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KUMMER'S THEOREM AND ITS CONTIGUOUS IDENTITIES

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Abstract. Recently Lavoie, Grondin and Rathie obtained ten results closely related to the classical Kummer's theorem as special cases from generalized Whipple's theorem on the sum of a ${}_3F_2$ with unit argument. The aim of this paper is to provide general summation formulas contiguous to the Kummer's theorem by simply using a known integral representation of ${}_2F_1$. As by-product, two classes of summation formulas closely related to the Kummer's theorem were obtained. Some simplified special cases were also given for later easy use.

1. Introduction and Preliminaries

In almost all the books of special functions (*e.g.*, Bailey [1], Rainville [4]) is given the well-known and useful Kummer's theorem:

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ 1+a-b \end{array} \,\Big|\, -1 \right) = \frac{\Gamma(1+a-b)\,\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\,\Gamma\left(1+\frac{1}{2}a-b\right)} \qquad (\Re(b)<1), \tag{1.1} $$

where the denominator parameter (here, $1+a-b$) in ${}_2F_1$ (in what follows) is not a nonnegative integer. (1.1) can be obtained with the help of the following integral representation of ${}_2F_1$ (see, *e.g.*, [2, p. 114, Eq.(1)]):

$$ {}_2F_1\,(a,\, b\,;\, c\,;\, z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\,(1-zt)^{-a}\,dt \tag{1.2} $$

$$ (\Re(c) > \Re(b) > 0;\ |\arg(1-z)| < \pi) $$

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by letting $z = -1$, $c = 1 + b - a$, and using

$$\int_0^1 t^{a-1}\left(1-t^2\right)^{b-1}\,dt = \frac{\Gamma\left(\frac{1}{2}a\right)\Gamma(b)}{2\,\Gamma\left(\frac{1}{2}a+b\right)} \quad (\Re(a)>0;\ \Re(b)>0). \tag{1.3}$$

Consider the known result (see [1]):

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\ \middle|\ \frac{1}{2}\right) = 2^a\,{}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix}\ \middle|\ -1\right), \tag{1.4}$$

whose right-hand side series can be summed by means of Kummer's theorem (1.1) when $c = \frac{1}{2}(a+b+1)$ or $a+b=1$. Both cases, respectively, produce Kummer's theorems:

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+1) \end{matrix}\ \middle|\ \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{1.5}$$

and

$$ {}_2F_1\left(\begin{matrix} a,\ 1-a \\ c \end{matrix}\ \middle|\ \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}c\right)\Gamma\left(\frac{1}{2}c+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}c+\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}\right)}. \tag{1.6}$$

The Kummer's theorem (1.1) can also be obtained by letting $c \to \infty$ in the classical Dixon's theorem (see [1]):

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix}\ \middle|\ 1\right) \\ &= \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)} \\ &\qquad (\Re(a-2b-2c)>-2). \end{aligned} \tag{1.7}$$

The aim of this paper is to provide general summation formulas contiguous to the Kummer's theorem (1.1) by using a known integral representation of ${}_2F_1$. As by-product, two classes of summation formulas closely related to the Kummer's theorem (1.5) were presented. Some simplified special cases were also considered for later easy use.

2. Main Summation Formulas

We can present general summation formulas contiguous to Kummer's theorem (1.1) by simply using a known integral representation of ${}_2F_1$:

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a-b+n \end{matrix} \,\middle|\, -1\right) \\
&= \frac{\Gamma(1+a-b+n)}{\Gamma(a)\,\Gamma(1-b+n)} \int_0^1 t^{a-1}\,(1-t)^{-b+n}\,(1+t)^{-b}\,dt \\
&\qquad \left(\Re(a)>0;\ \Re(b)<1+\frac{n}{2};\ n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right),
\end{aligned} \tag{2.1}
$$

where $\mathbb{N}$ denotes (as usual) the set of positive integers. In fact, (2.1) is an obvious special case of (1.2) when $z=-1$ and $c=1+b-a+n$ by noting that a and b are interchangeable in a Gauss hypergeometric function ${}_2F_1(a,\,b\,;\,c\,;\,z)$.

