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# THE EQUITABLE COLORINGS OF KNESER GRAPHS

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Dedicated to Professor Ko-Wei Lih on the occasion of his 60th birthday.

**Abstract.** An $m$-coloring of a graph $G$ is a mapping $f : V(G) \to \{1, 2, \ldots, m\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices $x$ and $y$ in $G$. The chromatic number $\chi(G)$ of $G$ is the minimum number $m$ such that $G$ is $m$-colorable. An equitable $m$-coloring of a graph $G$ is an $m$-coloring $f$ such that any two color classes differ in size by at most one. The equitable chromatic number $\chi_=(G)$ of $G$ is the minimum number $m$ such that $G$ is equitably $m$-colorable. The equitable chromatic threshold $\chi_=^*(G)$ of $G$ is the minimum number $m$ such that $G$ is equitably $r$-colorable for all $r \geq m$. It is clear that $\chi(G) \leq \chi_=(G) \leq \chi_=^*(G)$. For $n \geq 2k+1$, the Kneser graph $\mathsf{KG}(n,k)$ has the vertex set consisting of all $k$-subsets of an $n$-set. Two distinct vertices are adjacent in $\mathsf{KG}(n,k)$ if they have empty intersection as subsets. The Kneser graph $\mathsf{KG}(2k+1,k)$ is called the Odd graph, denoted by $O_k$. In this paper, we study the equitable colorings of Kneser graphs $\mathsf{KG}(n,k)$. Mainly, we obtain that $\chi_=(\mathsf{KG}(n,k)) \leq \chi_=^*(\mathsf{KG}(n,k)) \leq n-k+1$ and $\chi(O_k) = \chi_=(O_k) = \chi_=^*(O_k) = 3$. We also show that $\chi_=(\mathsf{KG}(n,k)) = \chi_=^*(\mathsf{KG}(n,k))$ for $k = 2$ or 3 and obtain their exact values.

## 1. Introduction

An $m$-coloring of a graph $G$ is a mapping $f : V(G) \to \{1, 2, \ldots, m\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices $x$ and $y$ in $G$. A color class $f^{-1}(i)$ under $f$ is a subset of $V(G)$ in which every vertex is assigned the same color $i$. A graph $G$ is $m$-colorable if it admits an $m$-coloring. The chromatic number $\chi(G)$ of $G$ is the minimum number $m$ such that $G$ is $m$-colorable. The well-known Brooks' Theorem is stated as following.

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**Theorem 1.** ([2]). *Suppose $G$ is a graph different from a complete graph and an odd cycle. Then $\chi(G) \leq \Delta(G)$.*

An equitable $m$-coloring of a graph $G$ is an $m$-coloring such that any two color classes differ in size by at most one. A graph $G$ is equitably $m$-colorable if it admits an equitable $m$-coloring. The equitable chromatic number $\chi_=(G)$ of $G$ is the minimum number $m$ such that $G$ is equitably $m$-colorable. One can also consider the minimum number $m$ such that $G$ is equitably $r$-colorable for all $r \geq m$. Such a number $m$ is called the equitable chromatic threshold of $G$, denoted by $\chi_=^*(G)$. It is clear that $\chi(G) \leq \chi_=(G) \leq \chi_=^*(G)$. Since $\chi(G) \leq \chi_=(G)$, Meyer then posed the following conjecture which, if true, is stronger than the Brooks' Theorem.

**Conjecture 1.** ([15]). Suppose $G$ is a connected graph different from a complete graph and an odd cycle. Then $\chi_=(G) \leq \Delta(G)$.

One well-known result of Hajnal and Szemerédi, when rephrased in terms of the equitable colorability, has already been shown as follows.

**Theorem 2.** ([6, 9]). *A graph $G$, not necessary connected, is equitably $m$-colorable if $m \geq \Delta(G) + 1$.*

Theorem 2 says that $\chi_=(G) \leq \chi_=^*(G) \leq \Delta(G) + 1$ for all graphs $G$. Since the graphs $G$ that require at least $\Delta(G) + 1$ colors to color the vertices equitably are complete graphs and odd cycles, Chen, Lih and Wu put forth the following.

**Conjecture 2.** ([4]). Equitable $\Delta$-Coloring Conjecture.

A connected graph $G$ is equitably $\Delta(G)$-colorable if and only if $G$ is different from the complete graph $K_n$, the odd cycle $C_{2n+1}$ and the complete bipartite graph $K_{2n+1,2n+1}$ for all $n \geq 1$.

They also verified this conjecture for a graph with $\Delta(G) \geq |V(G)|/2$ or $\Delta(G) \leq 3$. Yap and Zhang [18] obtained a finer bound when $|V(G)|/2 > \Delta(G) \geq (|V(G)|/3) + 1$. Moreover, some particular cases have been studied, such as trees $[1, 3]$, bipartite graphs [13], $d$-degenerate graphs $[11, 12]$ and planar graphs $[10, 16, 17]$. However, Conjecture 1 and Conjecture 2 are still open in general.

For $n \geq 2k + 1$, the Kneser graph $\mathsf{KG}(n, k)$ has the vertex set consisting of all $k$-subsets of an $n$-set. Two distinct vertices are adjacent in $\mathsf{KG}(n, k)$ if they have empty intersection as subsets. The Odd graph $O_k$ is the Kneser graph $\mathsf{KG}(2k+1, k)$. The chromatic number of $\mathsf{KG}(n, k)$ was obtained by Lovász.

**Theorem 3.** ([14]). $\chi(\mathsf{KG}(n, k)) = n - 2k + 2$.

In this paper, we study the equitable colorings of $\mathsf{KG}(n,k)$. Since $\mathsf{KG}(n,1) = K_n$, it is easy to see that $\chi(\mathsf{KG}(n,1)) = \chi_=(\mathsf{KG}(n,1)) = \chi_=^*(\mathsf{KG}(n,1)) = n$. Throughout this paper, we assume $k \geq 2$. For convenience, we introduce some notation. For integers $i < j$, let $[i,j]$ be the set of all integers $i, i+1, \ldots, j$ and $[n] = [1,n]$. If $X$ is a set, then the collection of all $k$-subsets of $X$ is denoted by $\binom{X}{k}$. Hence, the vertex set $V(\mathsf{KG}(n,k))$ is denoted by $\binom{[n]}{k}$ and $|V(\mathsf{KG}(n,k))| = C(n,k) = \binom{n}{k}$. An $i$-flower $\mathcal{F}$ of $\binom{X}{k}$ is a subcollection of $\binom{X}{k}$ in which all $k$-subsets have a common element $i$, i.e., $i \in \bigcap_{A \in \mathcal{F}} A$. It is clear that every $i$-flower is an independent set of $\mathsf{KG}(n,k)$. An independent set $\mathcal{F}$ of $\mathsf{KG}(n,k)$ is also called an intersection family of $\binom{[n]}{k}$, i.e., $A \cap B \neq \emptyset$ for all $A$ and $B$ in $\mathcal{F}$. The independence number $\alpha(\mathsf{KG}(n,k))$ of $\mathsf{KG}(n,k)$ was obtained by Erdős, Ko and Rado.

