TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 7, pp. 1691-1705, October 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

NEW ACCURACY CRITERIA FOR MODIFIED APPROXIMATE PROXIMAL POINT ALGORITHMS IN HILBERT SPACES

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Abstract. This paper proposes a modified approximate proximal point algorithm to solve the problem of finding zeros of a maximal monotone operator in a Hilbert space. New accuracy criteria are imposed. Weak and strong convergence results are established.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $T : H \longrightarrow 2^H$ be a maximal monotone operator. The problem of finding an element $x \in H$ such that $0 \in T(x)$ is very important in the area of optimization and related fields. For example, if T is the subdifferential ∂f of a proper lower semicontinuous convex functional $f : H \to (-\infty, \infty]$, then T is a maximal monotone operator and the inclusion $0 \in \partial f(x)$ is reduced to the following optimization problem:

$$f(x) = \min\{f(z) : z \in H\}.$$

One of the most efficient and enforceable methods for solving $0 \in T(x)$ is the proximal point algorithm which, staring with any vector $x_0 \in H$, iteratively updates x_{n+1} conforming to the following recursion:

$$x_n \in x_{n+1} + c_n T(x_{n+1}) \tag{1}$$

Received January 20, 2007, accepted May 30, 2007.
Communicated by M. H. Shih.
2000 *Mathematics Subject Classification*: 47J25, 47H05.
Key words and phrases: Proximal point algorithms, Monotone operators, Inexact methods.
[1]This research was partially supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China and the Dawn Program Foundation in Shanghai.
[2]This research was partially supported by grant from the National Science Council of Taiwan.
[3]This research was partially supported by grant from the National Science Council of Taiwan.
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where $\{c_n\}_{n=0}^{\infty} \subset [c, \infty), c > 0,$ is a sequence of scalars. However, as pointed out in [13], the ideal form of the method is often impractical, since in many cases solving problem (1) exactly is either impossible or as difficult as solving the original problem $0 \in T(x)$. On the other hand, there seems to be little justification of the effort required to solve the problem accurately when the iterate is far away from the solution point. In [20], Rockafellar gave an inexact variant of the method:

$$x_n + e_{n+1} \in x_{n+1} + c_n T(x_{n+1}) \tag{2}$$

where $\{e_{n+1}\}$ is regarded as an error sequence. This method is called an inexact proximal point algorithm. Rockafellar [20] proved that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \to z \in R^n$ with $0 \in T(z)$.

Because of its relaxed accuracy requirement, the inexact proximal point algorithm is more practical than the exact one. Thus it has been studied widely and various forms of the method have been developed; see, e.g., [3, 4, 8, 10-12, 17-19, 25]. In most of these papers, the condition that the error term being summable is essential for the convergence of the method. In [20] and some sequel papers (e.g., [5]) the accuracy criterion is

$$\|e_{n+1}\| \leq \eta_n \|x_{n+1} - x_n\| \quad \text{with} \quad \sum_{n=0}^{\infty} \eta_n < \infty. \tag{3}$$

Recently, Eckstein [13] extended the method to Bregman-function-based inexact proximal methods and proved that the sequence $\{x_n\}$ generated by the algorithm converges to a root of T under the conditions

$$\sum_{n=1}^{\infty} \|e_n\| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \langle e_n, x_n \rangle \text{ exists and is finite} \tag{4}$$

(see Eqs. (18) and (19) in [13]). Condition (4) is an assumption on the whole generated sequence $\{x_n\}$ and the error term sequence $\{e_n\}$, and thus seems to be slightly stronger, but it can be checked and enforced in practice more easily than those that existed earlier. On the other hand, as in He [15], Han and He [14] gave another inexact criterion

$$\|e_{n+1}\| \leq \eta_n \|x_{n+1} - x_n\| \quad \text{with} \quad \sum_{n=0}^{\infty} {\eta_n}^2 < \infty, \tag{5}$$

