix},$$

where e_i^λ and n_i^λ are the same as in the unscaled frame F_λ. The unscaled f_λ is now the $(k+1)$'st unit normal vector. The Maurer-Cartan form for $\tilde{F}_\lambda$ is

$$\tilde{A}_\lambda = \begin{bmatrix} \omega & \frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda - \lambda^{-1})\beta & \frac{\sqrt{\epsilon c}}{2}(\lambda + \lambda^{-1})\theta & \theta \\ \frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda - \lambda^{-1})\alpha & \eta & 0 & 0 \\ -\epsilon\frac{\sqrt{\epsilon c}}{2}(\lambda + \lambda^{-1})\theta^t J_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.2}$$

The frame $\tilde{F}_\lambda$ is obtained by integrating this with the initial condition

$$\tilde{F}(p) = \begin{bmatrix} I_{m+k+1} & [0, \dots, 0, \frac{2}{\sqrt{\epsilon c}(\lambda+\lambda^{-1})}]^t \\ 0 & 1 \end{bmatrix}.$$

Now $\tilde{F}_\lambda$ is not defined at $\lambda = \pm i$, because $\tilde{f}_\lambda$ is not. However, $\tilde{A}_\lambda$ *is* defined at $\lambda = \pm i$, and at that point reduces to

$$\tilde{A}_{\pm i} = \begin{bmatrix} \omega & \frac{(\pm i)\sqrt{\epsilon c}}{\sqrt{1-\epsilon c}}\beta & 0 & \theta \\ \frac{(\pm i)\sqrt{\epsilon c}}{\sqrt{1-\epsilon c}}\alpha & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we apply a translation to our initial condition for $\tilde{f}_\lambda$, and integrate $\tilde{A}_\lambda$ with the initial condition $\hat{F}(p) = I_{m+k+2}$, we get an adapted frame $\hat{F}_\lambda$ which satisfies

$$\hat{F}_{\pm i} = \begin{bmatrix} e_1^{\pm i} & \dots & e_m^{\pm i} & n_1^{\pm i} & \dots & n_k^{\pm i} & [0, \dots, 0, 1]^t & \hat{f}_{\pm i} \\ & 0 & & & 0 & & 0 & 1 \end{bmatrix},$$

and we see that $\hat{f}_{\pm i}$ maps into the hyperplane perpendicular to the vector $[0, \dots, 0, 1]^t$. In other words, it is an immersion into $\mathbf{R}_s^{n+k} \subset \mathbf{R}_{s+\delta}^{n+k+1}$, with flat normal bundle and constant curvature c. Comparing the Maurer-Cartan form of $\hat{F}_{\pm i}$ with (2.7), we have shown that the interpretation of the loop group map F_λ at i is just the immersion into pseudo-Euclidean space obtained from Theorem 2.1.

Finally, since $\tilde{f}_\lambda = \hat{f}_\lambda + [0, \dots 0, \frac{2}{\sqrt{\epsilon c}(\lambda+\lambda^{-1})}]^t$, we obtain, using (4.1), the formula

$$\hat{f}_\lambda = \frac{2}{\sqrt{\epsilon c}(\lambda + \lambda^{-1})}(f_\lambda - [0, \dots, 0, 1]^t). \tag{4.3}$$

Setting $\mu := \lambda + \lambda^{-1}$ and $g(\mu) := f_{\lambda(\mu)}$, we obtain a Sym-type formula for $\hat{f}_{\pm i}$:

$$\begin{aligned} \hat{f}_{\pm i} &= \frac{2}{\sqrt{\epsilon c}} \lim_{\mu \to 0} \frac{g(\mu) - g(0)}{\mu} \\ &= \frac{2}{\sqrt{\epsilon c}} \frac{\partial \lambda}{\partial \mu} \frac{\partial}{\partial \lambda} f_\lambda \Big|_{\lambda = \pm i} \\ &= \frac{1}{\sqrt{\epsilon c}} \frac{\partial}{\partial \lambda} f_\lambda \Big|_{\lambda = \pm i}. \end{aligned}$$

This formula is independent of the choice of adapted frame, F, and is therefore, by Remark 3.1, valid globally.