**Theorem 4.** ([5]). *Suppose $\mathcal{F}$ is an intersection family of $\binom{[n]}{k}$. Then $|\mathcal{F}| \leq C(n-1,k-1)$. Moreover, the equality holds if and only if $\mathcal{F} = \{A \in \binom{[n]}{k} : i \in A\}$ for some $i \in [n]$.*

There are independent sets of $\mathsf{KG}(n,k)$ which are not flowers. Denote by $\alpha_2(\mathsf{KG}(n,k))$, or simply by $\alpha_2(n,k)$, the maximum size of independent sets $\mathcal{H}$ of $\mathsf{KG}(n,k)$ satisfying $\bigcap_{A \in \mathcal{H}} A = \emptyset$. The following result was obtained by Hilton and Milner.

**Theorem 5.** ([8]). *Suppose $\mathcal{H}$ is an intersection family of $\binom{[n]}{k}$ with $\bigcap_{A \in \mathcal{H}} A = \emptyset$. Then $|\mathcal{H}| \leq C(n-1,k-1) - C(n-k-1,k-1) + 1$. Moreover, the equality holds if and only if $\mathcal{H} \cong \{A \in \binom{[n]}{3} : |A \cap [1,3]| \geq 2\}$ or $\mathcal{H} \cong \{A \in \binom{[n]}{k} : 1 \in A, |A \cap [2,k+1]| \geq 1\} \cup \{[2,k+1]\}$.*

We also need the following to prove our main results.

**Theorem 6.** ([7]). *A bipartite graph $G = G(X,Y)$ with bipartition $(X,Y)$ has a matching that saturates every vertex in $X$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$, where $N(S)$ denotes the set of neighbors of vertices in $S$.*

## 2. General Bounds

In this section, let $n \geq 2k+1$. Since every flower of $\binom{[n]}{k}$ is an independent set of $\mathsf{KG}(n,k)$, it is natural to partition flowers to form an equitable coloring of $\mathsf{KG}(n,k)$. In this case, every $k$-subset of $[n]$ is in some flower. Hence, if $f$ is an equitable $m$-coloring of $\mathsf{KG}(n,k)$ such that every color class under $f$ is contained in some flower, then $m \ge n-k+1$. Otherwise, suppose $m \le n-k$ and each color classe $f^{-1}(i)$ is contained in some $t_i$-flower for $1 \le i \le m$, respectively. Since $|[n] \setminus \{t_1, t_2, \ldots, t_m\}| \ge n-m \ge k$, we may choose a $k$-subset $A \subseteq [n] \setminus \{t_1, t_2, \ldots, t_m\}$. Since $f$ is an equitable $m$-coloring, $A \in f^{-1}(i)$ for some $i$, i.e., $t_i \in A$. It is a contradiction. Hence, we have the following.

**Lemma 7.** *If $f$ is an equitable $m$-coloring of $\mathsf{KG}(n,k)$ such that every color class under $f$ is contained in some flower of $\binom{[n]}{k}$, then $m \ge n-k+1$.*

In what follows, we should show that $\mathsf{KG}(n,k)$ is equitably $m$-colorable for all $m \ge n-k+1$ by partitioning flowers of $\binom{[n]}{k}$ into $m$ equitably independent sets. Precisely, letting $m = qn + r$, $0 \le r < n$, we will partition $\binom{[n]}{k}$ into $m$ subcollections $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ with $a_i = |\mathcal{V}_i| = \lceil (C(n,k) - i + 1)/m \rceil$, $1 \le i \le m$, such that $\mathcal{V}_i$ is contained in a $\pi(i)$-flower, where $\pi(i) = i (\text{mod } n)$ if $1 \le i \le qn$ and $\pi(i) = i + n - m$ if $qn + 1 \le i \le m$. The notation $i (\text{mod } n)$ denotes the residue of $i$ modulo $n$ taken in the set $[n]$. To do this, we construct a bipartite graph $G = G(X, Y)$ with bipartition $(X, Y)$, where $X$ is the disjoint union of the sets $X_i = \{x_{i,j} : 1 \le j \le a_i\}$, $1 \le i \le m$, and $Y = \binom{[n]}{k}$. Two vertices $x_{i,j} \in X$ and $A \in Y$ are adjacent if and only if $\pi(i) \in A$. It is easy to see that $|X| = |Y| = \binom{n}{k}$. If $G$ has a perfect matching $M = \{\{x_{i,j}, A_{i,j}\} : 1 \le i \le m, 1 \le j \le a_i\}$, letting $\mathcal{V}_i = \{A_{i,j} : 1 \le j \le a_i\}$, $1 \le i \le m$, then the partition $(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m)$ forms an equitable $m$-coloring of $\mathsf{KG}(n,k)$. By Theorem 6, $G$ has a perfect matching if $|N(S)| \ge |S|$ for all $S \subseteq X$. Hence, we need to show the inequality $|N(S)| \ge |S|$. Suppose $S \subseteq X$. Let $I(S) = \{\pi(j) : S \cap X_j \ne \emptyset\}$. Note that if $|I(S)| \ge n-k+1$, then $N(S) = Y$ and $|N(S)| \ge |S|$. For $|I(S)| = i \le n-k$, let $S_i = \bigcup_{\pi(j)=n-r+1}^{n-r+i} X_j$ if $i \le r$ and $S_i = (\bigcup_{\pi(j)=n-r+1}^{n} X_j) \cup (\bigcup_{\pi(j)=1}^{i-r} X_j)$ if $i > r$. Then $|S| \le |S_i|$. Moreover, the set $I(S_i) = \{\pi(j) : S_i \cap X_j \ne \emptyset\}$ has the same size as $I(S)$. It follows that $|N(S)| = |N(S_i)| = C(n,k) - C(n-i,k)$ and then $|N(S)| - |S| \ge |N(S_i)| - |S_i|$. The following lemmas are used to show the inequality $|N(S_i)| \ge |S_i|$ that implies $|N(S)| - |S| \ge 0$.

