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OSCILLATION THEOREMS RELATED TO AVERAGING TECHNIQUE FOR SECOND ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. Some oscillation theorems are established by the averaging technique for the second order nonlinear neutral delay differential equation

$$(r(t)|x'(t)|^{\gamma-1}x'(t)\,)' + q_1(t)|y(t-\sigma_1)|^{\alpha-1}y(t-\sigma_1) + q_2(t)|y(t-\sigma_2)|^{\beta-1}y(t-\sigma_2) = 0, \quad t \geq t_0,$$

where $x(t) = y(t) + p(t)y(t-\tau)$, τ, σ_1 and σ_2 are nonnegative constants, α, β and γ are positive constants, and r, p, q_1, $q_2 \in C([t_0,\infty),\mathbb{R})$. The results obtained here essentially improve some known results in the literature. In particular, two interesting examples that point out the applications of our results are also included.

1. Introduction

In this paper, we study the problem of oscillation of the second order nonlinear neutral delay differential equation

$$(r(t)|x'(t)|^{\gamma-1}x'(t)\,)' + q_1(t)|y(t-\sigma_1)|^{\alpha-1}y(t-\sigma_1) + q_2(t)|y(t-\sigma_2)|^{\beta-1}y(t-\sigma_2) = 0, \quad t \geq t_0, \tag{1.1}$$

where $x(t) = y(t) + p(t)y(t-\tau)$.

In what follows we will assume that

$(A1)$ τ, σ_1 and σ_2 are nonnegative constants, α, β and γ are positive constants with $0 < \alpha < \gamma < \beta$;

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$(A2)$ $r, q_1, q_2 \in C([t_0, \infty), \mathbb{R}^+ = (0, \infty))$, and $\int^{\infty} r^{-1/\gamma}(s)ds = \infty$;
$(A3)$ $p \in C([t_0, \infty), \mathbb{R})$, and $-1 < p_0 \le p(t) \le 1$, p_0 constant.

Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ with $\sup\{|y(t)| : t \ge T\} > 0$ for any $T \ge t_x$, and satisfy (1.1). We make the standing hypothesis that (1.1) does possess such solution [11]. As usual, a solution of (1.1) is said to be oscillatory if the set of its zeros is unbounded from above, otherwise it is called nonoscillatory. (1.1) is called oscillatory if all of its solutions are oscillatory.

We note that second order neutral delay differential equations are used in many fields such as vibrating masses attached to an elastic bar and some variational problems, etc., see, for example [11].

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and /or nonoscillation of second order linear and nonlinear neutral delay differential equations (see, for example, 1-3,5,7-10,14,15,17-21,24 and the references therein). For the second order neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)f(y(t-\sigma)) = 0. \tag{1.2}$$

To the best of our knowledge, almost all of the known results obtained for (1.2) required the assumption that the function $f(y)$ satisfies $f'(y) \ge k > 0$ or $f(y)/y \ge k > 0$ for $y \ne 0$, (see 2, 5, 7-10, 14, 15, 17, 18, 24), which is not applicable for $f(y) = |y|^{\nu-1}y$, the classical Emden-Fowler case. Very recently, the results of Atkinson [4] and Belohorec [6] for second order ordinary differential equation have been extended to (1.2) by Wong [21] under the assumption that the nonlinear function f satisfies the sublinear condition

$$0 < \int_{0+}^{\varepsilon} \frac{du}{f(u)}, \quad \int_{0-}^{-\varepsilon} \frac{du}{f(u)} < \infty \quad \text{for all } \varepsilon > 0,$$

as well as the superlinear condition

$$0 < \int_{\varepsilon}^{\infty} \frac{du}{f(u)}, \quad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty \quad \text{for all } \varepsilon > 0.$$

Also it will be of great interest to find some oscillation criteria for the special case for (1.2), even for the Emden-Fowler equation

$$(y(t) + p(t)y(t-\tau))'' + q(t)|y(t-\sigma)|^{\nu-1}y(t-\sigma) = 0, \quad \nu > 0. \tag{1.3}$$

This problem was posed by Wong [21, Remark d]. As an affirmative answer to it, Saker [19], Saker and Manojlović [20] have established some oscillation criteria for (1.2) and (1.3). However, these results cannot be applied to (1.1). Therefore, in the present paper, by using the averaging technique [13, 16, 22, 23], we shall establish several Kamenev-type oscillation criteria for (1.1). Our theorems essentially improve some known results in [19, 20]. In particular, two interesting examples that point out the application of our results are also included.

