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TAUBERIAN THEOREMS IN THE STATISTICAL SENSE FOR THE WEIGHTED MEANS OF DOUBLE SEQUENCES

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Abstract. Let $p := \{p_j\}_{j=0}^{\infty}$ and $q := \{q_k\}_{k=0}^{\infty}$ be complex sequences with $p, q \in$ SVA. Assume that $\{s_{mn}\}_{m,n=0}^{\infty}$ is a double sequence in **C** (or one of **R**, a Banach space, and an ordered linear space), with $s_{mn} \overset{st}{\to} s \ \ (\bar{\mathrm{N}}, p, q; \alpha, \beta)$, where $(\alpha, \beta) = (1, 1)$, $(1, 0)$ or $(0, 1)$. We give sufficient and/or necessary conditions under which $s_{mn} \overset{st}{\to} s$. The theory developed here is the statistical version of the work of Chen and Hsu in [*Anal. Math.*, **26** (2000), 243-262]. Our results generalize Móricz, [*J. Math. Anal. Appl.*, **286** (2003), 340-350].

1. INTRODUCTION

Let $p := \{p_j\}_{j=0}^{\infty}$ and $q := \{q_k\}_{k=0}^{\infty}$ be two complex sequences with $P_m := \sum_{j=0}^{m} p_j \neq 0$ for all $m \geq 0$ and $Q_n := \sum_{k=0}^{n} q_k \neq 0$ for all $n \geq 0$. The weighted means $t_{mn}^{\alpha\beta}$ of a given complex double sequence $\{s_{mn}\}_{m,n=0}^{\infty}$ are defined by

$$t_{mn}^{11} = \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k s_{jk},$$
$$t_{mn}^{10} = \frac{1}{P_m} \sum_{j=0}^{m} p_j s_{jn}, \quad t_{mn}^{01} = \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_{mk},$$

where $m, n \geq 0$. We say that the sequence $\{s_{mn}\}_{m,n=0}^{\infty}$ is $(\bar{\mathrm{N}}, p, q; \alpha, \beta)$ summable to s, in symbol, $s_{mn} \to s \ \ (\bar{\mathrm{N}}, p, q; \alpha, \beta)$, if $\lim_{m,n\to\infty} t_{mn}^{\alpha\beta} = s$, where $(\alpha, \beta) = (1, 1)$, $(1, 0)$ or $(0, 1)$. The symbol $\lim_{m,n\to\infty} t_{mn}^{\alpha\beta} = s$ means that $t_{mn}^{\alpha\beta}$ converges to s as $\min(m, n) \to \infty$.

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In [1], C.-P. Chen and J.-M. Hsu considered those p and q from the class SVA or its subclasses SVA$_+$ and SVA$_r$, which are defined as follows. We write $p \in$ SVA, if $P_m \neq 0$ for all $m \geq 0$ and

$$\liminf_{m\to\infty} \left| \frac{P_{\lambda_m}}{P_m} - 1 \right| > 0 \text{ for all } \quad \lambda > 0 \text{ with } \lambda \neq 1.$$

Here $\lambda_m := [\lambda m]$ denotes the integral part of λm. The space SVA$_r$ is the set of all real sequences in SVA. Analogously, SVA$_+$ stands for the set of all nonnegative sequences p with $p \in$ SVA. Obviously, SVA$_+ \subset$ SVA$_r \subset$ SVA. As shown in [1], Chen and Hsu presented some kind of Tauberian conditions under which the convergence of $t_{mn}^{\alpha\beta}$ implies that of s_{mn}. These conditions can be Landau's conditions, Schmidt's slow decrease conditions, or more general conditions involving the concept of deferred means. The work of Chen-Hsu extends Móricz's and Baron-Stadtmüller's results from p, q with $p_j = q_k = 1$ to $p, q \in$ SVA, SVA$_r$ or SVA$_+$. Their results also hold for Banach spaces and ordered linear spaces.

The purpose of this paper is to give the statistical version of the results of Chen-Hsu given in [1]. The concept of statistical convergence was originally introduced by H. Fast [2] for single sequences, and extended to double sequences by F. Móricz [4]. We say that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically convergent to some number ξ, in symbols,

$$\text{st-lim } s_{mn} = \xi \qquad \text{or} \qquad s_{mn} \xrightarrow{st} \xi,$$

if for each $\epsilon > 0$,

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \leq M \text{ and } n \leq N : |s_{mn} - \xi| \geq \epsilon \right\} \right| = 0,$$

where $m \leq M$ means $m = 0, 1, 2, \ldots, M$ and $|S|$ denotes the cardinality of the set $S \subseteq \mathbf{N} \times \mathbf{N}$ with $\mathbf{N} = \{0, 1, 2, \ldots\}$. We also say that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically $(\bar{\mathrm{N}}, p, q; \alpha, \beta)$ summable to s, in symbol, $s_{mn} \xrightarrow{st} s \ (\bar{\mathrm{N}}, p, q; \alpha, \beta)$, if st-lim $t_{mn}^{\alpha\beta} = s$. We are interested in finding conditions on p, q and $\{s_{mn}\}_{m,n=0}^{\infty}$ under which the statistical convergence of $t_{mn}^{\alpha\beta}$ implies that of s_{mn}. Our results extend [3] from (C,1,1) summability to $(\bar{\mathrm{N}}, p, q; \alpha, \beta)$ summability with $p, q \in$ SVA. We also establish Tauberian theorems for double sequences in a Banach space or in an ordered linear space.

