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A SPECIAL IDENTITY OF (α, β)-DERIVATIONS AND ITS CONSEQUENCES

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Abstract. In this note, we shall give a description of (α, β)-derivations δ, g and h of a prime ring R satisfying $\delta(x) = ag(x) + h(x)b$ for all $x \in U$, where a and b are some fixed noncentral elements of R and U a nonzero ideal of R. This result generalizes some known results.

Recently, Brešar [2] proved a theorem which generalizes a result in Herstein's paper [7]. Indeed, he gave a description of derivations d, g and h of a prime ring R satisfying $d(x) = ag(x) + h(x)b,\ x \in R$, where a and b are some fixed noncentral elernents in R. In [1], Aydin-Kaya proved that if d is a nonzero (α, β)-derivation of a prime ring, U is an ideal of R and $a \in R$ such that $\alpha(a)d(u) - d(u)\beta(a) = 0$ for all $u \in U$, then $a \in Z$ provided that char $R \neq 2$. In [3], the author proved that if $\delta \neq 0$ is an (α, β)-derivation of a prime ring R and if $a \in R$ is such that $[a, \delta(R)] = 0$, then (i) $a \in Z$ provided that char $R \neq 2$ and $\alpha\delta = \delta\alpha$, $\beta\delta = \delta\beta$; (ii) $a^2 + \eta a \in C$ for some $\eta \in C$ provided that char $R = 2$ and $\alpha\delta = \delta\alpha$ (or $\beta\delta = \delta\beta$). Moreover, if $a \notin Z$, then there exists an invertible element b in Q and $\lambda \in C$ such that $\delta(x) = \lambda[a, x]b$ for all $x \in R$.

In this note, we shall give a description of (α, β)-derivations δ, g and h of a prime ring R satisfying $\delta(x) = ag(x) + h(x)b$ for all $x \in U$, where a and b are some fixed noncentral elernents of R and U a nonzero ideal of R. This result generalizes those results mentioned above simultaneously.

Throughout, R will be a prime ring with center Z. Q will denote the symmetric Martindale quotient ring of R and C will be the extended centroid of R. α, β will be automorphisms of R. In the following, the (α, β)-derivation f of U into R means $f(x+y) = f(x) + f(y)$ and $f(xy) = \alpha(x)f(y) + f(x)\beta(y)$ for all $x, y \in U$.

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We begin with a lemma.

Lemma 1. *Let R be a prime ring and U a nonzero ideal of R. Let g and h be (α, β)-derivations of U into R, f an (α, α)-derivation of U into R and k an (β, β)-derivation of U into R. Suppose that*

$$f(x)g(y) = h(x)k(y) \quad \text{for all } x, y \in U. \tag{1}$$

Then there exists an invertible element $u \in Q$ such that $\beta(x) = u^{-1}\,\alpha(x)u, h(x) = f(x)u$ and $g(x) = uk(x)$ for all $x \in U$.

Proof. From (1) it follows that $g = 0$ if and only if $h = 0$. Hence if either $g = 0$ or $h = 0$, then the result follows. So we may assume that both g and h are not 0. Substituting zy for y in (1), we obtain $f(x)\alpha(z)g(y) + f(x)g(z)\beta(y) = h(x)\beta(z)k(y) + h(x)k(z)\beta(y)$. Applying (1) we then get

$$f(x)\alpha(z)g(y) = h(x)\beta(z)k(y) \quad \text{for all } x, y, z, \in U \tag{2}$$

Applying α^{-1} on both sides of (2) and let $f_1 = \alpha^{-1}f, g_1 = \alpha^{-1}g, h_1 = \alpha^{-1}h, k_1 = \alpha^{-1}k$, and $\alpha_1 = \alpha^{-1}\beta$, then we have

$$f_1(x)zg_1(y) = h_1(x)\alpha_1(z)k_1(y) \quad \text{for all } x, y, z \in U, \tag{3}$$