2. Main Results

In this section, we shall establish Kamemev-type oscillation theorems for (1.1) under the cases when $0 \le p(t) \le 1$ and $-1 < p_0 \le p(t) \le 0$, which extend the results in [13, 16, 22, 23] to (1.1). It will be convenient to make the following notations in the remainder of this paper. Let $\phi \in C^1([t_0,\infty),\mathbb{R}^+)$. Define

$$\mu = \min\left\{\frac{\beta-\alpha}{\beta-\gamma}, \frac{\beta-\alpha}{\gamma-\alpha}\right\}, \quad g(s) = \phi(s)r(s-\sigma_1), \quad \theta = \frac{1}{(\gamma+1)^{\gamma+1}},$$

$$Q_1(t) = \mu\phi(t)\left[q_1^{\beta-\gamma}(t)q_2^{\gamma-\alpha}(t)(1-p(t-\sigma_1))^{\alpha(\beta-\gamma)}(1-p(t-\sigma_2))^{\beta(\gamma-\alpha)}\right]^{1/(\beta-\alpha)},$$

$$Q_2(t) = \mu\phi(t)[\,q_1^{\beta-\gamma}(t)q_2^{\gamma-\alpha}(t)]^{1/(\beta-\alpha)}.$$

In order to present our theorems, we first introduce, following Philos [16], the function class $\Im$ which will be extensively used in the sequel. Namely, let $D_0 = \{(t,s) \in \mathbb{R}^2 : t > s \ge t_0\}$ and $D = \{(t,s) \in \mathbb{R}^2 : t \ge s \ge t_0\}$. We will say that the function $H \in C(D,\mathbb{R})$ belongs to the class $\Im$, denoted by $H \in \Im$, if

$(H1)$ $H(t,t) = 0$ for $t \ge t_0$, $H(t,s) > 0$ on $(t,s) \in D_0$;

$(H2)$ H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable such that

$$\frac{\partial}{\partial s}H(r,s) = -h(t,s)H(t,s) \quad \text{for } (t,s) \in D_0,$$

where $h \in C(D,\mathbb{R})$.

For $\varphi \in C([t_0,\infty),\mathbb{R})$, we take an operator $A(\,\cdot\,;T,t)$, which is defined in [22], in terms of H, as follows

$$A(\,\varphi;T,t) = \int_T^t H(r,s)\varphi(s)ds \quad \text{for } t > T. \tag{2.1}$$

It is easy to verify that $A(\cdot;T,t)$ is an linear operator.

For the sake of simplicity, throughout this paper, we always assume $H \in \Im$, $\phi \in C^1([t_0, \infty), \mathbb{R}^+)$, and the operator A defined by (2.1) without further mentioning.

Theorem 2.1. (1.1) is oscillatory provided that one of the following conditions holds.

(1). $0 \le p(t) \le 1$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(Q_1 - \theta g|h - \phi'\phi^{-1}|^{\gamma+1}; t_0, t) = \infty. \tag{2.2}$$

(2). $-1 < p_0 \le p(t) \le 0$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(Q_2 - \theta g|h - \phi'\phi^{-1}|^{\gamma+1}; t_0, t) = \infty. \tag{2.3}$$

Proof. **(1).** Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y(t) \ne 0$ for $t \ge t_0$. Further, we suppose that there exists a $t_1 > t_0$ such that

$$y(t) > 0, \quad y(t-\tau) > 0, \quad y(t-\sigma_1) > 0, \text{ and } y(t-\sigma_2) > 0 \text{ for } t \ge t_1, \tag{2.4}$$

since the substitution $u = -y$ transforms (1.1) into an equation of the same form subject to the assumptions of Theorem. As an analogous proof of Lemma 1 (1) [24], see also [15], then, for some $T_0 \ge t_1$, we have immediately that

$$x(t) > 0, \quad x'(t) > 0, \quad \text{and } (r(t)(x'(t))^\gamma)' \ge 0 \text{ for } t \ge T_0. \tag{2.5}$$