2. Preliminaries

The first lemma gives another representation of SVA and was proven in [1, Lemma 2.1].

Lemma 2.1. *Let $p := \{p_j\}_{j=0}^{\infty}$ be a complex sequence with $P_m \neq 0$ for all $m \geq 0$. Then for $\lambda > 0$ with $\lambda \neq 1$,*

$$\liminf_{m\to\infty}\left|\frac{P_{\lambda_m}}{P_m}-1\right| > 0 \iff \liminf_{m\to\infty}\left|\frac{P_m}{P_{\lambda_m}}-1\right| > 0.$$

The next lemma shows that the statistical limit relation is linear. It was given in [5] for single sequences.

Lemma 2.2. *Let $c \in \mathbf{C}$, st-lim $x_{jk} = L_1$ and st-lim $y_{jk} = L_2$. Then*

(i) st-lim $(x_{jk} + y_{jk}) = L_1 + L_2$,

(ii) st-lim $(cx_{jk}) = cL_1$.

Proof. The proof is based on the following two facts:

$$\begin{aligned}&\{j \leq m \text{ and } k \leq n : |(x_{jk}+y_{jk}) - (L_1+L_2)| \geq \epsilon\}\\ &\quad \subseteq \left\{j \leq m \text{ and } k \leq n : |x_{jk} - L_1| \geq \tfrac{\epsilon}{2}\right\}\\ &\qquad \cup \left\{j \leq m \text{ and } k \leq n : |y_{jk} - L_2| \geq \tfrac{\epsilon}{2}\right\},\end{aligned}$$

and for $c \neq 0$,

$$|cx_{jk} - cL_1| \geq \epsilon \iff |x_{jk} - L_1| \geq \epsilon/|c|. \qquad \blacksquare$$

The following lemma plays a key role in the proofs of the main results in §3 - §5. This lemma was stated in [3, Lemma 1] with $p_j = q_k = 1$.

Lemma 2.3. *Let $(\alpha, \beta) = (1,1)$, $(1,0)$ or $(0,1)$, and $s_{mn} \xrightarrow{st} s$ $(\bar{\mathrm{N}}, p, q; \alpha, \beta)$. Then for $\lambda > 0$, we have $t^{\alpha\beta}_{\lambda_m,\lambda_n} \xrightarrow{st} s$, $t^{\alpha\beta}_{\lambda_m,n} \xrightarrow{st} s$, and $t^{\alpha\beta}_{m,\lambda_n} \xrightarrow{st} s$.*

Proof. We only show $t^{\alpha\beta}_{\lambda_m,\lambda_n} \xrightarrow{st} s$ in the case $(\alpha,\beta) = (1,1)$. The proofs for other cases are similar and are left to the readers. The case $\lambda = 1$ is trivial. Let $\lambda > 1$ and $\epsilon > 0$. We know that $f(m,n) = (\lambda_m, \lambda_n)$ defines a one to one mapping, so

$$\begin{aligned}&\frac{1}{(M+1)(N+1)}\left|\left\{m \leq M \text{ and } n \leq N : |t^{11}_{\lambda_m,\lambda_n} - s| \geq \epsilon\right\}\right|\\ &\leq \frac{\lambda^2}{(\lambda_M+1)(\lambda_N+1)}\left|\left\{m \leq \lambda_M \text{ and } n \leq \lambda_N : |t^{11}_{mn} - s| \geq \epsilon\right\}\right|\\ &\longrightarrow 0 \quad \text{as} \quad \min(M,N) \longrightarrow \infty.\end{aligned}$$

The last step follows from the fact that $s_{mn} \xrightarrow{st} s$ $(\bar{\mathrm{N}}, p, q; 1, 1)$. For $0 < \lambda < 1$, the sequence $\{\lambda_j\}_{j=0}^{\infty}$ is nondecreasing. If for some integers j and l with

$$m = \lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+l-1} < \lambda_{j+l},$$

then $m \le \lambda j < \lambda(j+1) < \cdots < \lambda(j+l-1) < m+1 \le \lambda(j+l)$. This implies

$$m + \lambda(l-1) \le \lambda j + \lambda(l-1) = \lambda(j+l-1) < m+1,$$

and so $l < 1 + \lambda^{-1}$. Hence, the pairs (j,k) with $\lambda_j = m$ and $\lambda_k = n$ occur at most $(1+\lambda^{-1})^2$ times. Moreover, we have $(\lambda_N + 1)/(N+1) \le 2\lambda$ for $N \ge \max\{\lambda^{-1} - 2, 0\}$. This leads us to