where f_1 is a derivation, g_1 and h_1 are $(1, \alpha_1)$-derivation and k_1 is an (α_1, α_1)-derivation. Since $f \neq 0, k \neq 0, g \neq 0$ and $h \neq 0$, it follows that f, g, h and k are not 0 on U. Therefore there exists $x_0, y_0 \in U$ such that $f(x_0) \neq 0, h(x_0) \neq 0$ and $g(y_0) \neq 0, k(y_0) \neq 0$. Consequently, $f_1(x_0) \neq 0, h_1(x_0) \neq 0$ and $g_1(y_0) \neq 0, k_1(y_0) \neq 0$. By Kharchenko's result [8], (3) cannot hold for such x_0 and y_0 unless α_1 is Q-inner. This says, there exists $s \in Q$ such that $\alpha_1(x) = s^{-1}xs$ for all $x \in R$. Then (3) becomes

$$f_1(x)zg_1(y) = h_1(x)s^{-1}zsk_1(y) \quad \text{for all } x, y, z \in U,$$

and thus

$$f_1(x)zg_1(y)s^{-1} = h_1(x)s^{-1}zsk_1(y)s^{-1} \quad \text{for all } x, y, z \in U$$

Note that $f_1, g_1s^{-1}, h_1s^{-1}$ and sk_1s^{-1} are derivations of Q which maps some nonzero ideal I of R into R. Let $K = I \cap U$, then K is a nonzero ideal of R and we have

$$f_1(x)zg_1(y)s^{-1} = h_1(x)s^{-1}zsk_1(y)s^{-1} \quad \textit{for all } x, y, z \in K. \tag{4}$$

Substituting $zsk_1(\omega)s^{-1}$ for z in (4), where $z, \omega \in K$, we get $f_1(x)zsk_1(\omega)s^{-1}g_1(y)s^{-1} = h_1(x)s^{-1}zsk_1(\omega)s^{-1}sk_1(y)s^{-1}$. By (4), $h_1(x)s^{-1}zsk_1(\omega)s^{-1} =$

$f_1(x)zg_1(\omega)s^{-1}$ and so we have $f_1(x)z[sk_1(\omega)s^{-1}g_1(y)s^{-1}-g_1(\omega)s^{-1}sk_1(y)s^{-1}] = 0$. Since $f_1 \neq 0$ on K and R is prime, we have $sk_1(\omega)s^{-1}g_1(y)s^{-1} = g_1(\omega)s^{-1} sk_1(y)s^{-1}$ for all $\omega, y \in K$. Note that Lemma 2.2 in [2] still holds if one replaces the condition given there by the following one

$$d(x)g(y) = g(x)d(y) \quad \text{for all } x, y \in K,$$

K a nonzero ideal of R. By this result, we see that there exists $\eta \in C$ such that $g_1(\omega)s^{-1} = \eta sk_1(\omega)s^{-1}$ for all $\omega \in K$. Hence $\eta f_1(x)zsk_1(\omega)s^{-1} = h_1(x)s^{-1}zsk_1(\omega)s^{-1}$ and $(\eta f_1(x) - h_1(x)s^{-1})zsk_1(\omega)s^{-1} = 0$ for all $x, z, w \in K$. Therefore, $h_1(x)s^{-1} = \eta f_1(x)$ for all $x \in K$ and hence $h_1(x)s^{-1} = \eta f_1(x)$ for all $x \in R$. Thus $h(x)t^{-1} = \lambda f(x)$ for $x \in R$, where $t = \alpha(s), \lambda = \alpha(\eta)$. Similarly, from $g_1(\omega)s^{-1} = \eta sk_1(\omega)s^{-1}$ we get $t^{-1}g(x) = \lambda k(x)$ for all $x \in R$. Also, $t^{-1}\alpha(x)t = \alpha(s)^{-1}\alpha(x)\alpha(s) = \alpha(s^{-1}xs) = \alpha(\alpha^{-1}\beta(x)) = \beta(x)$. Now, set $u = \lambda t$, we obtain our lemma.

Now we are ready to prove our main result.

Theorem 1. *Let R be a prime ring, U a nonzero ideal of R, Q the symmetric Martindale quotient ring of R and C the extended centroid of R. Further, let δ, g and h be (α,β)-derivations of U into R and $a, b \in Q \setminus C$. Suppose that either $g \neq 0$ or $h \neq 0$. Then the following conditions are equivalent:*

(i) $\delta(x) = ag(x) + h(x)b$ *for all* $x \in U$.