Using (2.5), nothing that $x(t) \ge y(t)$, we have

$$y(t) = x(t) - p(t)y(t-\tau) \ge x(t) - p(t)x(t-\sigma) \ge (1-p(t))x(t).$$

Thus, for all $t > T_0$,

$$y(t-\sigma_1) \ge (1-p(t-\sigma_1))x(t-\sigma_1), \quad \text{and } y(t-\sigma_2) \ge (1-p(t-\sigma_2))x(t-\sigma_2).$$

Then (1.1) implies that

$$\begin{aligned} &(r(t)|x'(t)|^{\gamma-1}x'(t))' + q_1(t)(1-p(t-\sigma_1))^\alpha x^\alpha(t-\sigma_1) \\ &\quad + q_2(t)(1-p(t-\sigma_2))^\beta x^\beta(t-\sigma_2) \le 0, \quad t \ge T_0. \end{aligned} \tag{2.6}$$

Define

$$w(t) = \phi(t)\frac{r(t)|x'(t)|^{\gamma-1}x'(t)}{x^\gamma(t-\sigma_1)}, \tag{2.7}$$

clearly, $w(t) > 0$. Differentiating (2.7) and using (2.6), we have

$$\begin{aligned} w'(t) \le\ & \frac{\phi'(t)}{\phi(t)} w(t) - \phi(t)\Big[q_1(t)(1-p(t-\sigma_1))^{\alpha} x^{\alpha-\gamma}(t-\sigma_1) \\ & + q_2(t)(1-p(t-\sigma_2))^{\beta} \frac{x^{\beta}(t-\sigma_2)}{x^{\gamma}(t-\sigma_1)}\Big] \\ & - \frac{\gamma\phi(t) r(t) |x'(t)|^{\gamma-1} x'(t) x'(t-\sigma_1)}{x^{\gamma+1}(t-\sigma_1)}. \end{aligned} \tag{2.8}$$

For simplicity, we consider the case $\sigma_1 \ge \sigma_2$, (a similar argument holds for $\sigma_1 < \sigma_2$), then $x(t-\sigma_1) \le x(t-\sigma_2)$. Hence, (2.8) yields that

$$\begin{aligned} w'(t) \ \le\ & \frac{\phi'(t)}{\phi(t)} w(t) - \phi(t)[q_1(t)(1-p(t-\sigma_1))^{\alpha} x^{\alpha-\gamma}(t-\sigma_1) \\ & + q_2(t)(1-p(t-\sigma_2))^{\beta} x^{\beta-\gamma}(t-\sigma_1)] - \frac{\gamma w^{(\gamma+1)/\gamma}(t)}{g^{1/\gamma}(t)}, \end{aligned} \tag{2.9}$$

since $r(t)(x'(t))^{\gamma} \le r(t-\sigma_1)(x'(t-\sigma_1))^{\gamma}$. The Young inequality [12, Theorem 61] implies that

$$\begin{aligned} & \frac{\beta-\gamma}{\beta-\alpha} q_1(t)(1-p(t-\sigma_1))^{\alpha} x^{\alpha-\gamma}(t-\sigma_1) + \frac{\gamma-\alpha}{\beta-\alpha} q_2(t)(1-p(t-\sigma_2))^{\beta} x^{\beta-\gamma}(t-\sigma_1) \\ & \quad \ge \Big[q_1^{\beta-\gamma}(t) q_2^{\gamma-\alpha}(t)(1-p(t-\sigma_1))^{\alpha(\beta-\gamma)}(1-p(t-\sigma_2))^{\beta(\gamma-\alpha)}\Big]^{1/(\beta-\alpha)}. \end{aligned}$$

Consequently,

$$\begin{aligned} & q_1(t)(1-p(t-\sigma_1))^{\alpha} x^{\alpha-\gamma}(t-\sigma_1) + q_2(t)(1-p(t-\sigma_2))^{\beta} x^{\beta-\gamma}(t-\sigma_2) \\ & \quad \ge \mu\Big[q_1^{\beta-\gamma}(t) q_2^{\gamma-\alpha}(t)\,(1-p(t-\sigma_1))^{\alpha(\beta-\gamma)}(1-p(t-\sigma_2))^{\beta(\gamma-\alpha)}\Big]^{1/(\beta-\alpha)}. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), we have

$$w'(t) \le -Q_1(t) + \frac{\phi'(t)}{\phi(t)} w(t) - \frac{\gamma}{g^{1/\gamma}(t)} w^{(\gamma+1)/\gamma}(t), \quad t \ge T_0. \tag{2.11}$$