to recursion (2) for solving the equation $0 \in T(x)$ in R^n and studied the resulting convergence properties. It is clear that the accruacy criterion (5) is weaker than the one in [20] (see (3)). It is remarkable that da Silva e Silva et al. [9] and Solodov and Svaiter [21-23] recently proposed some new accuracy criteria for proximal point algorithms. Their criteria, rather than requiring inequality (5), require only $\sup_{n\geq 0}\eta_n < 1$. However, in [21-23], this comes at the cost of adding an additional projection or "extragradient" step to the algorithm, and the applicable portion of [9] applies only to convex minimization. Let H be a real Hilbert space and T be a maximal monotone operator on H. Throughout this paper, we assume that the equation $0 \in T(x)$ has a solution and let S be the solution set:

$$S = \{x \in H : 0 \in T(x)\} = T^{-1}(0).$$

Then S is a nonempty closed convex subset of H and thus the projection P_s from H onto S is well defined. Recall [2,16] that for any given $c > 0, J_c = (I + cT)^{-1}$ is called the resolvent operator associated with T, where I denotes the identity mapping on H. It is known that the mapping T is maximal monotone if and only if the resolvent operator J_c is defined everywhere on the space for each $c > 0$. It is also known [7] that T is maximal monotone if and only if T is monotone and $(I + cT)(D(T)) = H$ for each $c > 0$ (equivalently, T is monotone and $(I+T)(D(T)) = H$). Furthermore the resolvent operator J_c is single-valued and nonexpansive, that is, for all $u, v \in H$,

$$\|J_c(u) - J_c(v)\| \leq \|u - v\|.$$

In 2002, Xu [26] introduced and studied the following modified proximal point algorithms for solving the equation $0 \in T(x)$ in a real Hilbert space H.

Algorithm 1.1. [26, Algorithm 5.1].

(i) $x_0 \in H$ is chosen arbitrarily.

(ii) Choose a regularization parameter $c_n > 0$ with error $e_n \in H$ and relaxation parameter $\alpha_n \in [0, 1]$ and compute

$$y_n := (I + c_nT)^{-1}(x_n) + e_n.$$

(iii) Compute the $(n + 1)$th iterate:

$$x_{n+1} := \alpha_n x_0 + (1 - \alpha_n)y_n.$$

Algorithm 1.2. [26, Algorithm 5.2]. (Relaxed proximal point algorithm)

(i) Select $x_0 \in H$ arbitrarily.

(ii) Choose a regularization parameter $c_n > 0$ with error $e_n \in H$ and compute

$$y_n := (I + c_nT)^{-1}(x_n) + e_n.$$

(iii) Select a relaxation parameter $\alpha_n \in [0, 1]$ and compute the $(n+1)$th iterate:

$$x_{n+1} := \alpha_n x_n + (1 - \alpha_n) y_n.$$

Moreover, Xu [26] gave the following convergence criteria for the above algorithms.

Theorem 1.1. [26, Theorem 5.1]. *Let $\{x_n\}$ be generated by Algorithm 1.1. Assume that (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$; (iii) $\lim_{n\to\infty} c_n = \infty$; (iv) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.*
Then $\{x_n\}$ converges strongly to $P_S(x_0)$.

Theorem 1.2. [26, Theorem 5.2]. *Let $\{x_n\}$be the sequence generated by Algorithm 1.2. Assume that (i) $\{\alpha_n\}$ is bounded away from 1, namely $0 \le \alpha_n \le 1 - \delta$ for some $\delta \in (0, 1)$; (ii) $\lim_{n\to\infty} c_n = \infty$; (iii) $\sum_{n=0}^{\infty} \|e_n\| < \infty$. Then $\{x_n\}$ converges weakly to a point in S.*

In this paper, motivated and inspired by Xu [26], we propose modified approximate proximal point algorithms for finding approximate solutions of zeros of a maximal monotone operator in real Hilbert spaces. We introduce new accuracy criteria for these modified approximate proximal point algorithms. Under the suggested enforceable accuracy restrictions which are easy to verify, the convergence results of these modified approximate proximal point algorithms are established. In particular, results in this paper improve and extend corresponding results in [14] which were in finite dimensional space setting.