(ii) *There exists an invertible element $s \in Q$ such that*

$$\begin{aligned}
\beta(x) &= s^{-1}\alpha(x)s,\\
\delta(x) &= [asbs^{-1}, \alpha(x)]s,\\
g(x) &= s[b, \beta(x)] = [sbs^{-1}, \alpha(x)]s,\\
h(x) &= [a, \alpha(x)]s
\end{aligned}$$

for all $x \in U$.

Proof. It is easy to see that (ii) implies (i). So we only need to show that (i) implies (ii). Asumme (i) holds. Replacing x by xy in (i), we have

$$\begin{aligned}
&a\alpha(x)g(y) + ag(x)\beta(y) + \alpha(x)h(y)b + h(x)\beta(y)b\\
&\quad = ag(xy) + h(xy)b = \delta(xy) = \alpha(x)\delta(y) + \delta(x)\beta(y)\\
&\quad = \alpha(x)ag(y) + \alpha(x)h(y)b + ag(x)\beta(y) + h(x)b\beta(y).
\end{aligned}$$

Hence

$$[a, \alpha(x)]g(y) = h(x)[b, \beta(y)] \quad \text{for all } x, y \in U.$$

Since $a \notin C$ and $b \notin C$, without loss of generality, we may assume that $f(x) = [a, \alpha(x)]$ is a nonzero (α, α)-derivation of U into R and $k(y) = [b, \beta(y)]$ is a nonzero (β, β)-derivation of U into R. By Lemma 1, there exists an invertible element $s \in Q$ such that $g(x) = s[b, \beta(x)], h(x) = [a, \alpha(x)]s$ and $s^{-1}\alpha(x)s = \beta(x)$ for all $x \in U$. Substituting these into (i) we have

$$\begin{aligned} \delta(x) &= ag(x) + h(x)b \\ &= as[b, \beta(x)] + [a, \alpha(x)]sb \\ &= as[b, \beta(x)]s^{-1}s + [a, \alpha(x)]sbs^{-1}s \\ &= (a[sbs^{-1}, s\beta(x)s^{-1}] + [a, \alpha(x)]sbs^{-1})s \\ &= (a[sbs^{-1}, \alpha(x)] + [a, \alpha(x)]sbs^{-1})s \\ &= [asbs^{-1}, \alpha(x)]s \end{aligned}$$

So $\delta(x) = [asbs^{-1}, \alpha(x)]s$ for all $x \in U$.

The first corollary of this theorem is to generalize Theorem 1 in [3] mentioned in the introduction.

Corollary. *Let R be a prime ring, U a nonzero ideal of R, g and h (α, β)-derivations of U into R and $a, b \in Q \setminus C$. Suppose that either $g \neq 0$ or $h \neq 0$. Then the following conditions are equivalent:*

(i) *$ag(x) + h(x)b = 0$ for all $x \in U$.*

(ii) *There exists an invertible element $s \in Q$ such that*

$$\begin{aligned} &\beta(x) = s^{-1}\alpha(x)s, \\ &g(x) = s[b, \beta(x)] = [sbs^{-1}, \alpha(x)]s, \\ &h(x) = [a, \alpha(x)]s, \\ &\alpha^{-1}(a)\beta^{-1}(b) \in C \end{aligned}$$

for all $x \in U$.

Proof. (i)$\Longrightarrow$(ii). The first part follows immediately from Theorem 1. Setting $\delta(x) = ag(x) + h(x)b, x \in U$, we see that $\delta = 0$ and so $asbs^{-1} \in C$ by Theorem 1. As $\beta(x) = s^{-1}\alpha(x)s$, $\beta^{-1}(x) = \alpha^{-1}(sxs^{-1})$ for all $x \in U$. Therefore

$$\alpha^{-1}(a)\beta^{-1}(b) = \alpha^{-1}(asbs^{-1}) \in C$$

The inverse implication is obvious.

Theorem 2. *Let R be a prime ring, U a nonzero ideal of R, Q the symmetric Martindale quotient ring of R and C the extended centroid of R. Further, let δ: $U \to R$ be an (α,β)-derivation and $a \in Q\backslash C$. Then the following conditions are equivalent:*

(i) $[a,\delta(x)] = 0$ *for all* $x \in U$.