**Lemma 8.** *Suppose $m = qn + r$, where $q \ge 1$ and $0 < r < n$. Let $S_i$ be defined as above. Then $|S_i| \le \dfrac{2i}{n+i} C(n,k)$.*

*Proof.* For $1 \le j \le n$, let $W_j = \bigcup_{\pi(t)=j,\, t \le qn} X_t$. Then $|W_{j+1}| \le |W_j| \le |W_{j+1}| + 1$, $|X_{qn+t}| \le |W_{n-r+t}|$ and $|W_j| \le |W_{n-r+t}| + |X_{qn+t}|$ for $1 \le j \le n$ and $1 \le t \le r$.

If $i \le r$, then $|S_i| = \sum_{j=1}^{i} (|W_{n-r+j}| + |X_{qn+j}|) \le 2 \sum_{j=1}^{i} |W_{n-r+j}| \le 2i |W_{n-r+1}|$, or $\frac{|S_i|}{2i} \leq |W_{n-r+1}|$. On the other hand, $C(n,k)-|S_i| = \sum_{j=1}^{n-r}|W_j| + \sum_{j=i+1}^{n}(|W_{n-r+j}| + |X_{qn+j}|) \geq (n-i)|W_{n-r}|$, or $\frac{C(n,k)-|S_i|}{n-i} \geq |W_{n-r}|$. Hence, $\frac{|S_i|}{2i} \leq |W_{n-r+1}| \leq |W_{n-r}| \leq \frac{C(n,k)-|S_i|}{n-i}$. It follows that $\frac{|S_i|}{2i} \leq \frac{|S_i|+C(n,k)-|S_i|}{2i+n-i} = \frac{C(n,k)}{n+i}$.

If $i > r$, then $|S_i| = \sum_{j=1}^{i-r}|W_j| + \sum_{j=1}^{r}(|W_{n-r+j}| + |X_{qn+j}|) \leq (i-r)(|W_{n-r+1}| + |X_{qn+1}|) + r(|W_{n-r+1}| + |X_{qn+1}|) \leq 2i|W_{n-r+1}|$, or $\frac{|S_i|}{2i} \leq |W_{n-r+1}|$. On the other hand, $C(n,k) - |S_i| = \sum_{j=i-r+1}^{n-r}|W_j| \geq (n-i)|W_{n-r}|$, or $\frac{C(n,k)-|S_i|}{n-i} \geq |W_{n-r}|$. Hence, $\frac{|S_i|}{2i} \leq |W_{n-r+1}| \leq |W_{n-r}| \leq \frac{C(n,k)-|S_i|}{n-i}$. It follows that $\frac{|S_i|}{2i} \leq \frac{|S_i|+C(n,k)-|S_i|}{2i+n-i} = \frac{C(n,k)}{n+i}$.

Therefore, $|S_i| \leq \frac{2i}{n+i}C(n,k)$ as desired. ∎

**Lemma 9.** *Suppose that $k \leq n-i$.*

(1) $C(n, k-1) \geq C(n-i, k-1) + ik$ *for* $k \geq 3$.

(2) $C(n,k) - C(n-i,k) \geq \frac{2i}{n+i}C(n,k)$ *for* $k \geq 2$.

*Proof.*

(1) By direct computation, we have

$$\begin{aligned} C(n,k-1) &= C(n-1,k-1) + C(n-1,k-2) \\ &= C(n-i,k-1) + C(n-i,k-2) + C(n-i+1,k-2) \\ &\quad + \cdots + C(n-1,k-2) \\ &\geq C(n-i,k-1) + iC(k,1) \\ &= C(n-i,k-1) + ik. \end{aligned}$$

(2) By direct computation, we have

$$\begin{aligned}
\frac{C(n,k)}{C(n-i,k)} &= \frac{n(n-1)\cdots(n-k+1)}{(n-i)(n-i-1)\cdots(n-i-k+1)} \\
&= \left(1+\frac{i}{n-i}\right)\left(1+\frac{i}{n-i-1}\right)\cdots\left(1+\frac{i}{n-i-k+1}\right) \\
&> \left(1+\frac{i}{n-i}\right)^k \\
&\geq \left(1+\frac{i}{n-i}\right)^2 \\
&> \left(1+\frac{2i}{n-i}\right) \\
&= \frac{n+i}{n-i}.
\end{aligned}$$

Hence, $C(n,k) - C(n-i,k) \geq \frac{2i}{n+i}C(n,k)$ as desired. ■

Now, we are ready to show our main results.

**Lemma 10.** *Suppose that* $n-k+1 \leq m \leq n$. *Then* $\mathsf{KG}(n,k)$ *is equitably* $m$*-colorable.*

*Proof.* Let the bipartite graph $G = G(X,Y)$, $S$ and $S_i$ be defined as before. It suffices to show that $|N(S_i)| - |S_i| \geq 0$ for $i \leq n-k$. First, we consider $k = 2$. Then $m = n-1$ or $n$ and $i \leq n-2$. If $i = n-2$, then $|S_i| = |X| - |X_m|$ for $m = n-1$ or $|S_i| = |X| - |X_{m-1}| - |X_m|$ for $m = n$. Hence, $|N(S_i)| - |S_i| \geq C(n,2) - 1 - \left(C(n,2) - \left\lfloor \frac{C(n,2)}{n-1} \right\rfloor\right) = \left\lfloor \frac{n}{2} \right\rfloor - 1 > 0$. If $i \leq n-3$, then $|N(S_i)| - |S_i| \geq C(n,2) - C(n-i,2) - i\left\lfloor \frac{C(n,2)}{m} \right\rfloor - i \geq C(n,2) - C(n-i,2) - i\left\lfloor \frac{C(n,2)}{n-1} \right\rfloor - i \geq \frac{i}{2}(n-i-3) \geq 0$.

