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A CLASS OF LYM ORDERS IN DIVISOR LATTICES

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Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

Abstract. We present a new class of LYM orders, which generalizes Lih's result and is a common generalization of Griggs' result and a result of West, Harper and Daykin.

1. Introduction

A *partially ordered set* (or *poset*) is a set equipped with a reflexive, antisymmetric, and transitive relation. A poset P is *ranked* if there is a rank function $r : P \to \mathbb{N}$ such that $r(x) = 0$ if x is a minimal element of P and $r(z) = r(y) + 1$ if z covers y in P. We call $r(x)$ the *rank* of x. The rank of P is the maximum value of $r(x)$ taken over all $x \in P$. Let P_i denote the set of elements of rank i in P. Its cardinality $|P_i|$ is called the *ith Whitney number* of P. We say that P is *LC* if the Whitney numbers of P form a log-concave sequence, that is,

$$|P_i|^2 \geq |P_{i-1}| \cdot |P_{i+1}|$$

for all $i > 0$. An *antichain* is a subset of pairwise incomparable elements of P. We say that P has the *Sperner property* if the maximum size of an antichain in P equals the largest Whitney number of P. We say that P has the *LYM property* if

$$\sum_i |A \cap P_i|/|P_i| \leq 1$$

for every antichain A of P. It is well known that the LYM property implies the Sperner property ([5]).

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The subset lattice or the Boollean lattice B_n is the poset of all subsets of an n-element set, ordered by inclusion. In 1928, Sperner [13] showed, in current terminology, that the subset lattice has the Sperner property. In 1967, Rota [12] made a famous conjecture that the partition lattice has the Sperner property. Although the conjecture was shown to be invalid in general by Canfield [1] in 1978, efforts to prove analogues of Sperner's theorem for other posets have led to the emergence of an entire theory (see [4] for details). In 1980, Lih [11] discovered a generalization of Sperner's theorem. Let $X = \{1, 2, \ldots, n\}$ be an n-element set and Y a subset of k elements of X where $k \leq n$. Let $C(n, k)$ be the collection of all subsets of X which intersect Y, ordered by inclusion. Lih showed that $C(n, k)$ has the Sperner property. Griggs [6] further showed, among other things, that $C(n, k)$ has the LYM property before long. He also generalized this result as follows.

Theorem 1. ([6]). *Let $X = \{1, 2, \ldots, n\}$ be partitioned into parts X_1, X_2, …, X_r. Suppose that $I_i \subseteq \{0, 1, \ldots, |X_i|\}$ is an arithmetic progression for each i. Then*

$$P = \{Z \subseteq X : |Z \cap X_i| \in I_i, 1 \leq i \leq r\},$$

ordered by inclusion, is LYM and LC.

On the other hand, West, Harper and Daykin [16] gave a different generalization of Lih's result.

Theorem 2. ([16]). *Let $C_1 \subset C_2 \subset \cdots \subset C_s$ be a chain of subsets of $X = \{1, 2, \ldots, n\}$. Suppose that $\{a_i\}$ and $\{b_i\}$ are two nondecreasing sequences with $a_i \leq b_i$ for $1 \leq i \leq s$. Then*

$$P = \{Z \subseteq X : a_i \leq |Z \cap C_i| \leq b_i, 1 \leq i \leq s\},$$

ordered by inclusion, is LYM and LC.

They also hoped to find out a common generalization of their result and that of Griggs. Indeed, there are similarities in the statements and the proofs of Theorem 1 and Theorem 2. In this note we broaden these results to the divisor lattice and give a common generalization.

Let $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ be a positive integer, where the p_i are distinct primes and $e_i \in \mathbb{N}$. The *divisor lattice* $D(n)$ is the poset of all (positive) divisors of n, ordered by divisibility. As usual, let $\sigma(n) = e_1 + e_2 + \cdots + e_t$ denote the number of prime divisors of n counted according to multiplicity. Then $D(n)$ is a ranked poset with the rank function σ. Clearly, $D(n)$ reduces to B_n when n is square-free. Denote by (m, k) the largest common divisor of two positive integers m, k and replace $\sigma((m, k))$ by $\sigma(m, k)$. Given two nonnegative integers $a \leq b$, denote $[a, b] = \{i \in \mathbb{N} : a \leq i \leq b\}$. Our main result is the following.