When n is an integer satisfying the condition in (2.1), we see that, with the help of (1.3), the integral in (2.1) can be expressed in terms of Gamma function Γ:

$$
\begin{aligned}
\int_0^1 t^{a-1}\,(1-t)^n\,\left(1-t^2\right)^{-b}\,dt &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^1 t^{a-1+k}\,\left(1-t^2\right)^{-b}\,dt \\
&= \frac{\Gamma(1-b)}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}a-b+1+\frac{1}{2}k\right)}.
\end{aligned}
$$

We thus obtain a general summation formula contiguous to (1.1):

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a-b+n \end{matrix} \,\middle|\, -1\right) \\
&= \frac{\Gamma(1+a-b+n)\,\Gamma(1-b)}{2\,\Gamma(a)\,\Gamma(1-b+n)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}a-b+1+\frac{1}{2}k\right)} \\
&\qquad \left(\Re(b)<1+\frac{n}{2};\ n\in\mathbb{N}_0\right).
\end{aligned} \tag{2.2}
$$

When $n\in\mathbb{N}$, similarly, we also see that the integral in (2.1) can be expressed in terms of Gamma function Γ:

$$
\begin{aligned}
\int_0^1 t^{a-1}\,(1-t)^{-b-n}\,(1+t)^{-b}\,dt &= \int_0^1 t^{a-1}\,(1+t)^n\,\left(1-t^2\right)^{-b-n}\,dt \\
&= \sum_{k=0}^{n}\binom{n}{k}\int_0^1 t^{a-1+k}\,\left(1-t^2\right)^{-b-n}\,dt \\
&= \frac{\Gamma(1-b-n)}{2}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}a-b+1-n+\frac{1}{2}k\right)}.
\end{aligned}
$$

We, therefore, obtain another summation formula contiguous to (1.1):

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a-b-n \end{matrix} \;\middle|\; -1\right) \\
&= \frac{\Gamma(1+a-b-n)\,\Gamma(1-b-n)}{2\,\Gamma(a)\,\Gamma(1-b-n)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}a-b+1-n+\frac{1}{2}k\right)} \\
&\qquad \left(\Re(b) < 1-\frac{n}{2};\ n \in \mathbb{N}_0\right).
\end{aligned} \tag{2.3}
$$

As by-product, we can also provide two classes of summation formulas of ${}_2F_1$ with the argument $1/2$. It easily follows from (2.2) and (2.3) by considering (1.4), respectively, that

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, \quad 1+a-2b-n \\ 1+a-b-n \end{matrix} \;\middle|\; \tfrac{1}{2}\right) \\
&= \frac{\Gamma(1+a-b+n)\,\Gamma(1-b)}{2^{1-a}\,\Gamma(a)\,\Gamma(1-b+n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}a-b+1+\frac{1}{2}k\right)} \\
&\qquad (n \in \mathbb{N}_0)
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, \quad 1+a-2b-n \\ 1+a-b-n \end{matrix} \;\middle|\; \tfrac{1}{2}\right) \\
&= \frac{\Gamma(1+a-b-n)\,\Gamma(1-b-n)}{2^{1-a}\,\Gamma(a)\,\Gamma(1-b-n)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}a-b+1-n+\frac{1}{2}k\right)} \\
&\qquad (n \in \mathbb{N}).
\end{aligned} \tag{2.5}
$$

Note that formulas (2.4) and (2.5) always hold unless $1+a-b-n$ is a nonnegative integer.

We conclude this paper, for later easy reference, by presenting ten explicitly simplified formulas. Setting $n = 1, 2, 3, 4, 5$ in (2.2) and (2.3), and simplifying the resulting identities by mainly using the well-known functional relation for the Gamma function Γ, we obtain

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, \quad b \\ 2+a-b \end{matrix} \;\middle|\; -1\right) = \frac{\Gamma(2+a-b)\,\Gamma\left(\frac{1}{2}\right)}{2^a\,(1-b)} \\
&\qquad \cdot \left\{ \frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b+1\right)} - \frac{1}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b+\frac{3}{2}\right)} \right\} \\
&\qquad \left(\Re(b) < \frac{3}{2}\right);
\end{aligned} \tag{2.6}
$$

$$
(2.7) \quad {}_2F_1\left(\begin{matrix} a, & b \\ a-b \end{matrix} \;\middle|\; -1\right) = \frac{\Gamma(a-b)\,\Gamma\left(\frac{1}{2}\right)}{2^a} \cdot \left\{\frac{1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b\right)} + \frac{1}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}\right)}\right\} \quad \left(\Re(b) < \frac{1}{2}\right);
$$