Suppose $k \geq 3$. Then, $|S_i| = \sum_{j=1}^{i} \left\lceil \frac{C(n,k)-j+1}{m} \right\rceil \leq i\left(\frac{C(n,k)}{n-k+1} + 1\right)$. By Lemma 9(1), we have

$$\begin{aligned}
|N(S)| - |S| &\geq |N(S_i)| - |S_i| \\
&\geq \frac{n-k+1-i}{k}(C(n,k-1) - C(n-i,k-1)) - i \\
&\geq \frac{n-k+1-i}{k} ik - i \\
&= (n-k-i)i \geq 0.
\end{aligned}$$

Therefore, we complete the proof. ■

**Lemma 11.** *Suppose that $m > n$. Then $\mathsf{KG}(n, k)$ is equitably $m$-colorable.*

*Proof.* First, consider $m = qn$, $q \geq 1$. By Lemma 10, $\binom{[n]}{k}$ can be partitioned equitably into $n$ subcollections $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$, where each $\mathcal{X}_i$ is an $i$-flower. For each $i \geq 1$, we can partition $\mathcal{X}_i$ into $q$ equitable subcollections $\mathcal{X}_{i,1}, \mathcal{X}_{i,2}, \ldots, \mathcal{X}_{i,q}$. Hence the collection $\{\mathcal{X}_{i,j} : 1 \leq i \leq n, 1 \leq j \leq q\}$ forms an equitable $m$-coloring of $\mathsf{KG}(n, k)$.

Now, suppose $m$ is not divisible by $n$. Let the bipartite graph $G = G(X, Y)$, $S$ and $S_i$ be defined as before. It suffices to show that $|N(S_i)| - |S_i| \geq 0$ for $i \leq n-k$. By Lemma 8 and Lemma 9(2), $|N(S_i)| - |S_i| \geq C(n, k) - C(n-i, k) - \dfrac{2i}{n+i} C(n, k) \geq 0$.

Therefore, we complete the proof. ■

Combining Lemma 10 and Lemma 11, the following is easy to see.

**Theorem 12.** *Suppose that $m \geq n - k + 1$. Then $\mathsf{KG}(n, k)$ is equitably $m$-colorable, i.e., $\chi_=(\mathsf{KG}(n, k)) \leq \chi_=^*(\mathsf{KG}(n, k)) \leq n - k + 1$.*

Suppose $m \leq n - k$ and $\mathsf{KG}(n, k)$ is equitably $m$-colorable. Let $f$ be an equitable $m$-coloring of $\mathsf{KG}(n, k)$. By Lemma 7, there is some color class $f^{-1}(i)$ which is contained in no flowers of $\binom{[n]}{k}$. Moreover, the particular $f^{-1}(i)$ must satisfy that $|f^{-1}(i)| \leq \alpha_2(n, k) = C(n-1, k-1) - C(n-k-1, k-1) + 1$. Using this fact, we have the following.

**Lemma 13.** *Suppose that $m \leq n - k$ and $\left\lfloor \dfrac{C(n, k)}{m} \right\rfloor > \alpha_2(n, k)$. Then $\mathsf{KG}(n, k)$ is not equitably $r$-colorable for all $r \leq m$, i.e., $\chi_=^*(\mathsf{KG}(n, k)) \geq \chi_=(\mathsf{KG}(n, k)) \geq m + 1$.*

*Proof.* Suppose $\mathsf{KG}(n, k)$ has an equitable $r$-coloring $f$ for some $r \leq m$. Then there is some color class $f^{-1}(i)$ satisfying that $|f^{-1}(i)| \leq \alpha_2(n, k)$. Since $f$ is an equitable $r$-coloring, $|f^{-1}(i)| \geq \left\lfloor \dfrac{C(n, k)}{m} \right\rfloor > \alpha_2(n, k)$ which is a contradiction. Hence, $\mathsf{KG}(n, k)$ is not equitably $r$-colorable for all $r \leq m$ and then $\chi_=^*(\mathsf{KG}(n, k)) \geq \chi_=(\mathsf{KG}(n, k)) \geq m + 1$. ■

**Theorem 14.** *If $\left\lfloor \dfrac{C(n, k)}{n-k} \right\rfloor > \alpha_2(n, k)$. Then $\chi_=(\mathsf{KG}(n, k)) = \chi_=^*(\mathsf{KG}(n, k)) = n - k + 1$.*

*Proof.* It follows from Theorem 12 and Lemma 13. ■

## 3. Cases for $k = 2, 3$

By the same argument as in the proof of Lemma 10, the following is not difficult to see.

**Lemma 15.** *Suppose that $1 \le t \le m$. Then the collectiom $\binom{[m]}{t}$ can be partitioned equitably into $m$ subcollections $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$, such that each $\mathcal{F}_i$ is an $i$-flower.*

By Lemma 13, Theorem 14 and Lemma 15, we can show that $\chi_=(\mathsf{KG}(n,k)) = \chi_=^*(\mathsf{KG}(n,k))$ for $k = 2$ or 3 and obtain their exact values.

**Theorem 16.** *For $n \ge 5$,*

$$\chi_=(\mathsf{KG}(n,2)) = \chi_=^*(\mathsf{KG}(n,2)) = \begin{cases} n-1 & \text{if } n \ge 7, \\ n-2 & \text{if } n = 5 \text{ or } 6. \end{cases}$$

*Proof.* By Theorem 3 and Theorem 12,

$$n - 2 = \chi(\mathsf{KG}(n,2)) \le \chi_=(\mathsf{KG}(n,2)) \le \chi_=^*(\mathsf{KG}(n,2)) \le n - 1.$$

By direct computation, $\left\lfloor \dfrac{C(n,2)}{n-2} \right\rfloor > \alpha_2(n,2) = 3$ if and only if $n \ge 7$. Hence, by Theorem 14, $\chi_=(\mathsf{KG}(n,2)) = \chi_=^*(\mathsf{KG}(n,2)) = n - 1$ if $n \ge 7$.

For convenience, we use $ij$ to denote the 2-subset $\{i, j\}$. It is easy to see that the partition $(\{12, 13, 14, 15\}, \{23, 24, 25\}, \{34, 35, 45\})$ forms an equitable 3-coloring of $\mathsf{KG}(5,2)$ and the partition $(\{12, 14, 15, 16\}, \{23, 24, 25, 26\}, \{13, 34, 35, 36\}, \{45, 46, 56\})$ forms an equitable 4-coloring of $\mathsf{KG}(6,2)$. Hence, $\chi(\mathsf{KG}(n,2)) = \chi_=(\mathsf{KG}(n,2)) = \chi_=^*(\mathsf{KG}(n,2)) = n - 2$ if $5 \le n \le 6$. ■

**Lemma 17.** *For $7 \le n \le 15$, $\chi_=(\mathsf{KG}(n,3)) \le \chi_=^*(\mathsf{KG}(n,3)) \le n - 3$. Moreover, $\chi_=(\mathsf{KG}(n,3)) = \chi_=^*(\mathsf{KG}(n,3)) = n - 3$ if $14 \le n \le 15$.*