$$\begin{aligned}
&\frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : |t^{11}_{\lambda_m,\lambda_n} - s| \ge \epsilon\right\}\right| \\
\le\ & \frac{(1+\lambda^{-1})^2}{(M+1)(N+1)}\left|\left\{m \le \lambda_M \text{ and } n \le \lambda_N : |t^{11}_{mn} - s| \ge \epsilon\right\}\right| \\
\le\ & \frac{(1+\lambda^{-1})^2(2\lambda)^2}{(\lambda_M+1)(\lambda_N+1)}\left|\left\{m \le \lambda_M \text{ and } n \le \lambda_N : |t^{11}_{mn} - s| \ge \epsilon\right\}\right| \\
\longrightarrow\ & 0 \ \text{ as } \ \min(M,N) \longrightarrow \infty.
\end{aligned}$$

We complete the proof. ■

3. Double Complex Sequences

The following theorem gives a statistical version of [1, Theorem 3.1]. It generalizes [3, Lemma 2].

Theorem 3.1. *Assume that* $p, q \in$ *SVA and* $s_{mn} \xrightarrow{st} s$ $(\bar{\mathrm{N}}, p, q; 1, 1)$. *Then*

$$\text{st-lim}\ \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k s_{jk} = s \quad (\lambda > 1), \tag{3.1}$$

and

$$\text{st-lim}\ \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k s_{jk} = s \quad (0 < \lambda < 1). \tag{3.2}$$

Proof. Let $\lambda > 1$. Eq. (3.3) in [1] tells us that

$$
\begin{aligned}
& \frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k s_{jk} \\
= \quad & t^{11}_{\lambda_m,\lambda_n} + \frac{1}{\left(\frac{P_{\lambda_m}}{P_m}\right)-1}(t^{11}_{\lambda_m,\lambda_n} - t^{11}_{m,\lambda_n}) \\
& + \frac{1}{\left(\frac{Q_{\lambda_n}}{Q_n}\right)-1}(t^{11}_{\lambda_m,\lambda_n} - t^{11}_{\lambda_m,n}) \\
& + \frac{1}{\left(\frac{P_{\lambda_m}}{P_m}-1\right)} \frac{1}{\left(\frac{Q_{\lambda_n}}{Q_n}-1\right)}(t^{11}_{\lambda_m,\lambda_n} - t^{11}_{m,\lambda_n} - t^{11}_{\lambda_m,n} + t^{11}_{mn}).
\end{aligned} \tag{3.3}
$$

We have $p, q \in$ SVA and $s_{mn} \xrightarrow{st} s \ \ (\bar{\mathrm{N}}, p, q; 1, 1)$. By Lemmas 2.2 - 2.3 we obtain

$$
\begin{aligned}
\left| \frac{1}{\left(\frac{P_{\lambda_m}}{P_m}\right)-1}(t^{11}_{\lambda_m,\lambda_n} - t^{11}_{m,\lambda_n}) \right| \quad & \leq \quad \frac{1}{\inf_{k\geq m}\left|\left(\frac{P_{\lambda_k}}{P_k}\right)-1\right|}\,|t^{11}_{\lambda_m,\lambda_n} - t^{11}_{m,\lambda_n}| \\
& \xrightarrow{st} \quad 0 \ \text{ as } \ \min(m,n) \longrightarrow \infty.
\end{aligned}
$$

The last two terms in (3.3) also tend to zero statistically as $\min(m, n) \to \infty$. Therefore, (3.1) follows from (3.3). As for the case $0 < \lambda < 1$, by Lemma 2.1 we have

$$
\liminf_{m\to\infty} \left| \frac{P_m}{P_{\lambda_m}} - 1 \right| > 0 \ \text{ and } \ \liminf_{n\to\infty} \left| \frac{Q_n}{Q_{\lambda_n}} - 1 \right| > 0.
$$

By making the changes $\lambda_m \leftrightarrow m$ and $\lambda_n \leftrightarrow n$, an argument similar to the case $\lambda > 1$ leads us to (3.2). ∎

Theorem 3.1 says that the statistical convergence of $(\bar{\mathrm{N}}, p, q; 1, 1)$ means implies that of the corresponding deferred means. As a consequence of Theorem 3.1, we get the following generalization of [3, Theorem 2]. Our result is the statistical version of [1, Theorem 3.2].