Apply the operator $A(\,\cdot\,; T, t)$, $t > T \ge T_0$, to (2.11), and use $(H2)$ to find

$$\begin{aligned} A(Q_1; T, t) \le\ & H(t,T) w(T) + A(|h - \phi'\phi^{-1}| w; T, t) \\ & - \gamma A(g^{-1/\gamma} w^{(\gamma+1)/\gamma}; T, t). \end{aligned} \tag{2.12}$$

For given t and s, $(t \ne s)$, set

$$F(u) := \big|h - \phi'\phi^{-1}\big|\, u - \frac{\gamma}{g^{1/\gamma}} u^{(\gamma+1)/\gamma}, \quad u > 0.$$

$F(u)$ attains its maximum at

$$u = \frac{g}{(\gamma+1)^{\gamma}} \left|h - \phi'\phi^{-1}\right|^{\gamma},$$

and

$$F(u) \le F_{\max} = \theta g \left|h - \phi'\phi^{-1}\right|^{\gamma+1}. \tag{2.13}$$

Then by using (2.13) in (2.12), we get

$$A(Q_1; T, t) \le H(t, T)w(T) + \theta A(g|h - \phi'\phi^{-1}|^{\gamma+1}; T, t). \tag{2.14}$$

Set $T = T_0$ and in view of $(H2)$, so

$$A(Q_1 - \theta g|h - \phi'\phi^{-1}|^{\gamma+1}; T_0, t) \le H(t, T_0)w(T_0) \le H(t, t_0)w(T_0).$$

Thus, by $(H2)$, we obtain

$$\begin{aligned} &A(Q_1 - \theta g|h - \phi'\phi^{-1}|^{\gamma+1}; t_0, t) \\ &\quad = A(Q_1 - \theta g|h - \phi'\phi^{-1}|^{\gamma+1}; t_0, T_0) \\ &\qquad + A(Q_1 - \theta g|h - \phi'\phi^{-1}|^{\gamma+1}; T_0, t) \\ &\quad \le H(t, t_0) \left(\int_{t_0}^{T_0} Q_1(s)ds + w(T_0) \right). \end{aligned} \tag{2.15}$$

Divide (2.15) through by $H(t, t_0)$ and take limsup in it as $t \to \infty$. Condition (2.2) gives a desired contradiction in (2.15). This proves Case (1).

(2). Let $y(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y(t) \neq 0$ for $t \ge t_0$. Further, we suppose that there exists a $t_1 > t_0$ such that (2.4) holds. As an analogous proof of Lemma 1 (2) [24]. Then, for some $T_0 \ge t_1$, we still have that (2.5) holds for $t \ge T_0$. Noting that $y(t) \ge x(t)$, we get

$$y(t - \sigma_1) \ge x(t - \sigma_1), \;\; y(t - \sigma_2) \ge x(t - \sigma_2) \quad \text{for } t \ge T_0.$$

Then, (1.1) changes into

$$(r(t)|x'(t)|^{\gamma-1}x'(t))' + q_1(t)x^{\alpha}(t - \sigma_1) + q_2(t)x^{\beta}(t - \sigma_2) \le 0, \;\; t \ge T_0.$$

Consider the function $w(t)$ defined by (2.7), similar to the proof of (2.9), we can obtain

$$\begin{aligned} w'(t) \le{}& \frac{\phi'(t)}{\phi(t)}w(t) - \phi(t)\left[q_1(t)x^{\alpha-\gamma}(t - \sigma_1) + q_2(t)x^{\beta-\gamma}(t - \sigma_1)\right] \\ &- \frac{\gamma}{g^{1/\gamma}(t)}w^{(\gamma+1)/\gamma}(t). \end{aligned} \tag{2.16}$$

The rest of the proof is similar to that of Case (1), hence, is omitted. ∎

Theorem 2.2. *(1.1) is oscillatory provided that one of the following conditions holds.*