2. Algorithms and Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. A set $T \subset H \times H$ with the property

$$(x, y), (x', y') \in T \Longrightarrow \langle x - x', y - y' \rangle \ge 0,$$

is called a monotone operator on H. T is maximal if (considered as a graph) it is not strictly contained in any other monotone operator on H. In this paper we consider the central problem associated with T : Find $z \in H$ such that $0 \in T(z)$, i.e., to find one of the roots of T. Here $T(\cdot)$ is defined as $T(x) = \{y \in H : (x, y) \in T\}$.

At first, we summarize some basic properties and related definitions of the monotone operator T. As is the custom, we regard T as the graph of a point-to-set mapping. The domain of the mapping T is

$$D(T) = \{x \in H : \exists y \in H, (x, y) \in T\} = \{x \in H : T(x) \neq \emptyset\}.$$

We say that T has full domain if $D(T) = H$. The range or image of T is

$$R(T) = \{y \in H : \exists x \in H, (x, y) \in T\}.$$

For all real numbers c, we let

$$cT = \{(x, cy) : (x, y) \in T\},$$

and for all operators $A, B \subset H \times H$, we defined $A + B$ via

$$A + B = \{(x, y + z) : (x, y) \in A, (x, z) \in B\}.$$

Motivated and inspired by Xu [26, Algorithms 5.1 and 5.2], we give the modified approximate proximal point algorithms for computing approximate solutions to the equation $0 \in T(x)$.

Algorithm 2.1. (Relaxed proximal point algorithm).

(i) $x_0 \in H$ is chosen arbitrarily.

(ii) Choose a regularization parameter $c_n > 0$ with error $e_{n+1} \in H$ and compute

$$\tilde{x}_{n+1} := (I + c_n T)^{-1}(x_n + e_{n+1}). \tag{6}$$

(iii) Select a relaxation parameter $\alpha_n \in [0, 1]$ and compute the $(n + 1)$th iterate:

$$x_{n+1} := \alpha_n x_n + (1 - \alpha_n)\tilde{x}_{n+1}. \tag{7}$$

Algorithm 2.2.

(i) Select $x_0 \in H$ arbitrarily.

(ii) Choose a regularization parameter $c_n > 0$ with error $e_{n+1} \in H$ and compute

$$\tilde{x}_{n+1} := (I + c_n T)^{-1}(x_n + e_{n+1}).$$

(iii) Select a relaxation parameter $\alpha_n \in [0, 1]$ and compute the $(n + 1)$th iterate:

$$x_{n+1} := \alpha_n x_0 + (1 - \alpha_n)\tilde{x}_{n+1}. \tag{8}$$

Remark 2.1. Obviously, for the above (6) we have

$$\begin{aligned} \tilde{x}_{n+1} = J_{c_n}(x_n + e_{n+1}) &\Leftrightarrow x_n + e_{n+1} \in [I + c_n T](\tilde{x}_{n+1}) \\ &\Leftrightarrow \frac{1}{c_n}(x_n - \tilde{x}_{n+1} + e_{n+1}) \in T(\tilde{x}_{n+1}). \end{aligned}$$

The following lemma can be proved by similar argument as that in [13] and hence the proof will be omitted.

Lemma 2.1. *Let $\{x_n\}_{n=0}^{\infty}$, $\{\tilde{x}_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ be sequences that conform to recursion (6). Then for any $x^* \in S$ (root of T) and all $n \geq 0$ we have*

$$\langle x_n - \tilde{x}_{n+1} + e_{n+1}, \tilde{x}_{n+1} - x^* \rangle \geq 0 \tag{9}$$

and

$$\|\tilde{x}_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \|\tilde{x}_{n+1} - x_n\|^2 + 2\langle e_{n+1}, \tilde{x}_{n+1} - x^* \rangle. \tag{10}$$

Lemma 2.2. (see [24, Lemma 1, p. 303].) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality:*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3. (see [26, Lemma 2.5, p. 243]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality:

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,or equivalently,$\prod_{n=0}^{\infty}(1 - \alpha_n) = 0$;

(ii) $\text{limsup}_{n\to\infty}\beta_n \leq 0$;

(iii) $\gamma_n \geq 0 (\forall n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

For a nonempty closed convex subset $K \subset H$ and a vector $x \in H$, the orthogonal projection of x onto K, i.e., $\arg\min\{\|y - x\| : y \in K\}$, is denoted by $P_K(x)$. We state some well-known properties of the projection operator which will be used in the sequel; see [27].