$$
(2.8) \quad {}_2F_1\left(\begin{matrix} a, & b \\ 3+a-b \end{matrix} \;\middle|\; -1\right) = \frac{\Gamma(3+a-b)\,\Gamma\left(\frac{1}{2}\right)}{2^a\,(b-1)(b-2)} \cdot \left\{\frac{a-b+1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b+2\right)} - \frac{2}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b+\frac{3}{2}\right)}\right\} \quad (\Re(b) < 2);
$$

$$
(2.9) \quad {}_2F_1\left(\begin{matrix} a, & b \\ a-b-1 \end{matrix} \;\middle|\; -1\right) = \frac{\Gamma(a-b-1)\,\Gamma\left(\frac{1}{2}\right)}{2^a} \cdot \left\{\frac{a-b-1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b\right)} + \frac{2}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}\right)}\right\} \quad (\Re(b) < 0);
$$

$$
(2.10) \quad {}_2F_1\left(\begin{matrix} a, & b \\ 4+a-b \end{matrix} \;\middle|\; -1\right) = \frac{\Gamma(4+a-b)\,\Gamma\left(\frac{1}{2}\right)}{2^a\,(1-b)(2-b)(3-b)} \cdot \left\{\frac{2a-b+1}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b+2\right)} + \frac{3b-2a-5}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b+\frac{5}{2}\right)}\right\} \quad \left(\Re(b) < \frac{5}{2}\right);
$$

$$
(2.11) \quad {}_2F_1\left(\begin{matrix} a, & b \\ a-b-2 \end{matrix} \;\middle|\; -1\right) = \frac{\Gamma(a-b-2)\,\Gamma\left(\frac{1}{2}\right)}{2^a} \cdot \left\{\frac{2a-b-2}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b-1\right)} + \frac{2a-3b-4}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b-\frac{1}{2}\right)}\right\} \quad \left(\Re(b) < -\frac{1}{2}\right);
$$

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, \quad b \\ 5+a-b \end{matrix} \;\middle|\; -1\right) &= \frac{\Gamma(5+a-b)\,\Gamma\left(\frac{1}{2}\right)}{2^a\,(1-b)(2-b)(3-b)(4-b)} \\
&\quad \cdot \left\{ \frac{2a^2+b^2-4ab+8a-3b+2}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b+3\right)} - \frac{4(a-b+2)}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b+\frac{5}{2}\right)} \right\} \\
&\quad (\Re(b) < 3)\,;
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, \quad b \\ a-b-3 \end{matrix} \;\middle|\; -1\right) &= \frac{\Gamma(a-b-3)\,\Gamma\left(\frac{1}{2}\right)}{2^a} \\
&\quad \cdot \left\{ \frac{2a^2+b^2-4ab-8a+5b+6}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b-1\right)} + \frac{4(a-b-2)}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b-\frac{3}{2}\right)} \right\} \\
&\quad (\Re(b) < -1)\,;
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, \quad b \\ 6+a-b \end{matrix} \;\middle|\; -1\right) &= \frac{\Gamma(6+a-b)\,\Gamma\left(\frac{1}{2}\right)}{2^a\,(1-b)(2-b)(3-b)(4-b)(5-b)} \\
&\quad \cdot \left\{ \frac{A_5}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b+3\right)} - \frac{B_5}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b+\frac{7}{2}\right)} \right\} \\
&\quad \left(\Re(b) < \frac{7}{2}\right),
\end{aligned} \tag{2.14}
$$

where

$$
A_5 = 4(a-b+6)^2 + 2b(a-b+6) - b^2 - 34(a-b+6) - b + 62
$$

and

$$
B_5 = 4(a-b+6)^2 - 2b(a-b+6) - b^2 - 22(a-b+6) + 13b + 20;
$$

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a, \quad b \\ a-b-4 \end{matrix} \;\middle|\; -1\right) &= \frac{\Gamma(a-b-4)\,\Gamma\left(\frac{1}{2}\right)}{2^a} \\
&\quad \cdot \left\{ \frac{A_{-5}}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a-b-2\right)} + \frac{B_{-5}}{\Gamma\left(\frac{1}{2}a\right)\,\Gamma\left(\frac{1}{2}a-b-\frac{3}{2}\right)} \right\} \\
&\quad \left(\Re(b) < -\frac{3}{2}\right),
\end{aligned} \tag{2.15}
$$

where

$$
A_{-5} = 4(a-b-4)^2 + 2b(a-b-4) - b^2 + 16(a-b-4) - b + 12
$$

and

$$B_{-5} = 4(a-b-4)^2 - 2b(a-b-4) - b^2 + 8(a-b-4) - 7b.$$

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