*Proof.* Let $\mathcal{H} = \{A \in \binom{[n]}{3} : |A \cap \{n-2, n-1, n\}| \ge 2\}$. Then $\binom{[n-3,n]}{3} \subseteq \mathcal{H}$ and $|\mathcal{H}| = 3n - 8 \ge \left\lfloor \dfrac{C(n,3)}{n-3} \right\rfloor \ge 4$ for $n \le 15$. Note that if $A \notin \mathcal{H}$, then $A$ is in some $i$-flower, $1 \le i \le n - 4$. Let $\mathcal{F} = \bigcup_{i=1}^{n-4} (\{A \in \binom{[n]}{3} : i \in A\} \setminus \mathcal{H})$

and $\mathcal{G}_t = \{\{i,j,t\} : 1 \le i < j \le n-4\}$ for $n-3 \le t \le n$. Then $\mathcal{F} = \binom{[n-4]}{3} \cup (\bigcup_{t=n-3}^{n} \mathcal{G}_t)$. By Lemma 15, $\binom{[n-4]}{3}$ can be partitioned equitably into $n-4$ subcollections $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{n-4}$ such that each $\mathcal{X}_i$ is an $i$-flower. Since $\{A \setminus \{t\} : A \in \mathcal{G}_t\} = \binom{[n-4]}{2}$ for $n-3 \le t \le n$, by Lemma 15, $\mathcal{G}_t$ can be partitioned equitably into $n-4$ subcollections $\mathcal{X}_{1,t}, \mathcal{X}_{2,t}, \ldots, \mathcal{X}_{n-4,t}$ such that each $\mathcal{X}_{i,t}$ is an $i$-flower. By adjusting the sizes of $\mathcal{X}_i$ and $\mathcal{X}_{i,t}$, $\mathcal{F}$ can be partitioned equitably into $n-4$ subcollections $\mathcal{V}_i = \mathcal{X}_i \cup (\bigcup_{t=n-3}^{n} \mathcal{X}_{i,t})$, $1 \le i \le n-4$ such that each $\mathcal{V}_i$ is an $i$-flower.

It is easy to see that the set $\{i,s,t\} \in \mathcal{H}$ for $1 \le i \le n-4$ and $n-2 \le s < t \le n$. For each pair $(s,t)$, remove the sets $\{i,s,t\}$ from $\mathcal{H}$ and add them one by one into $\mathcal{V}_i$ to obtain new $\mathcal{V}'_i$, respectively, and preserve the equality of sizes of $\mathcal{V}'_i$'s. Continuing this process, $\mathcal{H}$ can be reduced to $\mathcal{H}'$ such that $|\mathcal{H}'| = \left\lfloor \frac{C(n,3)}{n-3} \right\rfloor$. In this case, the $\mathcal{V}'_i$'s satisfy $||\mathcal{V}'_i| - |\mathcal{V}'_j|| \le 1$. Hence, the partition $(\mathcal{V}'_1, \mathcal{V}'_2, \ldots, \mathcal{V}'_{n-4}, \mathcal{H}')$ forms an equitable $(n-3)$-coloring of $\mathsf{KG}(n,3)$. Therefore, $\chi_=(\mathsf{KG}(n,3)) \le \chi^*_=(\mathsf{KG}(n,3)) \le n-3$ for $7 \le n \le 15$.

Moreover, since $\left\lfloor \frac{C(n,3)}{n-4} \right\rfloor > \alpha_2(n,3) = 3n-8$ if and only if $n \ge 14$, by Lemma 13, $\chi^*_=(\mathsf{KG}(n,3)) \ge \chi_=(\mathsf{KG}(n,3)) \ge n-3$ if $n \ge 14$. Therefore, $\chi_=(\mathsf{KG}(n,3)) = \chi^*_=(\mathsf{KG}(n,3)) = n-3$ for $14 \le n \le 15$. ■

**Lemma 18.** *For* $7 \le n \le 13$, $\chi(\mathsf{KG}(n,3)) = \chi_=(\mathsf{KG}(n,3)) = \chi^*_=(\mathsf{KG}(n,3)) = n-4$.

*Proof.* By Theorem 3 and Lemma 17,

$$n-4 = \chi(\mathsf{KG}(n,3)) \le \chi_=(\mathsf{KG}(n,3)) \le \chi^*_=(\mathsf{KG}(n,3)) \le n-3.$$

It suffices to show that $\mathsf{KG}(n,3)$ is equitably $(n-4)$-colorable for $7 \le n \le 13$.

Let $\mathcal{H}_1 = \{A \in \binom{[n]}{3} : |A \cap \{n-2, n-1, n\}| \ge 2\}$ and $\mathcal{H}_2 = \{A \in \binom{[n]}{3} : |A \cap \{n-5, n-4, n-3\}| \ge 2\}$. Then $|\mathcal{H}_1| = |\mathcal{H}_2| = 3n-8 \ge \left\lfloor \frac{C(n,3)}{n-4} \right\rfloor \ge \frac{1}{2}\left|\binom{[n-5,n]}{3}\right| = \frac{C(6,3)}{2} = 10$ for $7 \le n \le 13$. By the same argument as in Lemma 17, $\mathcal{H}_1$ and $\mathcal{H}_2$ can be reduced to $\mathcal{H}'_1$ and $\mathcal{H}'_2$ such that $|\mathcal{H}'_1| = \left\lceil \frac{C(n,3)-(n-5)+1}{n-4} \right\rceil$ and $|\mathcal{H}'_2| = \left\lceil \frac{C(n,3)-(n-4)+1}{n-4} \right\rceil$. Moreover, $\binom{[n]}{3} \setminus (\mathcal{H}'_1 \cup \mathcal{H}'_2)$ can be partitioned equitably into $n-6$ subcollections $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n-6}$ such that each $\mathcal{V}_i$ is an $i$-flower and $|\mathcal{V}_i| = \left\lceil \frac{C(n,3)-i+1}{n-4} \right\rceil$. Hence, $\mathsf{KG}(n,3)$ is equitably $(n-4)$-colorable. Therefore, $\chi(\mathsf{KG}(n,3)) = \chi_=(\mathsf{KG}(n,3)) = \chi_=^*(\mathsf{KG}(n,3)) = n-4$ for $7 \le n \le 13$. ■