(ii) *There exists an invertible element* $s \in Q$ *such that*

(*a*) $\delta(x) = [a,\alpha(x)]s$ *for all* $x \in U$;

(*b*) $\alpha^{-1}(a) + \beta^{-1}(a) \in C$ ($\tau = a + sas^{-1} \in C$ *equivalently*);

(*c*) $\alpha^{-1}(a)\beta^{-1}(a) \in C$ ($\mu = asas^{-1} \in C$ *equivalently*);

(*d*) $a^2 - \tau a + \mu = 0$.(*In particular, if char* $R = 2$, *then* $\tau = [a,s]s^{-1}$).

Proof. (i)$\Longrightarrow$(ii). Since $a\delta(x) - \delta(x)a = [a,\delta(x)] = 0$ for all $x \in U$, we can appeal to Corollary 1 to conclude that $\alpha^{-1}(a)\beta^{-1}(a) \in C$ and $s[-a,\beta(x)] = \delta(x) = [a,\alpha(x)]s$ for all $x \in U$. The first part implies $\mu = asas^{-1} \in C$ and the last part, as before, implies

$$[a + sas^{-1},\alpha(x)]s = 0$$

for all $x \in U$. Since s is invertible in Q, we get $[a + sas^{-1},\alpha(x)] = 0$ for all $x \in U$. Therefore, $a + sas^{-1} \in C$. But again, $sas^{-1} = \alpha\beta^{-1}(a)$, so $a + \alpha\beta^{-1}(a) \in C$ and hence $\alpha^{-1}(a) + \beta^{-1}(a) \in C$. Put $\tau = a + sas^{-1}$. Then $a^2 - \tau a = a^2 - a(a + sas^{-1}) = a^2 - a^2 - asas^{-1} = -asas^{-1} = -\mu$ and hence $a^2 - \tau a + \mu = 0$. In particular, if char $R = 2$, then $\tau = a + sas^{-1} = (as - sa)s^{-1} = [a,s]s^{-1}$.

The inverse implication is obvious.

Remarks:

1. In Theorem 2 we don't need to assume any commutativity between δ and α,β as it did in [3] mentioned above.
2. Example 1 in [3] shows that $a\beta^{-1}(a) = 0 \in Z$ and $a + \beta^{-1}(a) = I_{2k} \in Z$. Moreover, $a^2 - a = 0 \in Z$.
3. If, in addition, we assume that δ commutes with α and β, then it is easy to show that $\beta^{-1}\alpha(a) - a \in C$. But then we have $2a \in C$. If char $\neq 2$, then $a \in C$, which is contrary to the assumption that $a \notin C$. So char $R = 2$.

The following result is a generalization of Theorem 1 in [1]. Before statting this theorem, we set $[x, y]_{\alpha,\beta} = \alpha(x)y - y\beta(x)$.

Theorem 3. *Let R be a prime ring, U a nonzero ideal of R, Q the symmetric Martindale quotient ring of R and C the extended centroid of R. Further, let δ be an (α, β)-derivation of U into R and $a \in Q\backslash C$. Then the following conditions are equivalent:*

(i) $[x, y]_{\alpha,\beta} = 0$ *for all* $x \in U$.

(ii) char $R = 2$, $a^2 \in C$ *and* $\alpha(a^2) = \beta(a^2)$. *Moreover, there exists an invertible element $t \in Q$ such that $\beta(x) = t^{-1}\alpha(x)t$ and $\delta(x) = [\alpha(a), \alpha(x)]t$ for all $x \in U$.*

Proof. (i)$\Longrightarrow$ (ii). Setting $h(x) = \alpha^{-1}(\delta(x)), x \in U$, $\gamma = \alpha^{-1}\beta$, we note that $h : U \to R$ is an $(1, \gamma)$-derivation of U into R and $[a, h(x)]_{1,\gamma} = 0$ for all $x \in U$. Since $ah(x) - h(x)\gamma(a) = [a, h(x)]_{1,\gamma} = 0$, by Corollary 1, there exists an invertible element $s \in Q$ such that

$$\begin{aligned}
&\gamma(x) = s^{-1}xs,\\
&h(x) = s[-\gamma(a), \gamma(x)] = [-s\gamma(a)s^{-1}, x]s = [-a, x]s,\\
&h(x) = [a, x]s,\\
&-a^2 = a\gamma^{-1}(-\gamma(a)) \in C
\end{aligned}$$