Theorem 3. *Let $n = n_1 n_2 \cdots n_r$ where n_i are positive integers of pairwise coprime. Suppose that $I_i \subseteq [0, \sigma(n_i)]$ is an arithmetic progression and $J_i = [a_i, b_i]$ where $a_i \le b_i$ for each i. Then*

$$P = \{m \in D(n) : \ \sigma(m, n_j) \in I_j \text{ and } \sum_{i=1}^{j} \sigma(m, n_i) \in J_j \text{ for } 1 \le j \le r\},$$

ordered by divisibility, is LYM and LC.

When n is square-free, the corresponding result is the following.

Corollary 1. *Suppose that $X = \{1, 2, \ldots, n\}$ is partitioned into parts $X_1, X_2, \ldots, X_r$. Let $I_i \subseteq [0, |X_i|]$ be an arithmetic progression and $J_i = [a_i, b_i]$ where $a_i \le b_i$ for each i. Then*

$$P = \{Z \subseteq X : \ |Z \cap X_j| \in I_j \text{ and } \sum_{i=1}^{j} |Z \cap X_i| \in J_j \text{ for } 1 \le j \le r\},$$

ordered by inclusion, is LYM and LC.

It is not difficult to see that Theorem 1 and 2 follow immediately from Corollary 1. In fact, we can obtain Theorem 1 by putting each $J_i = [0, n]$ in Corollary 1. On the other hand, suppose that $C_1 \subset C_2 \subset \cdots \subset C_s$ is a chain of subsets of X. Let

$$X_1 = C_1, X_2 = C_2 \setminus C_1, \ldots, X_s = C_s \setminus C_{s-1}, X_{s+1} = X \setminus C_s.$$

Then $X_1, X_2, \ldots, X_s, X_{s+1}$ is a partition of X and $C_i = X_1 \cup X_2 \cup \cdots \cup X_i$ $(1 \le i \le s)$. We obtain Theorem 2 by putting $J_{s+1} = [0, n]$ and $I_i = [0, |X_i|]$ $(1 \le i \le s+1)$ in Corollary 1.

2. Proof of Theorem 3

We use the product theorem for LYM posets to prove Theorem 3. The (*direct*) *product* $Q_1 \times Q_2$ of two posets Q_1 and Q_2 is defined to be the set of all pairs $(q_1, q_2), q_1 \in Q_1, q_2 \in Q_2$, with the order given by $(q_1, q_2) \le (q_1', q_2')$ if and only if $q_1 \le q_1'$ in Q_1 and $q_2 \le q_2'$ in Q_2. Furthermore, the product of two ranked posets Q_1 and Q_2 is defined to be the poset together with the rank function r given by $r(q_1, q_2) = r_1(q_1) + r_2(q_2)$, where r_1 and r_2 are the rank functions of Q_1 and Q_2, respectively. The product of two LYM posets P and Q may not be LYM in general, but it will be true if P and Q are LC also. The following result is discovered by Harper [7] and later independently by Hsieh and Kleitman [10].

Product Theorem. *If two posets Q_1, Q_2 are both LYM and LC, then so is their product poset $Q_1 \times Q_2$.*

A subposet of a poset Q is a subset of Q whose elements are ordered as in Q. Let $Q = \bigcup_{i=0}^{n} Q_i$ be a poset of rank n. Given a subset I of $[0, n]$, let $Q_I = \bigcup_{i \in I} Q_i$ be the subposet of Q induced by I. Clearly, an antichain of Q_I is also antichain of Q. It follows that if the poset Q is LYM, then so is the subposet Q_I.