**Theorem 19.** *For $n \ge 7$,*

$$\chi_=(\mathsf{KG}(n,3)) = \chi_=^*(\mathsf{KG}(n,3)) = \begin{cases} n-2 & \text{if } n \ge 16, \\ n-3 & \text{if } 14 \le n \le 15, \\ n-4 & \text{if } 7 \le n \le 13. \end{cases}$$

*Proof.* By Theorem 3 and Theorem 12,

$$n-4 = \chi(\mathsf{KG}(n,3)) \le \chi_=(\mathsf{KG}(n,3)) \le \chi_=^*(\mathsf{KG}(n,3)) \le n-2.$$

Since $\left\lfloor \dfrac{C(n,3)}{n-3} \right\rfloor > \alpha_2(n,3) = 3n-8$ if and only if $n \ge 16$, by Theorem 14, $\chi_=(\mathsf{KG}(n,2)) = \chi_=^*(\mathsf{KG}(n,2)) = n-2$ if $n \ge 16$. The remaining two cases follow from Lemma 17 and Lemma 18. ■

## 4. The Odd Graphs

Since $O_1 = K_3$, we have $\chi(O_1) = \chi_=(O_1) = \chi_=^*(O_1) = 3$. By Theorem 16 and Theorem 19, $\chi(O_k) = \chi_=(O_k) = \chi_=^*(O_k) = 3$ for $k = 2$ or $3$. Suppose $k \ge 4$.

**Lemma 20.** *$O_k$ is equitably 3-colorable.*

*Proof.* Let $\mathcal{F}_1 = \{A : 1 \in A, 2 \notin A\}$, $\mathcal{F}_2 = \{A : 1 \notin A, 2 \in A\}$, $\mathcal{F}_{12} = \{A : 1 \in A, 2 \in A\}$ and $\mathcal{F}_3 = \{A : 1 \notin A, 2 \notin A\}$. Then $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{12}, \mathcal{F}_3)$ forms a partition(or 4-coloring) of $O_k$, $|\mathcal{F}_1| = |\mathcal{F}_2| = C(2k-1,k-1) = C(2k-1,k) = |\mathcal{F}_3|$, $|\mathcal{F}_{12}| = C(2k-1,k-2)$ and $C(2k+1,k) = 3C(2k-1,k-1) + C(2k-1,k-2)$. Let $a_i = \left\lfloor \dfrac{C(n,k)+i-1}{3} \right\rfloor$, $i = 1,2,3$ and $t = \dfrac{1}{3}C(2k+1,k) - C(2k-1,k-1) = \dfrac{1}{3}C(2k-1,k-2)$. Consider the two collections $\mathcal{H}_1 = \{A \in \mathcal{F}_3 : 3 \in A, 4 \in A\}$ and $\mathcal{H}_2 = \{A \in \mathcal{F}_{12} : |A \cap [3,4]| = 1\}$. By direct computation, $\dfrac{|\mathcal{H}_1|}{t} = \dfrac{3(k+1)k}{(2k-1)(2k-2)} > 1$ for $k \le 8$ and $\dfrac{|\mathcal{H}_2|}{t} = \dfrac{3(k-2)(k+1)}{(2k-1)(k-1)} > 1$ for $k \ge 4$. For $4 \le k \le 8$, choose $\mathcal{S} \subseteq \mathcal{H}_1$ with $|\mathcal{S}| = \lfloor t \rfloor = a_1 - C(2k-1,k-1)$ and $\mathcal{T} \subseteq \mathcal{H}_2$ with $|\mathcal{T}| = a_2 - C(2k-1,k-1)$. Let $\mathcal{S}_1 = \{A \in \mathcal{F}_1 : [3,2k+1] \setminus A \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{A \in \mathcal{F}_2 : [3,2k+1] \setminus A \in \mathcal{S}\}$. Then $|\mathcal{S}| = |\mathcal{S}_1| = |\mathcal{S}_2|$. Moreover, if $A \in \mathcal{F}_i$ where $i = 1,2$, then $A \cap B \ne \emptyset$ for all $B \in \mathcal{F}_3$ except $B = [3,2k+1] \setminus A$. Hence, $(\mathcal{F}_3 \setminus \mathcal{S}) \cup \mathcal{S}_1 \cup \mathcal{S}_2$, $(\mathcal{F}_2 \setminus \mathcal{S}_2) \cup \mathcal{S} \cup \mathcal{T}$ and $(\mathcal{F}_1 \setminus \mathcal{S}_1) \cup (\mathcal{F}_{12} \setminus \mathcal{T})$ are independent sets of sizes $a_1$, $a_2$ and $a_3$, respectively. Thus, the partition $((\mathcal{F}_1 \setminus \mathcal{S}_1) \cup (\mathcal{F}_{12} \setminus \mathcal{T}), (\mathcal{F}_2 \setminus \mathcal{S}_2) \cup \mathcal{S} \cup \mathcal{T}, (\mathcal{F}_3 \setminus \mathcal{S}) \cup \mathcal{S}_1 \cup \mathcal{S}_2)$ forms an equitable 3-coloring of $O_k$. Hence, $O_k$ is equitably 3-colorable for $4 \le k \le 8$.

Now, suppose $k \ge 9$. Consider the two collections $\mathcal{H}_3 = \{A \in \mathcal{F}_3 : |A \cap [3,5]| = 2\}$ and $\mathcal{H}_4 = \{A \in \mathcal{F}_{12} : |A \cap [3,5]| \ge 2\}$. By direct computation, $\frac{|\mathcal{H}_3|}{t} = \frac{9(k+1)k}{2(2k-1)(2k-3)} > 1$ and $\frac{|\mathcal{H}_4|}{t} = \frac{12k^3 - 63k^2 + 87k - 18}{8k^3 - 24k^2 + 22k - 6} > 1$. Choose $\mathcal{P} \subseteq \mathcal{H}_3$ with $|\mathcal{P}| = \lfloor t \rfloor = a_1 - C(2k-1, k-1)$ and $\mathcal{Q} \subseteq \mathcal{H}_4$ with $|\mathcal{Q}| = a_2 - C(2k-1, k-1)$. Let $\mathcal{P}_1 = \{A \in \mathcal{F}_1 : [3, 2k+1] \setminus A \in \mathcal{P}\}$ and $\mathcal{P}_2 = \{A \in \mathcal{F}_2 : [3, 2k+1] \setminus A \in \mathcal{P}\}$. Then $|\mathcal{P}| = |\mathcal{P}_1| = |\mathcal{P}_2|$. By the same argument as above, $(\mathcal{F}_3 \setminus \mathcal{P}) \cup \mathcal{P}_1 \cup \mathcal{P}_2$, $(\mathcal{F}_2 \setminus \mathcal{P}_2) \cup \mathcal{P} \cup \mathcal{Q}$ and $(\mathcal{F}_1 \setminus \mathcal{P}_1) \cup (\mathcal{F}_{12} \setminus \mathcal{Q})$ are independent sets of sizes $a_1$, $a_2$ and $a_3$, respectively. Thus, the partition $((\mathcal{F}_1 \setminus \mathcal{P}_1) \cup (\mathcal{F}_{12} \setminus \mathcal{Q}), (\mathcal{F}_2 \setminus \mathcal{P}_2) \cup \mathcal{P} \cup \mathcal{Q}, (\mathcal{F}_3 \setminus \mathcal{P}) \cup \mathcal{P}_1 \cup \mathcal{P}_2)$ forms an equitable 3-coloring of $O_k$. Hence, $O_k$ is equitably 3-colorable for $k \ge 9$. Therefore, we complete the proof. ■