for all $x \in U$. It follows that $a^2 \in C$ and $[a + a, x] = [2a, x] = 0$ for all $x \in U$. If char $R \neq 2$, then $a \in C$ which is not the case. Therefore, char $R = 2$. Since

$$a^2h(x) = a(ah(x)) = a(h(x)\gamma(a)) = (ah(x))\gamma(a) = h(x)\gamma(a^2)$$

for all $x \in R$ and since $a^2 \in C$, it follows that $h(x)(a^2 - \gamma(a^2)) = 0$ for all $x \in U$ and so $a^2 = \gamma(a^2)$. Therefore $\alpha(a^2) = \beta(a^2)$. Clearly, $\beta(x) = t^{-1}\alpha(x)t$ and $\delta(x) = [\alpha(a), \alpha(x)]t$ for all $x \in U$, where $t = \alpha(s)$.

(ii)$\Longrightarrow$(i). We have

$$[x, y]_{\alpha,\beta} = \alpha(a)[\alpha(a), \alpha(x)]t - [\alpha(a), \alpha(x)]t\beta(a).$$

Since $\beta(a) = t^{-1}\alpha(a)t, a^2 \in C$ and char $R = 2$, we see that

$$\begin{aligned}
[x, y]_{\alpha,\beta} &= \alpha(a)[\alpha(a), \alpha(x)]t - [\alpha(a), \alpha(x)]\alpha(a)t\\
&= [\alpha(a), [\alpha(a), \alpha(x)]]t = [\alpha(a^2), \alpha(x)]t\\
&= 0
\end{aligned}$$

for all $x \in U$. The proof is complete.

The next corollary generalizes Theorem 2 (ii) in [3].

Corollary. *Let R be a prime ring of characteristic 2, U a nonzero ideal of R and δ a nonzero (α,β)-derivation of R. If $[\delta(x),\delta(y)] = 0$ for all $x, y \in U$, then R is an S_4-ring.*

Proof. The argument used in [3] works as well here.

The following example shows that we can not strengthen Theorem 2(i) in [3].

Example 2. Let R be the complete matrix ring of 2×2 matrices over a field F. Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $[a,b] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Define $\beta(x) = b^{-1}xb$ and $\delta(x) = [a,x]b$ for all $x \in R$. Then δ is an $(1,\beta)$-derivation of R such that $[\delta(x),\delta(y)] = 0$ for all $x \in R$. If char $F \neq 2$, then $\beta\delta \neq \delta\beta$.

Before proving the next result, we need the following lemmas.

Lemma 2. *Let R be a prime ring, f, g be (α,β)-derivations of R. Suppose that*

$$f(x)g(y) = g(x)f(y) \quad \textit{for all } x, y \in U, \tag{4}$$

where U is a nonzero ideal of R. If $f \neq 0$, then there exists $\lambda \in C$ such that $g(x) = \lambda f(x)$ for all $x \in R$.

Proof. Substituting zy for y in (4), where $y, z \in U$, we get

$$f(x)\alpha(z)g(y) + f(x)g(z)\beta(y) = g(x)\alpha(z)f(y) + g(x)f(z)\beta(y).$$

According to (4), this relation reduces to

$$f(x)ug(y) = g(x)uf(y) \quad \text{for all } x, y \in U, \tag{5}$$

where $u = \alpha(z), z \in U$. Hence if $f(x) \neq 0$, then we have that $g(x) = \lambda(x)f(x)$ for some $\lambda(x) \in C$ by Lemma 1.3.2 in [6]. Thus if $f(x) \neq 0$ and $f(y) \neq 0$, then it follows from (5) that

$$(\lambda(x) - \lambda(y))f(x)uf(y) = 0 \quad \text{for all } u \in \alpha(U).$$

Since R is prime, this relation implies that $\lambda(x) = \lambda(y)$. Thus we have proved that there exists $\lambda \in C$ such that the relation $g(x) = \lambda f(x)$ holds for all $x \in U$

with the property $f(x) \neq 0$. On the other hand, if $f(x) = 0$, then we see from (4), since $f \neq 0$ and R is prime, that $g(x) = 0$ as well. Thus $g(x) = \lambda f(x)$ for all $x \in R$.