(3). $0 \le p(t) \le 1$, and there exist $\varphi_1,\ \varphi_2 \in C([t_0,\infty),\mathbb{R})$ such that for all $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A(Q_1;T,t) \ge \varphi_1(T), \tag{2.17}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A(g|h-\phi'\phi^{-1}|^{\gamma+1};T,t) \le \varphi_2(T), \tag{2.18}$$

where φ_1 and φ_2 satisfy

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} A(g^{-1/\gamma}(\varphi_1-\theta\varphi_2)_+^{(\gamma+1)/\gamma};T,t) = \infty. \tag{2.19}$$

(4). $-1 < p_0 \le p(t) \le 0$, and there exist $\varphi_1,\ \varphi_2 \in C([t_0,\infty),\mathbb{R})$ such that for all $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} A(Q_2;T,t) \ge \varphi_1(T), \tag{2.20}$$

and (2.18) hold, where φ_1 and φ_2 satisfy (2.19), and $\varphi_+ = \max\{0,\varphi\}$.

Proof. We only show Case (3). The proof of Case (4) is similar to that of Case (3). Proceeding as the proof of Case (1) of Theorem 2.1, we have (2.12) and (2.14) hold, again divide (2.14) by $H(t,T)$, and obtain

$$\frac{1}{H(t,T)} A(Q_1;T,t) - \frac{\theta}{H(t,T)} A(g|h-\phi'\phi^{-1}|^{\gamma+1};T,t) \le w(T), \quad t > T. \tag{2.21}$$

Take limsup in (2.21) as $t\to\infty$, and note from (2.17) and (2.18) that

$$\varphi_1(T) - \theta\varphi_2(T) \le w(T),$$

from which it follows that

$$\begin{aligned} &\frac{1}{H(t,T)} A(g^{-1/\gamma}(\varphi_1-\theta\varphi_2)_+^{(\gamma+1)/\gamma};T,t) \\ &\qquad \le \frac{1}{H(t,T)} A(g^{-1/\gamma} w^{(\gamma+1)/\gamma};T,t). \end{aligned} \tag{2.22}$$

On the other hand, by (2.12), we have

$$\frac{\gamma}{H(t,T)}A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T,t)-\frac{1}{H(t,T)}A(|h-\phi'\phi^{-1}|w;T,t)\\ \leq w(T)-\frac{1}{H(t,T)}A(Q_1;T,t).$$

Thus, by (2.17),

$$\liminf_{r\to\infty}\left\{\frac{\gamma}{H(t,T_0)}A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t)\right.\\ \left.-\frac{1}{H(t,T_0)}A(|h-\phi'\phi^{-1}|w;T_0,t)\right\}\leq w(T_0)-\varphi_1(T_0)\leq C_0, \tag{2.23}$$

where C_0 is a constant. According to (2.23), there exists a sequence $\{t_j\}_{j=1}^{\infty}\in[t_0,\infty)$ with $\lim_{j\to\infty}t_j=\infty$ such that

$$\frac{\gamma}{H(t_j,T_0)}A\left(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t_j\right)\\ -\frac{1}{H(t_j,T_0)}A(|h-\phi'\phi^{-1}|w;T_0,t_j)\leq C_0+1. \tag{2.24}$$

Now, we claim that

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t)<\infty. \tag{2.25}$$

If (2.25) does not hold, and noting that (2.24), we get

$$\lim_{j\to\infty}\frac{1}{H(r_j,T_0)}A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t_j)=\infty. \tag{2.26}$$

So, (2.24) and (2.26) gives, for j large enough,

$$\frac{A(|h-\phi'\phi^{-1}|w;T_0,t_j)}{A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t_j)}-\gamma\geq-\frac{\gamma}{2},$$

that is,

$$A(|h-\phi'\phi^{-1}|w;T_0,t_j)\geq\frac{\gamma}{2}A\left(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t_j\right). \tag{2.27}$$

The Hölder inequality follows

$$A(|h-\phi'\phi^{-1}|w;T_0,t_j)\leq(A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t_j))^{\gamma/(\gamma+1)}\\ (A(g|h-\phi'\phi^{-1}|^{\gamma+1};T_0,t_j))^{1/(\gamma+1)}. \tag{2.28}$$