Theorem 3.2. *Let $p, q \in$ SVA. Assume that $s_{mn} \xrightarrow{st} s \ \ (\bar{\mathrm{N}}, p, q; 1, 1)$. Then the following four assertions hold.*

(i) $s_{mn} \xrightarrow{st} s$ *if and only if for all $\epsilon > 0$,*

$$
\begin{aligned}
& \inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Bigg| \Bigg\{ m \leq M \text{ and } n \leq N : \\
& \left| \frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}) \right| \geq \epsilon \Bigg\} \Bigg| = 0.
\end{aligned} \tag{3.4}
$$

(ii) $s_{mn} \xrightarrow{st} s$ *if and only if for all* $\epsilon > 0$,

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Bigg| \Bigg\{ m \le M \text{ and } n \le N : \tag{3.5}$$
$$\left| \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (s_{mn} - s_{jk}) \right| \ge \epsilon \Bigg\} \Bigg| = 0.$$

(iii) *Condition* (3.4) *can be replaced by* (3.4^*) *for some* $\lambda > 1$, *and* (3.5) *can be replaced by* (3.5^*) *for some* $0 < \lambda < 1$*:*

$$\textit{st-lim} \; \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}) = 0, \tag{3.4*}$$

$$\textit{st-lim} \; \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (s_{mn} - s_{jk}) = 0. \tag{3.5*}$$

(iv) *Moreover, if* (3.4^*) *holds for some* $\lambda > 1$, *then it is true for all* $\lambda > 1$. *The same situation happens to* (3.5^*).

Proof. It is clear that $(3.4^*) \Rightarrow (3.4)$ and $(3.5^*) \Rightarrow (3.5)$. Assume that $s_{mn} \xrightarrow{st} s$. Let $\lambda > 1$. By Theorem 3.1 and Lemma 2.2, we have

$$\begin{aligned}
& \text{st-lim} \; \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}) \\
= \; & \text{st-lim} \; \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k s_{jk} - \text{st-lim} \; s_{mn} \\
= \; & s - s = 0,
\end{aligned}$$

and so (3.4^*) holds for all $\lambda > 1$. Conversely, we show that (3.4) implies $s_{mn} \xrightarrow{st} s$. We have

$$\begin{aligned}
s_{mn} \; = \; & \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k s_{jk} \\
& - \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}) \\
= \; & A - B \text{ , say.}
\end{aligned}$$

For $\epsilon > 0$, we know that

$$\begin{aligned}&\{m \le M \text{ and } n \le N : |s_{mn} - s| \ge \epsilon\}\\ \subseteq &\left\{m \le M \text{ and } n \le N : |A - s| \ge \frac{\epsilon}{2}\right\} \bigcup \left\{m \le M \text{ and } n \le N : |B| \ge \frac{\epsilon}{2}\right\}.\end{aligned}$$

This implies

$$\begin{aligned}&\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : |s_{mn} - s| \ge \epsilon\right\}\right|\\ \le &\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : |A - s| \ge \frac{\epsilon}{2}\right\}\right|\\ &+ \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : |B| \ge \frac{\epsilon}{2}\right\}\right|.\end{aligned}$$

By (3.4) and Theorem 3.1, we get $s_{mn} \overset{st}{\to} s$. This completes the proofs of (i) and the first parts of both of (iii) and (iv). For $0 < \lambda < 1$, it can be verified in a similar way. We leave it to the readers. ■

Following [3], we say that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating with respect to the first index if, for every $\epsilon > 0$,

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : \max_{m<j\le\lambda_m} |s_{jn} - s_{mn}| \ge \epsilon\right\}\right| = 0, \tag{3.6}$$

and that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating in the strong sense with respect to the first index if (3.6) is satisfied with

$$\max_{\substack{m<j\le\lambda_m\\ n<k\le\lambda_n}} |s_{jk} - s_{mk}| \qquad \text{in place of} \qquad \max_{m<j\le\lambda_m} |s_{jn} - s_{mn}|.$$

The statistically slow oscillation property with respect to the second index is defined analogously. Now consider $p, q \in \text{SVA}_+$. We have

$$\begin{aligned}&\left|\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn})\right|\\ \le &\max_{\substack{m<j\le\lambda_m\\ n<k\le\lambda_n}} |s_{jk} - s_{mk}| + \max_{n<k\le\lambda_n} |s_{mk} - s_{mn}|.\end{aligned}$$

Hence, if $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating with respect to the second index and statistically slowly oscillating in the strong sense with respect to the first index, then (3.4) holds for all $\epsilon > 0$. By Theorem 3.2, we obtain the following result, which generalizes [3, Corollary 3].

Corollary 3.3. *Let $p, q \in SVA_+$ and $s_{mn} \xrightarrow{st} s$ $(\bar{\mathrm{N}}, p, q; 1, 1)$. If $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating with respect to both indices, in addition, in the strong sense with respect to one of the indices, then $s_{mn} \xrightarrow{st} s$.*

Consider the following two-sided Landau's conditions for the complex case:

$$j|s_{jn} - s_{j-1,n}| \leq H \qquad (j, n > \tilde{n}), \tag{3.7}$$

$$k|s_{mk} - s_{m,k-1}| \leq H \qquad (m, k > \tilde{n}), \tag{3.8}$$

where $\tilde{n} > 0$ and H are suitable constants. For $\lambda > 1$ and m, $n > \tilde{n}$, we have

$$\begin{aligned} \max_{\substack{m<j\leq\lambda_m \\ n<k\leq\lambda_n}} |s_{jk} - s_{mk}| &\leq \max_{\substack{m<j\leq\lambda_m \\ n<k\leq\lambda_n}} \left\{ \left(\sum_{l=m+1}^{j} \frac{1}{l} \right) \left(\sup_{m<l\leq j} l\,|s_{lk} - s_{l-1,k}| \right) \right\} \\ &\leq \sum_{l=m+1}^{\lambda_m} \frac{H}{l} \leq H \log \lambda. \end{aligned}$$

This indicates that if (3.7) holds, then $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating in the strong sense with respect to the first index. Similarly, (3.8) implies the statistically slow oscillation property in the strong sense with respect to the second index. As a consequence of Corollary 3.3, we get the following generalization of [3, Corollary 4]. It is the statistical version of [1, Corollary 3.4].