Lemma 3. *Let R be a prime ring, U a nonzero ideal of R, δ_1 a nonzero (α_1, β_1)-derivation of U into R and of δ_2 an (α_2, β_2)-derivation of U into R. Then the following conditions are equivalent:*

(i) $\delta_1(x) = \delta_2(x)$ *for all* $x \in U$.

(ii) *Either $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, or there exists an invertible element $u \in Q$ such that $\beta_2(x) = u^{-1}\alpha_1(x)u, \beta_1(x) = u^{-1}\alpha_2(x)u, \delta_1(x) = (\alpha_1(x) - (\alpha_2(x))u$ and $\delta_2(x) = u(\beta_2(x) - \beta_1(x))$ for all $x \in U$.*

Proof. (i)$\Longrightarrow$ (ii). For $x, y \in U$, we have $\alpha_1(x)\delta_1(y) + \delta_1(x)\beta_1(y) = \delta_1(xy) = \delta_2(xy) = \alpha_2(x)\delta_2(y) + \delta_2(x)\beta_2(y)$ and hence

$$(6) \qquad (\alpha_1(x) - \alpha_2(x))\delta_1(y) = \delta_2(x)(\beta_2(y) - \beta_1(y)) \quad \text{for all } x, y \in U.$$

Substituting yz for y into (6), we obtain $(\alpha_1(x) - \alpha_2(x))\alpha_1(y)\delta_1(z) + (\alpha_1(x) - \alpha_2(x))\delta_1(y)\beta_1(z) = \delta_2(x)\delta_2(y)(\beta_2(z) - \beta_1(z)) + \delta_2(x)(\beta_2(y) - \beta_1(y))\beta_1(z)$. Using (6) and letting $\delta = \delta = \delta_2$, we have

$$(7) \qquad \begin{aligned} &(x_1 - \alpha_1^{-1}\alpha_2(x))y\alpha_1^{-1}\delta(z) = \alpha_1^{-1}\delta(x)\alpha_1^{-1}\beta_2(y)(\alpha_1^{-1}\beta_2(z) - \alpha_1^{-1}\beta_1(z)) \\ &\text{for all } x, y, z \in U. \end{aligned}$$

If $\alpha_1^{-1}\beta_2$ is Q-outer, then by Kharchenko's result [8], we have from (7) that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. If $\alpha_1^{-1}\beta_2$ is Q-inner, then there exists an invertible element $s \in Q$ such that $\alpha_1^{-1}\beta_2(x) = s^{-1}xs$ for all $x \in R$. In this case, we have $\beta_2(x) = u^{-1}\alpha_1(x)u$ for all $x \in R$, where $u = \alpha_1(s)$. From (7), we have $(x - \alpha_1^{-1}\alpha_2(x))y\alpha_1^{-1}\delta(z) = \alpha_1^{-1}\delta(x)s^{-1}ys(\alpha_1^{-1}\beta_2(z) - \alpha_1^{-1}\beta_1(z))$ for all $x, y, z \in U$. By a similar argument we did before, there exists $\lambda \in C$ such that $x - \alpha_1^{-1}\alpha_2(x) = \lambda\alpha_1^{-1}\delta(x)s^{-1}$ and $\alpha_1^{-1}\beta_2(z) - \alpha_1^{-1}\beta_1(z) = \lambda s^{-1}\alpha_1^{-1}\delta(z)$ for all $x, z \in R$. Therefore,

$$(8) \qquad (\alpha_1(x) - \alpha_2(x))u = \xi\delta(x) = u(\beta_2(x) - \beta_1(x)) \quad \text{for all } x \in R,$$

where $\xi = \alpha_1(\lambda)$. From (8), we have $\beta_2(x) - \beta_1(x) = u^{-1}(\alpha_1(x) - \alpha_2(x))u = u^{-1}\alpha_1(x)u - u^{-1}\alpha_2(x)u = \beta_2(x) - u^{-1}\alpha_2(x)u$ and hence $\beta_1(x) = u^{-1}\alpha_2(x)u$ for all $x \in U$.

The inverse implication is obvious.

