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CONTINUITY OF DISTANCE RELATED TO INCOME COMPARISONS

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Dedicated to Professor Wataru Takahashi on the occasion of his 65th birthday

Abstract. The author proposed a distance on R^n in [1] which measures improvement of social welfare. Continuity of the distance with respect to the Euclidean metric of R^n is proved in this note, and conversely, the Euclidean metric is continuous with respect to our distance on some subset of R^n.

1. Introduction

The author defined in [1] a distance on R^n related to Lorenz dominance which is one of the important concepts of inequality of income distributions in the research of social welfare theory in economics (see, for example, [3]). The distance is defined as follows: for any two elements x and y of R^n, we define a kind of zero-sum two-person games and we assign the value $\delta(x, y)$ of the game to the two elements x and y. The associated minimax equation of the game is

$$\delta(x, y) = \max_{\lambda \in \Delta_N} \min_{\pi \in \Pi} \langle \lambda, \pi x - y \rangle = \min_{D \in \Delta_\Pi} \max_{i \in N} (xD - y)_i.$$

The meaning of the notations in the minimax equation are as follows: We denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product of R^n, that is, $\langle \lambda, x \rangle = \sum_{i=1}^n \lambda_i x_i$ for any elements λ and x of R^n. The set N is all coordinates of elements of R^n, that is, $N = \{1, 2, \ldots, n\}$ and Δ_N denotes the standard simplex of R^n, that is, $\Delta_N = \{\lambda \in R^n : \lambda_i \geq 0, \ \sum_{i=1}^n \lambda_i = 1\}$. The set Π denotes all permutations of N. For any permutation $\pi \in \Pi$ and any $x \in R^n$, define $\pi x = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. Thus, if P is a permutation matrix corresponding to π, we have $\pi x = xP$. The set Δ_Π denotes the convex hull of the permutations and we can identify Δ_Π as the set of all doubly stochastic matrices by virtue of Birkohff's theorem([2, p. 19]). Thus we obtain a real-valued function δ on $R^n \times R^n$ where $\delta(x, y)$ is the value of the

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game. We have shown in [1] that δ satisfies the triangle inequality, and finally we define our distance $d(x, y)$ between x and y as

$$d(x, y) = \delta(x, y) \vee \delta(y, x).$$

The function d is actually almost a distance on R^n. The word "almost" means d satisfies all the axioms of distances except the axiom that $d(x, y) = 0$ implies $x = y$. Our function d satisfies only the property that $d(x, y) = 0$ implies $x = \pi y$ for some permutation $\pi \in \Pi$.

Our purpose of this note is making clear topological relationship between our distance d and the familiar Euclidean distance $d_E(x, y) = \sqrt{\langle x - y, x - y\rangle} = \|x - y\|$.

The topology induced by our distance d on R^n is not Hausdorff because of the lack of the axiom that $d(x, y) = 0$ implies $x = y$. We denote by τ the topology induced by our distance d, and by τ_E the topology induced by the Euclidean distance d_E.

2. Results

The following lemma is proved in [1], but it is a key lemma in this note and we show the proof here again. Let $M = \{\lambda \in R^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$.

Lemma 1. *For any two elements x and y of R^n, we have*

$$\delta(x, y) = \max_{\lambda \in M \cap \Delta_N} \langle \lambda, x^* - y^* \rangle,$$

where x^* denotes the increasing rearrangement of x, that is, $x^* = \pi x$ for some $\pi \in \Pi$ and the inequalities

$$x_{\pi(1)} \leq x_{\pi(2)} \leq \cdots \leq x_{\pi(n)}$$

hold.

Proof. Put $\beta = \max_{\lambda \in M \cap \Delta_N} \langle \lambda, x^* - y^* \rangle$. Firstly we show $\beta \leq \delta(x, y)$. Take $\lambda' \in M \cap \Delta_N$ such that $\beta = \langle \lambda', x^* - y^* \rangle = \langle \lambda', x^* \rangle - \langle \lambda', y^* \rangle$. Take $\pi' \in \Pi$ such that $\pi' y^* = y$, then we have $\beta = \langle \lambda', x^* \rangle - \langle \pi'\lambda', y \rangle$. Since $\langle \lambda', \pi x \rangle \geq \langle \lambda', x^* \rangle$ for all $\pi \in \Pi$ ([2, p. 141]), we have

$$\begin{aligned}
\beta &\leq \langle \lambda', \pi x \rangle - \langle \pi'\lambda', y \rangle \\
&= \langle \pi'\lambda', (\pi' \circ \pi)x \rangle - \langle \pi'\lambda', y \rangle \\
&= \langle \pi'\lambda', (\pi' \circ \pi)x - y \rangle.
\end{aligned}$$

Thus we have $\beta \leq \min_{\pi\in\Pi}\langle \pi'\lambda', (\pi'\circ\pi)x - y\rangle$ because π is arbitrary, and hence $\beta \leq \min_{\pi\in\Pi}\langle \pi'\lambda', \pi x - y\rangle$. Since $\pi'\lambda' \in \Delta_N$, we have

$$\beta \leq \max_{\lambda\in\Delta_N}\min_{\pi\in\Pi}\langle \lambda, \pi x - y\rangle = \delta(x, y).$$