Corollary 3.4. *Let $p, q \in SVA_+$ and $s_{mn} \xrightarrow{st} s$ $(\bar{\mathrm{N}}, p, q; 1, 1)$. If $\{s_{mn}\}_{m,n=0}^{\infty}$ satisfies (3.7) and (3.8) for some $\tilde{n} > 0$ and some H, then $s_{mn} \xrightarrow{st} s$.*

As in [1], we write $(\bar{\mathrm{N}}, p, *; 1, 0)$ in the place of $(\bar{\mathrm{N}}, p, q; 1, 0)$. For $\lambda > 1$ and $\lambda_m > m$, we have

$$\frac{1}{P_{\lambda_m} - P_m} \sum_{j=m+1}^{\lambda_m} p_j s_{jn} = t^{10}_{\lambda_m, n} + \frac{1}{\left(\frac{P_{\lambda_m}}{P_m}\right) - 1} (t^{10}_{\lambda_m, n} - t^{10}_{mn}). \tag{3.9}$$

To replace (3.3) by (3.9), the same argument as given above will lead us to the following two results, which are the statistical versions of those given in [1, Theorems 3.6, 3.7].

Theorem 3.5. *Assume that* $p \in$ *SVA and* $s_{mn} \xrightarrow{st} s \ (\bar{\mathrm{N}}, p, *; 1, 0)$. *Then*

$$\textit{st-lim}\ \frac{1}{P_{\lambda_m} - P_m} \sum_{j=m+1}^{\lambda_m} p_j s_{jn} = s \qquad (\lambda > 1),$$

and

$$\textit{st-lim}\ \frac{1}{P_m - P_{\lambda_m}} \sum_{j=\lambda_m+1}^{m} p_j s_{jn} = s \qquad (0 < \lambda < 1).$$

Theorem 3.6. *Assume that* $p \in$ *SVA and* $s_{mn} \xrightarrow{st} s \ (\bar{\mathrm{N}}, p, *; 1, 0)$. *Then the following four assertions hold.*

(i) $s_{mn} \xrightarrow{st} s$ *if and only if for all* $\epsilon > 0$,

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Bigg| \Bigg\{ m \le M \text{ and } n \le N : \left| \frac{1}{P_{\lambda_m} - P_m} \sum_{j=m+1}^{\lambda_m} p_j (s_{jn} - s_{mn}) \right| \ge \epsilon \Bigg\} \Bigg| = 0. \tag{3.10}$$

(ii) $s_{mn} \xrightarrow{st} s$ *if and only if for all* $\epsilon > 0$,

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Bigg| \Bigg\{ m \le M \text{ and } n \le N : \left| \frac{1}{P_m - P_{\lambda_m}} \sum_{j=\lambda_m+1}^{m} p_j (s_{mn} - s_{jn}) \right| \ge \epsilon \Bigg\} \Bigg| = 0. \tag{3.11}$$

(iii) *Condition* (3.10) *can be replaced by* (3.10^*) *for some* $\lambda > 1$, *and* (3.11) *can be replaced by* (3.11^*) *for some* $0 < \lambda < 1$*:*

$$\textit{st-lim}\ \frac{1}{P_{\lambda_m} - P_m} \sum_{j=m+1}^{\lambda_m} p_j (s_{jn} - s_{mn}) = 0, \tag{3.10*}$$

$$\textit{st-lim}\ \frac{1}{P_m - P_{\lambda_m}} \sum_{j=\lambda_m+1}^{m} p_j (s_{mn} - s_{jn}) = 0. \tag{3.11*}$$

(iv) *Moreover, if* (3.10^*) *holds for some* $\lambda > 1$, *then it is true for all* $\lambda > 1$. *The same situation happens to* (3.11^*).

For $p \in \text{SVA}_+$, we have

$$\left| \frac{1}{P_{\lambda_m} - P_m} \sum_{j=m+1}^{\lambda_m} p_j(s_{jn} - s_{mn}) \right| \leq \max_{m<j\leq\lambda_m} |s_{jn} - s_{mn}|.$$

Hence, (3.10) follows from the condition that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating with respect to the first index. As indicated in the paragraph before Corollary 3.4, the condition (3.7) ensures that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating in the strong sense with respect to the first index. As a consequence of Theorem 3.6, we get the following result, which is the statistical version of [1, Corollary 3.8].

