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ON THE STABILITY OF A FUNCTIONAL EQUATION OF PEXIDER TYPE

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Abstract. We study the Hyers-Ulam stability of a functional equation of Pexider type associated with a functional equation $f(xy) = xf(y) + f(x)y$ which defines derivations in algebras.

1. Introduction

The problem of stability of functional equations was originally raised by S. M. Ulam [9] in 1940: given a group V, a metric group W with metric $d(\cdot,\cdot)$, and a $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : V \to W$ satisfies $d(f(xy), f(x)f(y)) \le \delta$ for all $x,\ y \in V$, then a homomorphism $g : V \to W$ exists with $d(f(x), g(x)) \le \epsilon$ for all $x \in V$? For Banach spaces the Ulam problem was first solved by D. H. Hyers [1] in 1941, which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$||f(x+y) - f(x) - f(y)|| \le \delta \tag{1.1}$$

for all $x,\ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x) - T(x)|| \le \delta$$

for all $x,\ y \in X$. Due to this fact, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability property on (X, Y). This terminology is also applied to other functional equations which has been studied by many authors (see, for example, [2-4, 6]. During the 34th International Symposium on Functional Equations, G. Maksa [4] posed the problem concerning the Hyers-Ulam stability of the functional equation

$$f(xy) = xf(y) + f(x)y \tag{1.2}$$

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on the interval $(0, 1]$, which is usually called a derivation. Recently J. Tabor [8] gave an answer to the question of Maksa by proving the Hyers-Ulam stability of the functional equation (1.2) on the interval $(0, 1]$. In a similar way, Zs. Páles [5] proved that the functional equation (1.2) for real-valued functions on the interval $[1, \infty)$ has the Hyers-Ulam stability . In this note, by using an idea of Tabor [8], we deal with the Hyers-Ulam stability of the functional equation (1.2) of Pexider type:

$$f_1(xy) = xf_2(y) + f_3(x)y. \tag{1.3}$$

2. Hyers-ulam Stability of Eq. (1.3).

We first introduce a theorem of F. Skof [7] concerning the stability of the additive functional equation $f(x + y) = f(x) + f(y)$ on a restricted domain:

Theorem 2.1. *Let X be a Banach space. Given $c > 0$, let a mapping $f : [0, c) \to X$ satisfy the inequality*

$$||f(x + y) - f(x) - f(y)|| \le \delta$$

for some $\delta > 0$ and for all $x,\ y \in [0, c)$ with $x + y \in [0, c)$. Then there exists an additive mapping $A : \mathbb{R} \to X$ such that

$$||f(x) - A(x)|| \le 3\delta$$

for any $x \in [0, c)$, where $\mathbb{R}$ is the set of all real numbers.

Our main result is the following:

Theorem 2.2. *Let X be a Banach space, and let $f_1, f_2, f_3 : (0, \infty) \to X$ be mappings satisfying the inequality*

$$||f_1(xy) - xf_2(y) - f_3(x)y|| \le \delta \tag{2.1}$$

for some $\delta > 0$ and for all $x,\ y \in (0, \infty)$. Then there exists a solution $D : (0, \infty) \to X$ of the functional equation (1.2) *such that*

$$||f_1(x) - D(x) - (f_2(1) + f_3(1))x|| \le (12e)\delta \tag{2.2}$$

$$||f_2(x) - D(x) - f_2(1)x|| \le (12e + 1)\delta \tag{2.3}$$

$$\|f_3(x) - D(x) - f_3(1)x\| \le (12e+1)\delta \tag{2.4}$$

for all $x \in (0, \infty)$.

Proof.

Case 1. We first prove (2.2), (2.3) and (2.4) under the inequality (2.1) on the interval $(0, 1]$.

Let us define the mappings $F_1,\ F_2,\ F_3 : (0, 1] \to X$ by

$$F_1(x) = \frac{f_1(x)}{x},\ F_2(x) = \frac{f_2(x)}{x},\ F_3(x) = \frac{f_3(x)}{x}$$

for all $x \in (0, 1]$, respectively. Then, by (2.1), we see that $F_1,\ F_2,\ F_3$ satisfy the inequality

$$\|F_1(xy) - F_2(y) - F_3(x)\| \le \frac{\delta}{xy}$$

for all $x,\ y \in (0, 1]$. Define the mappings $G_1,\ G_2,\ G_3 : [0, \infty) \to X$ by

$$G_1(u) = F_1(e^{-u}),\ G_2(u) = F_2(e^{-u}),\ \text{and } G_3(u) = F_3(e^{-u}),$$