Next we show the reverse inequality $\delta(x, y) \leq \beta$. Take $\lambda' \in \Delta_N$ such that $\delta(x, y) = \min_{\pi\in\Pi}\langle \lambda', \pi x - y\rangle = \min_{\pi\in\Pi}\langle \lambda', \pi x\rangle - \langle \lambda', y\rangle$. Take $\pi' \in \Pi$ such that $\pi'\lambda'$ belongs to M, then $\langle \pi'\lambda', y^*\rangle \leq \langle \lambda', y\rangle$, and hence we have $\delta(x, y) \leq \min_{\pi\in\Pi}\langle \lambda', \pi x\rangle - \langle \pi'\lambda', y^*\rangle$. Let $\pi'' \in \Pi$ be the permutation such that $\pi'' x^* = x$. Then we have the following series of inequalities:

$$\begin{aligned}
\delta(x, y) &\leq \min_{\pi\in\Pi}\langle \lambda', (\pi\circ\pi'')x^*\rangle - \langle \pi'\lambda', y^*\rangle \\
&= \min_{\pi\in\Pi}\langle (\pi\circ\pi'')^{-1}\lambda', x^*\rangle - \langle \pi'\lambda', y^*\rangle \\
&\leq \langle \pi'\lambda', x^*\rangle - \langle \pi'\lambda', y^*\rangle \\
&= \langle \pi'\lambda', x^* - y^*\rangle \\
&\leq \max_{\lambda\in M\cap\Delta_N}\langle \lambda, x^* - y^*\rangle \\
&= \beta.
\end{aligned}$$

Firstly, we show the continuity of our distance d with respect to τ_E.

Proposition 1. *The distance $d : R^n \times R^n \to R$ is continuous when R^n is equipped with τ_E.*

Proof. At first, we show the continuity of the mapping $x \mapsto x^*$ with respect to τ_E. Take a sequence $\{x_m\}$ and an element x such that $d_E(x_m, x)$ converges to 0. Take a permutation matrix P with $x^* = xP$. If x^* has components with the same value, then they appear contiguously and let the sets of the contiguous indices be $E_1, \ldots, E_p$ and the sets be ordered such that if $i \in E_k$, $j \in E_l$ and $k < l$, then $x_i^* < x_j^*$. Put $x'_m = x_m P$. Then x'_m converges to x^*. On the other hand, fix an arbitrary $\varepsilon > 0$. For sufficiently large m, for all $i \notin \cup_{k=1}^{p} E_k$, we have $x'_{mi} = x^*_{mi}$. Thus we have $|x^*_{mi} - x^*_i| = |x'_{mi} - x^*_i| < \varepsilon$. Next, fix a k and consider the set E_k. For any $i \in E_k$, there is $j \in E_k$ such that $|x^*_i - x^*_{mi}| = |x^*_j - x'_{mj}| < \varepsilon$ for sufficiently large m. Thus, it follows that $d_E(x^*_m, x^*)$ converges to 0.

It is sufficient to show the continuity of δ with respect to τ_E. By Lemma 1, we have $\delta(x, y) = \max_{\lambda\in M\cap\Delta_N}\langle \lambda, x^* - y^*\rangle$. Since we have shown the continuity of the map $x \mapsto x^*$, it is sufficient to prove the function defined by $f(x) = \max_{\lambda\in M\cap\Delta_N}\langle \lambda, x\rangle$ is continuous with respect to τ_E. By the definition of f, it is lower semicontinuous, and hence we need to show it is upper semicontinuous. Suppose a sequence $\{x_m\}$ converges to an element x in τ_E, and fix $\varepsilon > 0$ arbitrarily.

Take $\lambda_m \in M \cap \Delta_N$ with $f(x_m) = \langle \lambda_m, x_m \rangle$. Since $\lambda_m \in M \cap \Delta_N$, we have $\|\lambda_m\| \leq 1$. Thus we have $\langle \lambda_m, x_m \rangle - \langle \lambda_m, x \rangle \leq \|\lambda_m\| \|x_m - x\| \leq \|x_m - x\|$. Therefore, for sufficiently large m, we have

$$f(x_m) = \langle \lambda_m, x_m \rangle < \langle \lambda_m, x \rangle + \varepsilon \leq f(x) + \varepsilon,$$

which means f is upper semicontinuous at x with respect to τ_E.

Next we investigate the continuity of d_E with respect to τ. For almost all $x \in R^n$, $d_E(x, \pi x) > 0$ holds if π is not the identity mapping, but we have $d(x, \pi x) = 0$ for all permutations π. Thus d_E is not continuous with respect to τ on R^n. However, if we restrict the space R^n to the subspace

$$M = \{x \in R^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\},$$

then the situation changes. Note that d becomes accurately a distance on the set M.

Proposition 2. *The distance $d_E : M \times M \to R$ is continuous when the set M is equipped with τ.*

Proof. Suppose a sequence $\{x_m\}$ in M converges to an element x in M with respect to τ. Since $d(x, y) = \delta(x, y) \vee \delta(y, x)$, for any $\lambda \in M \cap \Delta_N$, $\langle \lambda, x_m^* \rangle$ converges to $\langle \lambda, x^* \rangle$. For each $i = 1, \ldots, n$, let e_i be the element of R^n whose components are all 0 except for the ith component whose value is 1. If we take $\lambda = e_1$, then we have x_{m1}^* converges to x_1^*. If we take $\lambda = (e_1 + e_2)/2$, then we have $(x_{m1}^* + x_{m2}^*)/2$ converges to $(x_1^* + x_2^*)/2$. Since x_{m1}^* converges to x_1^*, it is easily seen x_{m2}^* converges to x_2^*. Similarly we have x_{mi}^* converges to x_i^* for all $i = 1, \ldots, n$. This means $d_E(x_m, x)$ converges to 0. Since any distance is continuous with respect to the topology induced by the distance, it follows that d_E is continuous with respect to τ on M.

Combining Proposition 1 and Proposition 2, we have the following corollary.

Corollary 1. *The distances d and d_E are topologically equivalent on the set M.*

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