Corollary 3.7. *Let $p \in SVA_+$ and $s_{mn} \overset{st}{\to} s$ $(\bar{\text{N}}, p, *; 1, 0)$. If $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly oscillating with respect to the first index or (3.7) is satisfied for some $\tilde{n} > 0$ and some H, then $s_{mn} \overset{st}{\to} s$.*

4. Double Real Sequences

The following result is the statistical version of [1, Corollary 4.3]. Our result generalizes [3, Theorem 1].

Theorem 4.1. *Let $p, q \in SVA_r$. Assume that $\{s_{mn}\}_{m,n=0}^{\infty}$ is a double real sequence and $s_{mn} \overset{st}{\to} s$ $(\bar{\text{N}}, p, q; 1, 1)$. Then $s_{mn} \overset{st}{\to} s$ if and only if the following two conditions are satisfied for all $\epsilon > 0$:*

$$\begin{aligned}(4.1)\quad &\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Bigg| \Bigg\{ m \leq M \text{ and } n \leq N : \\ &\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}) \leq -\epsilon \Bigg\} \Bigg| = 0,\end{aligned}$$

$$\begin{aligned}(4.2)\quad &\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Bigg| \Bigg\{ m \leq M \text{ and } n \leq N : \\ &\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (s_{mn} - s_{jk}) \leq -\epsilon \Bigg\} \Bigg| = 0.\end{aligned}$$

Proof. Assume $s_{mn} \overset{st}{\to} s$. For $\lambda > 1$, $\lambda_m > m$ and $\lambda_n > n$, let

$$B = \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}).$$

Then $\{m \le M$ and $n \le N : B \le -\epsilon\} \subseteq \{m \le M$ and $n \le N : |B| \ge \epsilon\}$. By Theorem 3.2 (i), we get (4.1). A similar argument will also lead us to (4.2). Conversely, assume that both of (4.1) and (4.2) hold. Let $\lambda > 1$, $\epsilon > 0$, and $\delta > 0$. Rewrite $s_{mn} - s$ in the following form:

$$\begin{aligned} s_{mn} - s &= \left[\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k s_{jk} - s \right] \\ &\quad - \left[\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}) \right] \\ &= C - B \text{ , say.} \end{aligned}$$

By (4.1), there exists $\lambda > 1$ such that

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \le M \text{ and } n \le N : B \le -\frac{\epsilon}{2} \right\} \right| < \delta. \tag{4.3}$$

For such λ, it follows from Theorem 3.1 that

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \le M \text{ and } n \le N : C \ge \frac{\epsilon}{2} \right\} \right| = 0. \tag{4.4}$$

Putting (4.3) and (4.4) together yields

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \le M \text{ and } n \le N : s_{mn} - s \ge \epsilon \right\} \right| < \delta.$$

Let $\delta \searrow 0$. We find that

$$\limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \le M \text{ and } n \le N : s_{mn} - s \ge \epsilon \right\} \right| = 0. \tag{4.5}$$

Next, consider $0 < \lambda < 1$. We have

$$\begin{aligned} s_{mn} - s &= \left[\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k s_{jk} - s \right] \\ &\quad + \left[\frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} p_j q_k (s_{mn} - s_{jk}) \right] \\ &= C' + B' \text{ , say.} \end{aligned}$$

Employing the argument given for the case $\lambda > 1$, we get

$$(4.6)\quad \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : s_{mn} - s \le -\epsilon\right\}\right| = 0.$$

Putting (4.5) and (4.6) together yields $s_{mn} \xrightarrow{st} s$. This completes the proof. ∎

Following [3], we say that a double real sequence $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing with respect to the first index if, for every $\epsilon > 0$,

$$(4.7)\quad \inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : \min_{m<j\le\lambda_m} (s_{jn} - s_{mn}) \le -\epsilon\right\}\right| = 0.$$

We also say that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing in the strong sense with respect to the first index if (4.7) is satisfied with

$$\min_{\substack{m<j\le\lambda_m \\ n<k\le\lambda_n}} (s_{jk} - s_{mk}) \qquad \text{in place of} \qquad \min_{m<j\le\lambda_m} (s_{jn} - s_{mn}).$$

The statistically slow decrease property with respect to the second index is defined in a similar way. As indicated in [3, Remark 2], condition (4.7) implies

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)}\left|\left\{m \le M \text{ and } n \le N : \min_{\lambda_m<j\le m} (s_{mn} - s_{jn}) \le -\epsilon\right\}\right| = 0,$$

and vice versa. Similar conclusions also hold for the second index or in the strong sense. Now, consider $p, q \in$ SVA$_+$. For $\lambda > 1$, we have

$$\begin{aligned}
&\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} p_j q_k (s_{jk} - s_{mn}) \\
\ge \quad & \min_{\substack{m<j\le\lambda_m \\ n<k\le\lambda_n}} (s_{jk} - s_{mk}) + \min_{n<k\le\lambda_n} (s_{mk} - s_{mn}).
\end{aligned}$$

Hence, if $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing with respect to the second index and statistically slowly decreasing in the strong sense with respect to the first index, then (4.1) holds for all $\epsilon > 0$. Similarly, (4.2) holds for all $\epsilon > 0$. By Theorem 4.1, we obtain the following generalization of [3, Corollary 1].