Lemma 2.4. *Let K be a nonempty closed convex subset of H. For any $x, y \in H$ and $z \in K$, the following statements hold:*

(*i*) $\langle P_K(x) - x, z - P_K(x) \rangle \geq 0$.

(*ii*) $\|P_K(x) - P_K(y)\|^2 \leq \|x - y\|^2 - \|P_K(x) - x + y - P_K(y)\|^2$.

Remark 2.2. As a matter of fact, (i) of Lemma 2.4 provides also a sufficient condition for a vector u to be the projection of the vector x; i.e., $u = P_K(x)$ if and only if

$$\langle u - x, z - u\rangle \geq 0, \quad z \in K.$$

Throughout the rest of the paper, we shall use the following notation: for a given sequence $\{x_n\}$, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \{x \in H : w - \lim_{j\to\infty} x_{n_j} = x \text{ for some } \{n_j\} \subset \{n\}, n_j \uparrow \infty\}$$

where $w\text{-}\lim_{j\to\infty} x_{n_j} = x$ means the weak convergence of $\{x_{n_j}\}$ to x, i.e., $x_{n_j} \to x$ weakly.

3. Main Results

Now we begin to investigate the convergence of Algorithms 2.1 and 2.2 under the condition

$$(11) \qquad \|e_{n+1}\| \leq \eta_n \|\tilde{x}_{n+1} - x_n\| \quad \text{with} \quad \sum_{n=0}^{\infty} \eta_n^2 < \infty.$$

Note that in the exact proximal point algorithm (1), x_n is a root of T if and only if $x_{n+1} = x_n$. Hence, roughly speaking, we can see the distance $\|\tilde{x}_{n+1} - x_n\|$ as an "error bound" which measures how much x_n fails to be in the roots set of T. If $\|\tilde{x}_{n+1} - x_n\|$ is small enough, it follows from Eq.(6) that $\tilde{x}_{n+1}$ is an approximate solution of the original problem $0 \in T(x)$. Thus there is no doubt that under the mild restrictions on $\{\alpha_n\}$ and $\{c_n\}$, the x_{n+1} generated by (7) or (8) is naturally an acceptable approximate solution of the original problem $0 \in T(x)$.

Theorem 3.1. *Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Algorithm 2.1.Assume that condition (11) is satisfied and that*

(i) $\{\alpha_n\}$ is bounded away form 1, namely $0 \leq \alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1)$;
(ii) $\{c_n\}_{n=0}^{\infty} \subset [c, \infty)$ for some $c > 0$.
Then the following statements are valid:

[a] there exists an integer $N_0 \geq 0$ such that for all $n \geq N_0$

$$\|x_{n+1} - x^*\|^2 \leq (1 + \frac{2\eta_n^2}{1 - 2\eta_n^2})\|x_n - x^*\|^2 - \frac{\delta}{2}\|\tilde{x}_{n+1} - x_n\|^2, \quad \forall x^* \in S;$$

[b] $\lim_{n\to\infty} \|\tilde{x}_{n+1} - x_n\| = 0$;

[c] *$\{x_n\}$ converges weakly to a point in S.*

Proof. We divide the proof into four steps where Steps 1 and 2 are similar to [14, Theorem 1].