Let $\mathcal{U} = \binom{[2k+1]}{k}$ and $\mathcal{X} = \binom{[4,2k+1]}{k}$. For $1 \le i \le 3$, let $\mathcal{F}_i = \{A \in \mathcal{U} : i \in A\}$ and $\mathcal{F}_{i0} = \{A \in \mathcal{U} : |A \cap \{1,2,3\}| = i\}$. For $1 \le i < j \le 3$, let $\mathcal{F}_{ij} = \{A \in \mathcal{U} : A \cap \{1,2,3\} = \{i,j\}\}$. Let $\mathcal{F}_{123} = \{A \in \mathcal{U} : \{1,2,3\} \subseteq A\}$. Then $\mathcal{U} = (\bigcup_{i=1}^{3} \mathcal{F}_{i0}) \cup (\bigcup_{1 \le i < j \le 3} \mathcal{F}_{ij}) \cup \mathcal{F}_{123} \cup \mathcal{X}$, $\mathcal{F}_i = \mathcal{F}_{i0} \cup \mathcal{F}_{is} \cup \mathcal{F}_{it} \cup \mathcal{F}_{123}$, $\{i,s,t\} = \{1,2,3\}$, $|\mathcal{X}| = C(2k-2, k)$, $|\mathcal{F}_{i0}| = C(2k-2, k-1)$, $|\mathcal{F}_{ij}| = C(2k-2, k-2)$ and $|\mathcal{F}_{123}| = C(2k-2, k-3)$. It is not difficult to see that $\mathcal{X} \cup \mathcal{F}_{i0}$ is an independent set. If $A$ and $B$ both are in $\mathcal{F}_{i0}$, then $|A \cap B| \ge 2$ except $(A \setminus \{i\}) \cup (B \setminus \{j\}) = [4, 2k+1]$. Hence, each $\mathcal{F}_{i0}$ can be partitioned into $\mathcal{S}_i$ and $\mathcal{T}_i$ such that if $A \in \mathcal{S}_i$, then $([4, 2k+1] \setminus A) \cup \{i\} \in \mathcal{T}_i$. Moreover, we may assume that $\{A \setminus \{i\} : A \in \mathcal{S}_i\} = \{A \setminus \{j\} : A \in \mathcal{S}_j\}$ for $1 \le i < j \le 3$. Hence, $|\mathcal{S}_i| = |\mathcal{S}_j| = |\mathcal{T}_i| = |\mathcal{T}_j| = \frac{|\mathcal{F}_{i0}|}{2}$ and $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{X}$ is an independent set. By direct computation, we have the following.

(I1) $|\mathcal{X}| < \frac{|\mathcal{U}|}{m} < |\mathcal{X} \cup \mathcal{S}_i \cup \mathcal{S}_j|$ if $4 \le m \le 7$.

(I2) $\frac{1}{6}|\mathcal{U}| < |\mathcal{X} \cup \mathcal{S}_i| < \frac{2}{6}|\mathcal{U}| \le |\mathcal{F}_i \setminus \mathcal{F}_{123}| \le \frac{2}{5}|\mathcal{U}| \le |\mathcal{F}_i|$.

The inequalities (I1) and (I2) will be used to guarantee that there are $\mathcal{P}_i \subseteq \mathcal{S}_i$ ($\mathcal{P}_i$ may be empty) for $1 \le i \le 3$ such that $|\mathcal{X} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3| = \left\lfloor \frac{|\mathcal{U}|}{m} \right\rfloor$ for $4 \le m \le 7$. Then we can partition $\bigcup_{i=1}^{3} (\mathcal{F}_i \setminus \mathcal{P}_i)$ equitably into $m-1$ subcollections so that $O_k$ is equitably $m$-colorable.

**Theorem 21.** $\chi(O_k) = \chi_=(O_k) = \chi_=^*(O_k) = 3$ *for* $k \geq 1$.

*Proof.* If $k = 1, 2$ or 3, then we are done. Suppose $k \geq 4$. By Lemma 20, $O_k$ is equitably 3-colorable. It suffices to show that $O_k$ is equitably $m$-colorable for all $m \geq 4$.

For $m = 4$, by (I1), we may choose $\mathcal{P}_i \subseteq \mathcal{S}_i$, $1 \leq i \leq 3$, such that $||\mathcal{P}_1| - |\mathcal{P}_2|| \leq 1$ and $|\mathcal{X} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3| = \left\lfloor \frac{|\mathcal{U}|}{4} \right\rfloor$. Partition $\mathcal{F}_{123}$ into three subcollections $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ such that $||\mathcal{R}_i| - |\mathcal{R}_j|| \leq 1$ and $|(\mathcal{F}_{i0} \setminus \mathcal{S}_i) \cup \mathcal{F}_{i,i+1} \cup \mathcal{R}_i| = \left\lceil \frac{|\mathcal{U}| - i + 1}{4} \right\rceil$ for $1 \leq i \leq 3$. Note that $\mathcal{F}_{34} = \mathcal{F}_{13}$. Hence, $O_k$ is equitably 4-colorable.