Corollary 4.2. *Let $p, q \in$ SVA$_+$ and $\{s_{mn}\}_{m,n=0}^{\infty}$ be a double real sequence with $s_{mn} \xrightarrow{st} s \quad (\bar{\mathrm{N}}, p, q; 1, 1)$. If $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing with* *respect to both indices, in addition, in the strong sense with respect to one of the indices, then* $s_{mn} \xrightarrow{st} s$.

Consider the following Landau's conditions:

$$j(s_{jn} - s_{j-1,n}) \geq -H \qquad (j, n > \tilde{n}), \tag{4.8}$$

$$k(s_{mk} - s_{m,k-1}) \geq -H \qquad (j, n > \tilde{n}), \tag{4.9}$$

where $\tilde{n} > 0$ and H are suitable constants. For $\lambda > 1$ and m, $n > \tilde{n}$, we have

$$\begin{aligned} \min_{\substack{m<j\leq\lambda_m \\ n<k\leq\lambda_n}} (s_{jk} - s_{mk}) &\geq \min_{\substack{m<j\leq\lambda_m \\ n<k\leq\lambda_n}} \left\{ \left(\sum_{l=m+1}^{j} \frac{1}{l} \right) \left(\inf_{m<l\leq j} l\left(s_{lk} - s_{l-1,k}\right) \right) \right\} \\ &\geq -\left(\sum_{l=m+1}^{\lambda_m} \frac{H}{l} \right) \geq -H \log \lambda. \end{aligned}$$

Therefore, under (4.8), $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing in the strong sense with respect to the first index. Similarly, (4.9) implies the statistically slow decrease property in the strong sense with respect to the second index. By Corollary 4.2, we get the following generalization of [3, Corollary 2]. It is the statistical version of [1, Corollary 4.4].

Corollary 4.3. *Let* $p, q \in SVA_+$ *and* $\{s_{mn}\}_{m,n=0}^{\infty}$ *be a double real sequence with* $s_{mn} \xrightarrow{st} s \quad (\bar{\mathrm{N}}, p, q; 1, 1)$. *If* $\{s_{mn}\}_{m,n=0}^{\infty}$ *satisfies* (4.8) *and* (4.9) *for some* $\tilde{n} > 0$ *and some* H, *then* $s_{mn} \xrightarrow{st} s$.

For the $(\bar{\mathrm{N}}, p, *; 1, 0)$ summability, a parallel theory to the one given above can be derived. We list these results below and omit their proofs. Our results are statistical versions of [1, Corollaries 4.9 - 4.11].

Theorem 4.4. *Let* $p \in SVA_r$ *and* $\{s_{mn}\}_{m,n=0}^{\infty}$ *be a double real sequence with* $s_{mn} \xrightarrow{st} s \quad (\bar{\mathrm{N}}, p, *; 1, 0)$. *Then* $s_{mn} \xrightarrow{st} s$ *if and only if the following two conditions are satisfied for all* $\epsilon > 0$:

$$\begin{aligned} \inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \Bigg| \Bigg\{ & m \leq M \text{ and } n \leq N : \\ & \frac{1}{P_{\lambda_m} - P_m} \sum_{j=m+1}^{\lambda_m} p_j (s_{jn} - s_{mn}) \leq -\epsilon \Bigg\} \Bigg| = 0, \end{aligned}$$

$$\inf_{0<\lambda<1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \le M \text{ and } n \le N : \frac{1}{P_m - P_{\lambda_m}} \sum_{j=\lambda_m+1}^{m} p_j (s_{mn} - s_{jn}) \le -\epsilon \right\} \right| = 0.$$

Corollary 4.5. *Let $p \in$ SVA$_+$ and $\{s_{mn}\}_{m,n=0}^{\infty}$ be a double real sequence with $s_{mn} \overset{st}{\to} s \quad (\bar{\mathrm{N}}, p, *; 1, 0)$. If $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing with respect to the first index or (4.8) is satisfied for some $\tilde{n} > 0$ and some H, then $s_{mn} \overset{st}{\to} s$.*

5. Double Sequences in Ordered Linear Spaces

The theory developed in §4 can be extended to the case of double sequences in ordered linear spaces. As defined in [1], let $(X, \le)$ be an ordered linear space over $\mathbf{R}$ and $\tau \in X$ be a nonnegative element. We assume that $\{s_{mn}\}_{m,n=0}^{\infty}$ is a double sequence in X. We say that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically convergent (in Pringsheim's sense) to s relative to $\tau \in X$, and we write

$$\text{st-lim } s_{mn} =_{(\tau)} s \qquad \text{or} \qquad s_{mn} \overset{st}{\to}_{(\tau)} s,$$

if for each $\epsilon > 0$,

$$\lim_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \le M \text{ and } n \le N : s_{mn} - s \le -\epsilon\tau \text{ or } s_{mn} - s \ge \epsilon\tau \right\} \right| = 0.$$