Step 1. Let x^* be any root of T. For $\eta_n > 0$, using the Cauchy-Schwartz inequality we have

$$2\langle e_{n+1}, \tilde{x}_{n+1} - x^*\rangle \leq \frac{1}{2\eta_n^2}\|e_{n+1}\|^2 + 2\eta_n^2\|\tilde{x}_{n+1} - x^*\|^2. \tag{12}$$

Since $\eta_n \to 0$, there exists $N_0 \geq 0$ such that for all $n \geq N_0, 1 - 2\eta_n^2 > 0$. Substituting (12) in (10) we obtain

$$\begin{aligned} \|\tilde{x}_{n+1} - x^*\|^2 &\leq (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2 - \frac{1}{2(1-2\eta_n^2)}\|\tilde{x}_{n+1} - x_n\|^2 \\ &\leq (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2 - \frac{1}{2}\|\tilde{x}_{n+1} - x_n\|^2. \end{aligned} \tag{13}$$

Note that for all x, y in H and $0 \leq \lambda \leq 1$, the following identity holds:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

Thus, it follows from (7) and (13) that

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 = \|\alpha_n(x_n - x^*) + (1-\alpha_n)(\tilde{x}_{n+1} - x^*)\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|\tilde{x}_{n+1} - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[(1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2 - \frac{1}{2}\|\tilde{x}_{n+1} - x_n\|^2] \\ &\leq (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2 - \frac{1}{2}(1-\alpha_n)\|\tilde{x}_{n+1} - x_n\|^2. \end{aligned}$$

Since $0 \leq \alpha_n \leq 1 - \delta$ for some $\delta \in (0,1)$, $\frac{1}{2}(1-\alpha_n) \geq \frac{1}{2}\delta$. Hence, we get

$$\|x_{n+1} - x^*\|^2 \leq (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2 - \frac{\delta}{2}\|\tilde{x}_{n+1} - x_n\|^2, \quad \forall n \geq N_0. \tag{14}$$

Step 2. It follows from (14) that

$$\|x_{n+1} - x^*\|^2 \leq (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2, \quad \forall n \geq N_0. \tag{15}$$

Since $\sum_{n=0}^{\infty} \eta_n^2 < \infty$, we get

$$C_0 := \sum_{n=N_0}^{\infty} \frac{2\eta_n^2}{1-2\eta_n^2} < \infty \quad \text{and} \quad C_1 := \prod_{n=N_0}^{\infty} (1 + \frac{2\eta_n^2}{1-2\eta_n^2}) < \infty,$$

and thus $\{x_n\}$ is bounded. Set $M = \sup_{n\geq 0} \|x_n - x^*\|$. Then from (15) we get

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \frac{2\eta_n^2}{1-2\eta_n^2}M^2, \quad \forall n \geq N_0.$$

Hence, by Lemma 2.2 we conclude that $\lim_{n\to\infty} \|x_n - x^*\|$exists. Also from (14) we obtain

$$\frac{\delta}{2}\|\tilde{x}_{n+1} - x_n\|^2 \leq (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Therefore, we deduce that $\lim_{n\to\infty} \|\tilde{x}_{n+1} - x_n\| = 0$.

Step 3. We claim that $\omega_w(x_n) \subset S$. Indeed, let $z \in \omega_w(x_n)$ and take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that w-$\lim_{j\to\infty} x_{n_j} = z$. Since $\|\tilde{x}_{n+1} - x_n\| \to 0$ as $n \to \infty$, we get

$$w - \lim_{j\to\infty} \tilde{x}_{n_j+1} = z.$$

Observe that

$$\begin{aligned}
\|\tfrac{1}{c_n}(x_n - \tilde{x}_{n+1} + e_{n+1})\| &\leq \frac{1}{c_n}\|\tilde{x}_{n+1} - x_n\| + \frac{1}{c_n}\|e_{n+1}\| \\
&\leq \frac{1}{c}\|\tilde{x}_{n+1} - x_n\| + \frac{1}{c}\eta_n\|\tilde{x}_{n+1} - x_n\| \\
&= \frac{1}{c}(1+\eta_n)\|\tilde{x}_{n+1} - x_n\| \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

and

$$\frac{1}{c_{n_j}}(x_{n_j} - \tilde{x}_{n_j+1} + e_{n_j+1}) \in T(\tilde{x}_{n_j+1}).$$

Taking the limit as $j \to \infty$, we infer by the maximality of T (hence T is demiclosed) that $0 \in T(z)$; that is, $z \in S$.