for all $u \in [0, \infty)$, respectively. Then

$$\|G_1(u+v) - G_2(u) - G_3(v)\| \le \delta e^{u+v} \tag{2.5}$$

for all $u,\ v \in [0, \infty)$. Putting $v = 0$ in (2.5) we get

$$\|G_1(u) - G_2(u) - G_3(0)\| \le \delta e^{u} \tag{2.6}$$

for all $u \in [0, \infty)$. Analogously, if we put $u = 0$ in (2.5), we have

$$\|G_1(v) - G_2(0) - G_3(v)\| \le \delta e^{v} \tag{2.7}$$

for all $v \in [0, \infty)$. We now define a mapping $F : [0, \infty) \to X$ by

$$F(u) = G_1(u) - G_2(0) - G_3(0) \tag{2.8}$$

for all $u \in [0, \infty)$. We claim that

$$\|F(u+v) - F(u) - F(v)\| \le 3\delta e^{u+v} \tag{2.9}$$

for all $u,\ v \in [0, \infty)$. In fact, it follows from (2.5), (2.6), (2.7) and (2.8) that for all $u,\ v \in [0, \infty)$,

$$\begin{aligned}
&\|F(u+v) - F(u) - F(v)\| \\
&\quad = \|G_1(u+v) - G_2(u) - G_3(v) + G_2(0) + G_3(0)\| \\
&\quad \le \|G_1(u+v) - G_2(u) - G_3(v)\| + \|G_2(u) - G_1(u) + G_3(0)\| \\
&\quad\quad + \|G_3(v) - G_1(v) + G_2(0)\| \\
&\quad \le \delta e^{u+v} + \delta e^{u} + \delta e^{v} \\
&\quad \le 3\delta e^{u+v}.
\end{aligned}$$

This means that

$$||F(u+v)-F(u)-F(v)|| \le 3\delta e^c$$

for all $u,\ v \in [0,c)$ with $u+v<c$, where $c>1$ is an arbitrary given constant. According to Theorem 2.1, there exists an additive mapping $A:\mathbb{R}\to X$ such that $||F(u)-A(u)|| \le 9\delta e^c$ for all $u\in[0,c)$. If we let $c\to 1$ in the last inequality, we then get

$$||F(u)-A(u)|| \le 9e\delta \tag{2.10}$$

for all $u\in[0,1]$. Moreover, it follows from (2.9) that

$$\begin{aligned}
&||F(u+1)-F(u)-F(1)|| \le 3\delta e^{u+1}\\
&||F(u+2)-F(u+1)-F(1)|| \le 3\delta e^{u+2}\\
&\qquad\qquad\vdots\\
&||F(u+k)-F(u+k-1)-F(1)|| \le 3\delta e^{u+k}
\end{aligned}$$

for all $u\in[0,1]$ and $k\in\mathbb{N}$. Summing up these inequalities we obtain

$$||F(u+k)-F(u)-kF(1)|| \le 3\delta e\cdot e^{u+k} \tag{2.11}$$

for all $u\in[0,1]$ and $k\in\mathbb{N}$. We claim that

$$||F(v)-A(v)|| \le 12\delta e\cdot e^v \tag{2.12}$$

for all $v\in[0,\infty)$. Indeed, let $v\ge 0$ and let $k\in\mathbb{N}\cup\{0\}$ be given with $v-k\in[0,1]$. Then, by (2.10) and (2.11), we have

$$\begin{aligned}
||F(v)-A(v)|| &\le ||F(v)-F(v-k)-kF(1)||\\
&\quad +||F(v-k)-A(v-k)||+||A(k)-kF(1)||\\
&\le 3\delta e\cdot e^v+9\delta e+||A(k)-kF(1)||\\
&\le 3\delta e\cdot e^v+9\delta e+k||A(1)-F(1)||\\
&\le 3\delta e\cdot e^v+9\delta e+9\delta ev\\
&\le 3\delta e(e^v+3(1+v))\\
&\le 12\delta e\cdot e^v.
\end{aligned}$$