For $(\alpha, \beta) = (1,1)$, $(1,0)$, or $(0,1)$, we write

$$s_{mn} \overset{st}{\to}_{(\tau)} s \ \ (\bar{\mathrm{N}}, p, q; \alpha, \beta) \qquad \text{if st-lim}\, t_{mn}^{\alpha\beta} =_{(\tau)} s.$$

The sequence $\{s_{mn}\}_{m,n=0}^{\infty}$ is said to be statistically slowly decreasing with respect to the first index relative to $\tau \in X$ if, for each $\epsilon > 0$,

$$\inf_{\lambda>1} \limsup_{M,N\to\infty} \frac{1}{(M+1)(N+1)} \left| \left\{ m \le M \text{ and } n \le N : \min_{m<j\le\lambda_m} (s_{jn} - s_{mn}) \le -\epsilon\tau \right\} \right| = 0. \tag{5.1}$$

We also say that $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing in the strong sense with respect to the first index relative to $\tau \in X$, if (5.1) is satisfied with

$$\min_{\substack{m<j\le\lambda_m\\ n<k\le\lambda_n}}(s_{jk}-s_{mk}) \quad \text{in place of} \quad \min_{m<j\le\lambda_m}(s_{jn}-s_{mn}).$$

The statistically slow decrease property with respect to the second index relative to $\tau\in X$ is defined in a similar way. Based on the above concepts, we can derive the following results, which are the statistical versions of the corresponding ones given in [1, §5].

Theorem 5.1. *The conclusions in Theorems 3.1 and 3.5 remain true, if we replace " SVA " by " SVA$_r$ ", " $\xrightarrow{st}$ " by " $\xrightarrow{st}_{(\tau)}$ ", and " $=$ " by " $=_{(\tau)}$ ".*

Theorem 5.2. *Let $p,q\in$ SVA$_r$ and $s_{mn}\xrightarrow{st}_{(\tau)} s$ $(\bar{\mathrm{N}},p,q;1,1)$. Then $s_{mn}\xrightarrow{st}_{(\tau)}$ s if and only if the following two conditions are satisfied for all $\epsilon>0$:*

$$\inf_{\lambda>1}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\Bigg|\Bigg\{m\le M \text{ and } n\le N:$$
$$\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}p_jq_k(s_{jk}-s_{mn})\le-\epsilon\tau\Bigg\}\Bigg|=0,$$

$$\inf_{0<\lambda<1}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\Bigg|\Bigg\{m\le M \text{ and } n\le N:$$
$$\frac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}p_jq_k(s_{mn}-s_{jk})\le-\epsilon\tau\Bigg\}\Bigg|=0.$$

Corollary 5.3. *Let $p,q\in$ SVA$_+$ and $s_{mn}\xrightarrow{st}_{(\tau)} s$ $(\bar{\mathrm{N}},p,q;1,1)$. If $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing with respect to both indices relative to $\tau\in X$, in addition, in the strong sense with respect to one of the indices, then $s_{mn}\xrightarrow{st}_{(\tau)} s$.*

Theorem 5.4. *Let $p\in$ SVA$_r$ and $s_{mn}\xrightarrow{st}_{(\tau)} s$ $(\bar{\mathrm{N}},p,*;1,0)$. Then $s_{mn}\xrightarrow{st}_{(\tau)} s$ if and only if the following two conditions are satisfied for all $\epsilon>0$:*

$$\inf_{\lambda>1}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\Bigg|\Bigg\{m\le M \text{ and } n\le N:$$
$$\frac{1}{P_{\lambda_m}-P_m}\sum_{j=m+1}^{\lambda_m}p_j(s_{jn}-s_{mn})\le-\epsilon\tau\Bigg\}\Bigg|=0,$$

$$\inf_{0<\lambda<1}\limsup_{M,N\to\infty}\frac{1}{(M+1)(N+1)}\Bigg|\Bigg\{m\le M \text{ and } n\le N:$$
$$\frac{1}{P_m-P_{\lambda_m}}\sum_{j=\lambda_m+1}^{m}p_j(s_{mn}-s_{jn})\le-\epsilon\tau\Bigg\}\Bigg|=0.$$

Corollary 5.5. *Let $p \in SVA_+$ and $s_{mn} \xrightarrow{st}_{(\tau)} s \ (\bar{\mathrm{N}}, p, *; 1, 0)$. If $\{s_{mn}\}_{m,n=0}^{\infty}$ is statistically slowly decreasing with respect to the first index relative to $\tau \in X$, then $s_{mn} \xrightarrow{st}_{(\tau)} s$.*

6. Double Sequences in Banach Spaces

It is easy to see that the theory developed in §3 can be extended to any Banach space $(X, \|\cdot\|)$ over $\mathbf{C}$ (or over $\mathbf{R}$), provided we make the following changes: replace $|\cdot|$ by $\|\cdot\|$ whenever it applies to elements of X, and regard st-lim $s_{mn} = s$ as st-lim $\|s_{mn} - s\| = 0$. We leave the proofs to the readers.

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