Step 4. We claim that $\{x_n\}$ converges weakly to some $z \in S$. Indeed, it suffices to show that $\omega_w(x_n)$ consists of one point. Let $z_1, z_2 \in \omega_w(x_n)$ and let

$$w - \lim_{i\to\infty} x_{n_i} = z_1 \quad \text{and} \quad w - \lim_{j\to\infty} x_{m_j} = z_2.$$

We deduce by Step 2 that

$$\begin{aligned}
&\lim_{n\to\infty} \|x_n - z_2\|^2 \\
&= \lim_{i\to\infty} \|x_{n_i} - z_2\|^2 = \lim_{i\to\infty} \|x_{n_i} - z_1 + z_1 - z_2\|^2 \\
&= \lim_{i\to\infty} (\|x_{n_i} - z_1\|^2 + 2\langle x_{n_i} - z_1, z_1 - z_2\rangle + \|z_1 - z_2\|^2) \\
&= \lim_{n\to\infty} \|x_n - z_1\|^2 + \|z_1 - z_2\|^2.
\end{aligned} \tag{16}$$

Interchanging z_1 and z_2, we obtain

$$\lim_{n\to\infty}\|x_n - z_1\|^2 = \lim_{n\to\infty}\|x_n - z_2\|^2 + \|z_2 - z_1\|^2. \tag{17}$$

Adding up (16) and (17) we obtain $z_1 = z_2$. ■

Corollary 3.1. *Let T be a maximal monotone operator on R^n and $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Algorithm 2.1. Assume that condition (11) is satisfied and that*

(i) *$\{\alpha_n\}$ is bounded away from 1, namely $0 \le \alpha_n \le 1-\delta$ for some $\delta \in (0,1)$;*

(ii) *$\{c_n\}_{n=0}^{\infty} \subset [c,\infty)$ for some $c > 0$.*

Then $\{x_n\}$ converges to a point in S.

Corollary 3.2. [14, Theorem 2]. *Let T be a maximal monotone operator on R^n and $\{x_n\}$ be the sequence generated by the inexact proximal point Algorithm* (2) *under the proposed accuracy criterion* (5) *where $\{c_n\}_{n=0}^{\infty} \subset [c,\infty)\,, c > 0$ is a sequence of scalars. Then $\{x_n\}$ converges to some x_∞ with $0 \in T(x_\infty)$.*

Theorem 3.2. *Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Algorithm 2.2. Assume that condition (11) is satisfied and that*

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty} c_n = \infty$.

Then $\{x_n\}$ converges strongly to $P_s(x_0)$.

Proof. We divide the proof into three steps.

Step 1. We claim that $\{x_n\}$ is bounded. Indeed, let x^* be any root of T. As in Step 1 of the proof of Theorem 3.1, we can infer that there exists $N_0 \ge 0$ such that for all $n \ge N_0, 1 - 2\eta_n^2 > 0$ and

$$\|\tilde{x}_{n+1} - x^*\|^2 \le (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2 - \frac{1}{2}\|\tilde{x}_{n+1} - x_n\|^2.$$

This immediately implies that

$$\|\tilde{x}_{n+1} - x^*\|^2 \le (1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|^2. \tag{18a}$$

Note that $\sqrt{1+t} \le 1 + \frac{1}{2}t,\ \forall t \in [0,\infty)$. Thus, we derive for all $n \ge N_0$,

$$\|\tilde{x}_{n+1} - x^*\| \le (1 + \frac{\eta_n^2}{1-2\eta_n^2})\|x_n - x^*\|. \tag{18b}$$

Now, we prove that

$$\|x_{N_0+n+1} - x^*\| \le \prod_{k=0}^{n}(1+\frac{\eta_{N_o+k}^2}{1-2\eta_{N_0+k}^2})\|x_0 - x^*\| + \prod_{k=0}^{n}(1+\frac{\eta_{N_0+k}^2}{1-2\eta_{N_0+k}^2})\|x_{N_0} - x^*\|. \tag{19}$$