Let $\{W_i\}_{i=0}^{n}$ be a log-concave sequence of positive numbers. Then the sequence $\{W_i/W_{i-1}\}_{i=1}^{n}$ is nonincreasing. Thus $W_j/W_{j-1} \geq W_k/W_{k-1}$ for $j \leq k$, or equivalently, $W_j W_{k-1} \geq W_{j-1} W_k$. It follows that

$$W_i^2 \geq W_{i-1}W_{i+1} \geq W_{i-2}W_{i+2} \geq \cdots \geq W_{i-d}W_{i+d}. \tag{2.1}$$

Let $I = \{a, a+d, a+2d, \ldots, a+md\}$ be an arithmetic progression in the closed interval $[0, n]$. Then the inequality (2.1) implies that the subsequence $\{W_i\}_{i \in I}$ is log-concave.

From the above discussion, we can conclude the following.

Lemma 1. *Let Q be a ranked poset of rank n and let I be an arithmetic progression in the closed interval $[0, n]$. If Q is LYM and LC, then so is the subposet Q_I induced by I.*

We now prove Theorem 3.

Proof of Theorem 3. We proceed by induction on r. If $r = 1$, then

$$P = \{m \in D(n) : \sigma(m, n) \in I \cap J\},$$

where $I \subseteq [1, \sigma(n)]$ is an arithmetic progression and $J = [a, b]$. Clearly, $I \cap J$ is still an arithmetic progression. Note that P consists of those elements of $D(n)$ with rank in $I \cap J$ and it is also well known that $D(n)$ is LYM and LC ([3]). Hence the subposet P of $D(n)$ is LYM and LC by Lemma 1.

Suppose next that $r > 1$. Consider the following two posets:

$$P_1 = \{m \in D(n_r) : \sigma(m, n_r) \in I_r\}$$

and

$$P_2 = \{m \in D(n_1 \cdots n_{r-1}) : \ \sigma(m, n_j) \in I_j \text{ and } \sum_{i=1}^{j} \sigma(m, n_i) \in J_j \text{ for } 1 \leq j \leq r-1\}.$$

By the induction hypotheses and Lemma 1, both P_1 and P_2 are LYM and LC. So $P_1 \times P_2$ is also LYM and LC by the Product Theorem. Note that $P_1 \times P_2$ is isomorphic to the subposet of $D(n)$

$$Q = \{m \in D(n) : \sigma(m, n_j) \in I_j \text{ for } 1 \leq j \leq r \text{ and } \sum_{i=1}^{j} \sigma(m, n_i) \in J_j \text{ for } 1 \leq j \leq r-1\}$$

and that P is the subposet Q_{J_r} of Q induced by J_r. Hence P is LYM and LC by Lemma 1. This completes the proof of Theorem 3. ■

3. Remarks

Let F be a collection of t-subsets of $X = \{1, \cdots, n\}$. Consider the filter generated by F:

$$P(F) = \{Y \subseteq X : Y \supseteq A \text{ for some } A \in F\},$$

which is a subposet of the Boolean lattice B_n. Lih [11] conjectured that $P(F)$ has the Sperner property. The case $t = 0$ is just the classical Sperner theorem and the case $t = 1$ is Lih's result about $C(n, k)$. However, Zhu [18] found counterexamples to the conjecture with $t > n/2$. Griggs [6] showed that the conjecture fails for $t = 4$ and Zha [17] constructed counterexamples for all $t \geq 4$ and $n \geq 2t - 1$. Horrocks [8, 9] gave a graph-theoretical interpretation for the $t = 2$ conjecture and left 116 exceptional graphs in his proof. Cheng and Lih [2] carried on further with Horrocks's reduction method to reduce the number of exceptional graphs and gave a complete proof for the $t = 2$ conjecture. The conjecture remains open for $t = 3$. An interesting problem is to consider analogue of Lih's conjecture for the divisor lattices and other posets. We also refer the reader to [14, 15] for a subspace lattice analogue of Lih's poset $C(n, k)$.

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