Conversely, given an isometric immersion with flat normal bundle $f : M \to \mathbf{R}_s^{m+k}$, it follows from the converse part of Theorem 2.1 that there are unique loop group maps F_λ, normalised at p, with Maurer-Cartan forms of the form (3.1), corresponding to f. We summarise this discussion as:

Theorem 4.1. *Let $f : M_{c,r}^m \to Q_s^{m+k}(\epsilon)$ be an isometric immersion with flat normal bundle of a simply connected m-dimensional pseudo-Riemannian manifold*

$M^m_{c,r}$, of signature r, constant curvature c and with base point p, into the pseudo-Riemannian sphere or hyperbolic space, for $\epsilon = 1$, or $\epsilon = -1$ respectively. Suppose that $c \in (-\infty, 0) \cup (1, \infty)$ if $\epsilon = 1$, and $c \in (-\infty, -1) \cup (0, \infty)$ if $\epsilon = -1$.

Let f_λ be the associated family of immersions given by the last column of the frame obtained by integrating the 1-form defined by (3.1), normalised at p. Then the corresponding isometric immersion $\hat{f} : M^m_{c,r} \to \mathbf{R}^{m+k}_s$ from Theorem 2.1 is given by the formula:

$$\hat{f} = \frac{1}{\sqrt{\epsilon c}} \pi_{m+k} \left\{ \frac{\partial}{\partial \lambda} f_\lambda \big|_{\lambda = i} \right\}, \tag{4.4}$$

where π_{m+k} is the projection onto the first $m+k$ coordinates.

Conversely, every isometric immersion with flat normal bundle $f : M^m_{c,r} \to \mathbf{R}^{m+k}_{s+\delta}$ is obtained in this way.

Remark 4.2. The formula (4.3) for $\hat{f}_\lambda$ gives a continuous deformation of the original immersion $f - [0, ..., 0, 1]^t$ into (the displaced) $Q^{m+k}_s(\epsilon)$, obtained at $\lambda = \lambda_0$, through to the immersion into $\mathbf{R}^{m+k}_s \subset \mathbf{R}^{m+k+1}_s$, obtained at $\lambda = \pm i$.

Example 4.3. As a simple test case, here is an example of a family, from the loop group construction described above, of immersions into H^3_1 of the de Sitter spaces $S^2_{c_\lambda,1}$ with constant sectional curvature $c_\lambda \in (0, \infty)$, for values of λ in $i\mathbf{R}^* \setminus \{\pm i\}$:

$$f_\lambda(u, v) = [-ia \cosh u \sin v,\ -ia \sinh u,\ ab(1 - \cos v \cosh u),\ a^2 \cos v \cosh u - b^2]^t,$$

$$F_\lambda = \begin{bmatrix} \cos v & \sin v \sinh u & -ib \cosh u \sin v & -ia \cosh u \sin v \\ 0 & \cosh u & -ib \sinh u & -ia \sinh u \\ ib \sin v & -ib \cos v \sinh u & a^2 - b^2 \cos v \cosh u & ab(1 - \cos v \cosh u) \\ -ia \sin v & ia \cos v \sinh u & ab(\cos v \cosh u - 1) & a^2 \cos v \cosh u - b^2 \end{bmatrix},$$

$$a := \frac{1}{2}(\lambda + \lambda^{-1}) \qquad b := \frac{1}{2}(\lambda - \lambda^{-1}).$$

For any $c \in (0, \infty)$, we apply the above formula at $\lambda = i$ to obtain

$$\hat{f}(u, v) = \frac{1}{\sqrt{c}}[-\cosh u \sin v,\ -\sinh u,\ 1 - \cos v \cosh u]^t,$$

an embedding of the de Sitter space $S^2_{c,1}$ of constant curvature c into $\mathbf{R}^3_1$.

Analogous test cases with other signatures can be similarly constructed. A Riemannian example can be computed using the example in [2].

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