When $n = 0$, from (8) and (18b) we get

$$\begin{aligned}\|x_{N_0+1} - x^*\| &= \|\alpha_{N_0}(x_0 - x^*) + (1-\alpha_{N_0})(\tilde{x}_{N_0+1} - x^*)\| \\ &\le \alpha_{N_0}\|x_0 - x^*\| + (1-\alpha_{N_0})(1+\frac{\eta_{N_0}^2}{1-2\eta_{N_0}^2})\|x_{N_0} - x^*\| \\ &\le (1+\frac{\eta_{N_0}^2}{1-2\eta_{N_0}^2})\|x_0 - x^*\| + (1+\frac{\eta_{N_0}^2}{1-2\eta_{N_0}^2})\|x_{N_0} - x^*\|.\end{aligned}$$

This shows that (19) holds for $n = 0$. Suppose that (19) holds for $n \ge 0$. Then from (8) and (18b) we obtain

$$\begin{aligned}&\|x_{N_0+n+2} - x^*\| \\ &= \|\alpha_{N_0+n+1}(x_0 - x^*) + (1-\alpha_{N_0+n+1})(\tilde{x}_{N_0+n+2} - x^*)\| \\ &\le \alpha_{N_0+n+1}\|x_0 - x^*\| + (1-\alpha_{N_0+n+1})(1+\frac{\eta_{N_0+n+1}^2}{1-2\eta_{N_0+n+1}^2})\|x_{N_0+n+1} - x^*\| \\ &\le \alpha_{N_0+n+1}\|x_0 - x^*\| + (1-\alpha_{N_0+n+1})(1+\frac{\eta_{N_0+n+1}^2}{1-2\eta_{N_0+n+1}^2}) \\ &\quad [\prod_{k=0}^{n}(1+\frac{\eta_{N_o+k}^2}{1-2\eta_{N_0+k}^2})\|x_0 - x^*\| + \prod_{k=0}^{n}(1+\frac{\eta_{N_0+k}^2}{1-2\eta_{N_0+k}^2})\|x_{N_0} - x^*\|] \\ &\le \alpha_{N_0+n+1}\|x_0 - x^*\| + (1-\alpha_{N_0+n+1})\prod_{k=0}^{n+1}(1+\frac{\eta_{N_o+k}^2}{1-2\eta_{N_0+k}^2})\|x_0 - x^*\| \\ &\quad +(1-\alpha_{N_0+n+1})\prod_{k=0}^{n+1}(1+\frac{\eta_{N_0+k}^2}{1-2\eta_{N_0+k}^2})\|x_{N_0} - x^*\| \\ &\le \prod_{k=0}^{n+1}(1+\frac{\eta_{N_o+k}^2}{1-2\eta_{N_0+k}^2})\|x_0 - x^*\| + \prod_{k=0}^{n+1}(1+\frac{\eta_{N_0+k}^2}{1-2\eta_{N_0+k}^2})\|x_{N_0} - x^*\|.\end{aligned}$$

This shows that (19) holds for $n+1$. Therefore by induction we conclude that (19) is true for all nonnegative integers $n \ge 0$. Since $\sum_{k=0}^{\infty} \eta_k^2 < \infty$, it follows that

$$C_0' := \sum_{k=N_0}^{\infty} \frac{\eta_k^2}{1-2\eta_k^2} < \infty \quad \text{and} \quad C_1' := \prod_{k=N_0}^{\infty} (1+\frac{\eta_k^2}{1-2\eta_k^2}) < \infty,$$

and thus $\{x_n\}$ is bounded.