Corollary. *Let R be a commutative domain and let $0 \neq \delta_1$ be an (α_1,β_1)-derivation, $0 \neq \delta_2$ an (α_2,β_2)-derivation of R. If $\delta_1 = \delta_2$, then we have either $\alpha_1 = \alpha_2$ $\beta_1 = \beta_2$ or$\alpha_1 = \beta_2,\ \alpha_2 = \beta_1$.*

Proof. The result follows easily from Lemma 3 since $R \subset C$, the center of Q.

Lemma 4. *Let $0 \neq \delta$ be an (α,β)-derivation of a commutative domain R. If $\alpha \neq \beta$, then there exists $\lambda \in C$ such that $\delta(x) = \lambda(\alpha(x) - \beta(x))$ for all $x \in R$.*

Proof. Since R is commutative, we have $\alpha(x)\delta(y) + \delta(x)\beta(y) = \delta(xy) = \delta(yx) = \alpha(y)\delta(x) + \delta(y)\beta(x)$ for all $x, y \in R$. Therefore,

$$(\alpha(x) - \beta(x))\delta(y) = \delta(x)(\alpha(y) - \beta(y)) \quad \text{for all } x, y \in R.$$

By Lemma 2, there exists $\eta \in C$ such that $\alpha(x) - \beta(x) = \eta\delta(x)$ for all $x \in R$. If $\eta = 0$, then $\alpha(x) = \beta(x)$ for all $x \in R$. If $\eta \neq 0$, then $\delta(x) = \eta^{-1}(\alpha(x) - \beta(x)) = \lambda(\alpha(x) - \beta(x))$ for all $x \in R$.

Theorem 4. *Let R be a prime ring, U a nonzero ideal of R, and δ a nonzero (α,β)-derivation of R. If $[\delta(U),\delta(U)]_{\alpha,\beta} = 0$, then*

- (i) *if char $R \neq 2$, then R is commutative. In this case, if $\alpha \neq \beta$, then $\alpha^2 = \beta^2, \alpha\beta = \beta\alpha$ and $\delta(x) = \lambda(\alpha(x) - \beta(x))$ for some $\lambda \in C$ such that $\alpha(\lambda) + \beta(\lambda) = 0$. Also $\delta^2 = 0$.*
- (ii) *if char $R = 2$, then R is an S_4-ring.*

Proof. If char $R \neq 2$, then $\delta(U) \subset Z$ by Theorem 3. So R is commutative. Also, we have $\alpha(\delta(x)) = \beta(\delta(x))$ for all $x \in U$. By Lemma 3, we have either $\alpha^2(x) = \beta\alpha(x)$ and $\alpha\beta(x) = \beta^2(x)$ for all $x \in U$ or $\alpha^2(x) = \beta^2(x)$ and $\alpha\beta(x) = \beta\alpha(x)$ for all $x \in U$. For the earlier case, $\alpha(x) = \beta(x)$ for all $x \in \alpha(U)$ and hence $\alpha = \beta$. For the latter case, $\alpha^2 = \beta^2$ and $\alpha\beta = \beta\alpha$.

Since R is commutative, by Lemma 4, $\delta(x) = \lambda(\alpha(x) - \beta(x))$ if $\alpha \neq \beta$. Note that $\lambda \neq 0$. Assume $\alpha \neq \beta$. Since $\alpha(\delta(x)) = \beta(\delta(x))$, we have $(\alpha(\lambda) + \beta(\lambda))(\alpha^2(x) - \alpha\beta(x)) = 0$ for all $x \in U$. If $\alpha(\lambda) + \beta(\lambda) \neq 0$ then $\alpha^2(x) - \alpha\beta(x) = 0$ for all $x \in U$ and hence $\alpha(x) = \beta(x)$ for all $x \in R$ which is not the case. So $\alpha(\lambda) + \beta(\lambda) = 0$. Also, we have $\delta^2(x) = \lambda(\alpha(\delta(x)) - \beta(\delta(x))) = 0$ for all $x \in R$ and hence $\delta^2 = 0$.

If char $R = 2$, then $\delta(u)^2 \in Z$ for all $u \in U$ by Theorem 3 (ii). Now we can appeal to Theorem A in [4] to conclude that R is an S_4-ring.

Remark. Q. Deng, M. S. Yenigül and N. Argac have obtained some partial result of this theorem in [5] by a different way.

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