For $m = 5$, by (I1), we may choose $\mathcal{P}_i \subseteq \mathcal{S}_i$, $2 \leq i \leq 3$, such that $||\mathcal{P}_2| - |\mathcal{P}_3|| \leq 1$ and $|\mathcal{X} \cup \mathcal{P}_2 \cup \mathcal{P}_3| = \left\lfloor \frac{|\mathcal{U}|}{5} \right\rfloor$. By (I2), we may choose $\mathcal{R} \subseteq \mathcal{F}_{123}$ such that $(\mathcal{F}_1 \setminus \mathcal{F}_{123}) \cup \mathcal{R} = \mathcal{V}_1 \cup \mathcal{V}_2$ with $|\mathcal{V}_1| = \left\lfloor \frac{|\mathcal{U}| + 4}{5} \right\rfloor$ and $|\mathcal{V}_1| = \left\lfloor \frac{|\mathcal{U}| + 3}{5} \right\rfloor$. It can be done since $(\mathcal{F}_1 \setminus \mathcal{F}_{123}) \cup \mathcal{R}$ is a 1-flower. Partition $(\mathcal{F}_{123} \setminus \mathcal{R}) \cup \mathcal{F}_{23}$ into two subcollections $\mathcal{R}_2$ and $\mathcal{R}_3$ such that $||\mathcal{R}_2| - |\mathcal{R}_3|| \leq 1$ and $|(\mathcal{F}_{i0} \setminus \mathcal{P}_i) \cup \mathcal{R}_i| = \left\lfloor \frac{|\mathcal{U}| + i - 1}{5} \right\rfloor$ for $2 \leq i \leq 3$. Hence, $O_k$ is equitably 5-colorable.

For $m = 6$, by (I2), we may choose $\mathcal{P}_3 \subseteq \mathcal{S}_3$ such that $|\mathcal{X} \cup \mathcal{P}_3| = \left\lfloor \frac{|\mathcal{U}|}{6} \right\rfloor$ and choose $\mathcal{Q}_1 \subseteq \mathcal{F}_{13}$ and $\mathcal{Q}_2 \subseteq \mathcal{F}_{23}$ such that $||\mathcal{Q}_1| - |\mathcal{Q}_2|| \leq 1$ and $|\mathcal{F}_3 \setminus (\mathcal{P}_3 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \mathcal{F}_{123})| = \left\lfloor \frac{|\mathcal{U}| + 1}{6} \right\rfloor$. Partition $\mathcal{F}_{12} \cup \mathcal{F}_{123}$ into two subcollections $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $||\mathcal{R}_1| - |\mathcal{R}_2|| \leq 1$ and $||\mathcal{F}_{10} \cup \mathcal{Q}_1 \cup \mathcal{R}_1| - |\mathcal{F}_{20} \cup \mathcal{Q}_2 \cup \mathcal{R}_2|| \leq 1$. Since $\mathcal{F}_{i0} \cup \mathcal{Q}_i \cup \mathcal{R}_i$ is an $i$-flower, it can be partitioned into $\mathcal{V}_{i,1}$ and $\mathcal{V}_{i,2}$ such that $|\mathcal{V}_{i,j}| = \left\lfloor \frac{|\mathcal{U}| + 8 - 2i - j}{6} \right\rfloor$ for $1 \leq i \leq 2$ and $1 \leq j \leq 2$. Hence, $O_k$ is equitably 6-colorable.

For $m = 7$, by (I1), we may choose $\mathcal{P}_i \subseteq \mathcal{S}_i$, $1 \leq i \leq 3$, such that $||\mathcal{P}_i| - |\mathcal{P}_j|| \leq 1$ and $|\mathcal{X} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3| = \left\lfloor \frac{|\mathcal{U}|}{7} \right\rfloor$. Partition $\mathcal{F}_{123}$ into three subcollections $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ such that $||\mathcal{R}_i| - |\mathcal{R}_j|| \leq 1$ and $||(\mathcal{F}_{i0} \setminus \mathcal{P}_i) \cup \mathcal{F}_{i,i+1} \cup \mathcal{R}_i| - |(\mathcal{F}_{j0} \setminus \mathcal{P}_j) \cup \mathcal{F}_{j,j+1} \cup \mathcal{R}_j|| \leq 1$. Note that $\mathcal{F}_{34} = \mathcal{F}_{13}$. Since each $(\mathcal{F}_{i0} \setminus \mathcal{P}_i) \cup \mathcal{F}_{i,i+1} \cup \mathcal{R}_i$ is an $i$-flower, it can partitioned into $\mathcal{V}_{i,1}$ and $\mathcal{V}_{i,2}$ such that $|\mathcal{V}_{i,j}| = \left\lfloor \frac{|\mathcal{U}| + 9 - 2i - j}{7} \right\rfloor$ for $1 \leq i \leq 3$ and $1 \leq j \leq 2$. Hence, $O_k$ is equitably 7-colorable.

From the foregoing argument, there are $\mathcal{P}_i \subseteq \mathcal{F}_i$ such that $|\mathcal{X} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3| = \left\lfloor \frac{|\mathcal{U}|}{m} \right\rfloor$ and $\mathcal{F}_i \setminus \mathcal{P}_i = \mathcal{V}_{i,1} \cup \mathcal{V}_{i,2}$ ($\mathcal{V}_{i,2}$ may be empty) with $|\mathcal{V}_{i,j}| = \left\lfloor \frac{|\mathcal{U}|}{m} \right\rfloor$ or $\left\lceil \frac{|\mathcal{U}|}{m} \right\rceil$

for $4 \le m \le 7$. Now, for $t \ge 1$, we can partition $\mathcal{X} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ into $t+1$ subcollections, partition $\mathcal{F}_i \setminus \mathcal{P}_i$ into $t+1$ or $t+2$ (if $\mathcal{V}_{i,2}$ is not empty) subcollections such that all of the subcollections are of size $\left\lfloor \frac{|\mathcal{U}|}{m+4t} \right\rfloor$ or $\left\lceil \frac{|\mathcal{U}|}{m+4t} \right\rceil$. Hence, $O_k$ is equitably $(m+4t)$-colorable. Therefore, we complete the proof. ■

## 5. A Conjecture

In this paper, we have shown that $\chi_=(\mathsf{KG}(n,k)) \le \chi_=^*(\mathsf{KG}(n,k)) \le n-k+1$ and $\chi(O_k) = \chi_=(O_k) = \chi_=^*(O_k) = 3$. We have also shown that $\chi_=(\mathsf{KG}(n,k)) = \chi_=^*(\mathsf{KG}(n,k))$ for $k = 2$ or $3$ and obtained their exact values. We conclude this paper by posing the following conjecture.

**Conjecture 3.** $\chi_=(\mathsf{KG}(n,k)) = \chi_=^*(\mathsf{KG}(n,k))$ for $k \ge 2$.

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