Now, from (2.12) and the definitions of F, F_i, G_i $(i=1,2,3)$, it follows that

$$||F_1(x)-F_2(1)-F_3(1)-A(-lnx)|| \le 12\delta e\cdot e^{-lnx}=\frac{12\delta e}{x}$$

for all $x \in (0,1]$, i.e.,

$$\left\|\frac{f_1(x)}{x} - f_2(1) - f_3(1) - A(-lnx)\right\| \leq \frac{12\delta e}{x} \tag{2.13}$$

for all $x \in (0,1]$. If we put $D(x) = xA(-lnx)$ for all $x \in (0,1]$, we can easily check that D is a solution of the functional equation (1.2). This and (2.13) yield that

$$||f_1(x) - D(x) - (f_2(1) + f_3(1))x|| \leq (12e)\delta$$

for all $x \in (0,1]$ which proves (2.2). It remains to show (2.3) and (2.4). From (2.6), (2.8) and (2.12), it follows that

$$\begin{aligned}||G_2(v) - A(v) - G_2(0)|| &= ||G_2(v) - A(v) + H(v) - G_1(v) + G_3(0)|| \\ &\leq ||F(v) - A(v)|| + ||G_1(v) - G_2(v) - G_3(0)|| \\ &\leq 12\delta e \cdot e^v + \delta e^v = (12e+1)\delta e^v\end{aligned}$$

for all $v \in [0,\infty)$, and hence this and the definitions of F_2, G_2 imply

$$\left\|\frac{f_2(x)}{x} - A(-lnx) - f_2(1)\right\| \leq (12e+1)\delta e^{-lnx} = \frac{(12e+1)\delta}{x}$$

for all $x \in (0,1]$, that is,

$$||f_2(x) - D(x) - f_2(1)x|| \leq (12e+1)\delta$$

for all $x \in (0,1]$ which verifies (2.3). Similarly, using (2.7), (2.8) and (2.12), we have

$$\begin{aligned}||G_3(v) - A(v) - G_3(0)|| &= ||G_3(v) - A(v) + F(v) - G_1(v) + G_2(0)|| \\ &\leq ||F(v) - A(v)|| + ||G_1(v) - G_2(0) - G_3(v)|| \\ &\leq 12\delta e \cdot e^v + \delta e^v = (12e+1)\delta e^v\end{aligned}$$

for all $v \in [0,\infty)$. By this and the definitions of F_3, G_3, we get

$$\left\|\frac{f_3(x)}{x} - A(-lnx) - f_3(1)\right\| \leq (12e+1)\delta e^{-lnx} = \frac{(12e+1)\delta}{x}$$

for all $x \in (0,1]$, that is,

$$||f_3(x) - D(x) - f_3(1)x|| \leq (12e+1)\delta.$$

Case 2. We now intend to prove (2.2), (2.3) and (2.4) under the inequality (2.1) on the interval $[1,\infty)$. But this is verified by using a similar way as the proof of Case 1.

In fact, defining the mappings $F_1,\ F_2,\ F_3 : [1,\infty) \to X$ as in the proof of Case 1, and defining the mappings $G_1,\ G_2,\ G_3 : [0,\infty) \to X$ by

$$G_1(u) = F_1(e^u),\ G_2(u) = F_2(e^u),\ \text{and } G_3(u) = F_3(e^u),$$

for all $u \in [0,\infty)$, respectively, we see that

$$||G_1(u+v) - G_2(u) - G_3(v)|| \le \delta e^{-(u+v)} \le \delta e^{u+v} \tag{2.14}$$

for all $u,\ v \in [0,\infty)$. Setting $v = 0$ in (2.14) we get

$$||G_1(u) - G_2(u) - G_3(0)|| \le \delta e^u \tag{2.15}$$

for all $u \in [0,\infty)$. Similarly, if we set $u = 0$ in (2.14), we have

$$||G_1(v) - G_2(0) - G_3(v)|| \le \delta e^v \tag{2.16}$$

for all $v \in [0,\infty)$. Introducing the mapping $F : [0,\infty) \to X$ defined as the identity (2.8) in the proof of Case 1, and making use of (2.14), (2.15) and (2.16), we see that

$$||F(u+v) - F(u) - F(v)|| \le 3\delta e^{u+v}$$

for all $u,\ v \in [0,\infty)$ by following the similar method to the proof of the inequality (2.9). The remainder follows the similar reasoning to the one of Case 1 by putting $D(x) = xA(lnx)$ for all $x \in [1,\infty)$. This completes the proof of the theorem. ■

The next corollary can be easily obtained from Theorem 2.2.

Corollary 2.3. *Let X be a Banach space and let $f_1, f_2, f_3 : (0,\infty) \to X$ be mappings satisfying the equation*

$$f_1(xy) - xf_2(y) - f_3(x)y = 0 \quad \textit{for all } x,\ y \in (0,\infty).$$

Then there exist a solution $D : (0,\infty) \to X$ of the functional equation (1.2) *and constants $a,\ b,\ c$ such that for all $x \in (0,\infty)$,*

$$f_1(x) = D(x) + ax$$

$$f_2(x) = D(x) + bx$$

$$f_3(x) = D(x) + cx$$

with $a = b + c$.

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