Step 2. We claim that $\limsup_{n\to\infty}\langle x_0 - y^*, x_n - y^*\rangle \le 0$ where $y^* = P_s(x_0)$. Pick a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ so that

$$\limsup_{n\to\infty}\langle x_0 - y^*, x_n - y^*\rangle = \lim_{j\to\infty}\langle x_0 - y^*, x_{n_j} - y^*\rangle. \tag{20}$$

We may also assume that w-$\lim_{j\to\infty} x_{n_j} = x_\infty$. It thus follows from (20) that

$$\limsup_{n\to\infty}\langle x_0 - y^*, x_n - y^*\rangle = \langle x_0 - y^*, x_\infty - y^*\rangle, \tag{21}$$

and it remains to show that $x_\infty \in S$. Since $\{x_n\}$ is bounded, it follows from (18b) that $\{\tilde{x}_n\}_{n=1}^\infty$ is bounded. Observe that

$$\begin{aligned}\|\tfrac{1}{c_n}(x_n - \tilde{x}_{n+1} + e_{n+1})\| &\le \frac{1}{c_n}\|x_n - \tilde{x}_{n+1}\| + \frac{1}{c_n}\|e_{n+1}\| \\ &\le \frac{1}{c_n}\|\tilde{x}_{n+1} - x_n\| + \frac{\eta_n}{c_n}\|\tilde{x}_{n+1} - x_n\| \to 0 \quad \text{as} \quad n\to\infty,\end{aligned}$$

and that (8) implies

$$\|x_{n+1} - \tilde{x}_{n+1}\| = \alpha_n\|x_0 - \tilde{x}_{n+1}\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, we get

$$\lim_{j\to\infty}\frac{1}{c_{n_j-1}}(x_{n_j-1} - \tilde{x}_{n_j} + e_{n_j}) = 0 \quad \text{and} \quad w - \lim_{j\to\infty}\tilde{x}_{n_j} = w - \lim_{j\to\infty} x_{n_j} = x_\infty.$$

Since (6) implies that $\frac{1}{c_{n_j-1}}(x_{n_j-1} - \tilde{x}_{n_j} + e_{n_j}) \in T(\tilde{x}_{n_j})$, taking the limit as $j \to \infty$, we obtain by the maximality of T (hence T is demiclosed) that $0 \in T(x_\infty)$; that is, $x_\infty \in S$. The claim of Step 2 now follows from Lemma 2.4.

Step 3. We claim that $\lim_{n\to\infty} x_n = y^* := P_s(x_0)$. Indeed, for each $n \ge N_0$, put

$$\beta_n := 2\langle x_0 - y^*, x_{n+1} - y^*\rangle, \quad \text{and} \quad \gamma_n := \frac{\eta_n^2}{1 - 2\eta_n^2}\cdot M,$$

where $M = \sup_{n\ge 0}\|x_n - y^*\|^2$. Then it follows from (18a) that

$$\begin{aligned}\|x_{n+1} - y^*\|^2 &= \|(1-\alpha_n)(\tilde{x}_{n+1} - y^*) + \alpha_n(x_0 - y^*)\|^2 \\ &\le (1-\alpha_n)\|\tilde{x}_{n+1} - y^*\|^2 + 2\alpha_n\langle x_0 - y^*, x_{n+1} - y^*\rangle \\ &\le (1-\alpha_n)(1 + \frac{2\eta_n^2}{1-2\eta_n^2})\|x_n - y^*\|^2 + 2\alpha_n\langle x_0 - y^*, x_{n+1} - y^*\rangle \\ &\le (1-\alpha_n)\|x_n - y^*\|^2 + \alpha_n\beta_n + \gamma_n.\end{aligned}$$

Combining Step 2, condition (11) and Lemma 2.3, we can see that $\|x_n - y^*\| \to 0$ as $n \to \infty$. ■

Remark 3.1. As in [26], it is unclear if the weak limit z of $\{x_n\}$ in Theorem 3.1 equals $P_s(x_0)$.

Remark 3.2. In this paper, we suggest and propose the new accuracy criteria for modified approximate proximal point algorithms. The accuracy conditions are easy to verify and to enforce. However, we would like to point out that the convergence analysis is based on the assumption that the roots set of T is nonempty. Note that T may have no root even if T is maximal monotone; that is, the roots set S may be empty. If S is empty, then the sequence $\{x_n\}$ conforming to Algorithm (2) (as a special case of Algorithm 2.1) is unbounded; see e.g., [1, 6, 11, 20].

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