From (2.27) and (2.28), we obtain

$$\begin{aligned}(2.29)\qquad &\Big(\frac{\gamma}{2}\Big)^{\gamma+1}\frac{1}{H(t_j,T_0)}A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T_0,t_j)\\ &\le \frac{1}{H(t_j,T_0)}A(g|h-\phi'\phi^{-1}|^{(\gamma+1)/\gamma};T_0,t_j).\end{aligned}$$

By (2.18), the right-hand side of (2.29) is bounded, which contradicts (2.26). Thus (2.25) holds. Hence, by (2.22),

$$\begin{aligned}&\liminf_{r\to\infty}\frac{1}{H(t,T)}A(g^{-1/\gamma}(\varphi_1-\theta\varphi_2)_+^{(\gamma+1)/\gamma};T,t)\\ &\qquad\le \liminf_{r\to\infty}\frac{1}{H(t,T)}A(g^{-1/\gamma}w^{(\gamma+1)/\gamma};T,t)<\infty.\end{aligned}$$

which contradicts (2.19). ■

Remark 2.1. Let $\gamma=1$. For the superlinear equation (1.2), Theorems 2.1-2.2 improve the main results in [20]. For the Emden-Fowler type equation (1.3), our results extend and improve the main results in [19].

3. Corollaries and Examples

As Theorems 2.1 and 2.2 are rather general, it is convenient for applications to derive a number of oscillation criteria with the appropriate choice of the functions H and ϕ. In this section, we will give some corollaries of Theorems 2.1 and 2.2, and show the applications of our main results in two interesting examples. Further, we will see that equations (3.10) and (3.11) are oscillatory based on Corollary 3.2 and Theorem 2.2, though the known results such as in [1-3,7-10,14,15,17-21,24] fail to apply to these equations.

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 3.1. (1.1) *is oscillatory provided that one of the following conditions holds.*

(5). $0\le p(t)\le 1$, and

$$(3.1)\qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}A(Q_1;t_0,t)=\infty,$$

and

$$(3.2)\qquad \limsup_{t\to\infty}\frac{1}{H(t,t_0)}A(g|h-\phi'\phi^{-1}|^{\gamma+1};t_0,t)<\infty.$$

(6). $-1 < p_0 \le p(t) \le 0$, and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(Q_2; t_0, t) = \infty, \tag{3.3}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} A(g|h - \phi'\phi^{-1}|^{\gamma+1}; t_0, t) < \infty. \tag{3.4}$$

Corollary 3.1. (1.1) *is oscillatory provided that one of the following conditions holds.*

(7). $0 \le p(t) \le 1$, and for some $\lambda > \gamma$,

$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_{t_0}^{t} [G(t) - G(s)]^\lambda Q_3(s) ds = \infty. \tag{3.5}$$

(8). $-1 < p_0 \le p(t) \le 0$, and for some $\lambda > \gamma$,

$$\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_{t_0}^{t} [G(t) - G(s)]^\lambda Q_4(s) ds = \infty. \tag{3.6}$$

Where

$$\begin{aligned}
G(t) &= \int_{t_0}^{t} \frac{ds}{r^{1/\gamma}(s-\sigma_1)}, \\
Q_3(t) &= \mu \left[q_1^{\beta-\gamma}(t) q_2^{\gamma-\alpha}(t) (1-p(t-\sigma_1))^{\alpha(\beta-\gamma)} (1-p(t-\sigma_2))^{\beta(\gamma-\alpha)} \right]^{1/(\beta-\alpha)}, \\
Q_4(t) &= \mu \left[q_1^{\beta-\gamma}(t) q_2^{\gamma-\alpha}(t) \right]^{1/(\beta-\alpha)}.
\end{aligned}$$

Proof. Let

$$H(t,s) = [G(t) - G(s)]^\lambda \quad \text{and} \quad \phi(t) = 1,$$

then

$$h(t,s) = \frac{\lambda}{r^{1/\gamma}(s-\sigma_1)[G(t) - G(s)]}.$$

Note that

$$A(g|h-\phi'\phi^{-1}|^{\gamma+1};T,t)=\frac{\lambda^{\gamma+1}}{\lambda-\gamma}[G^{\lambda-\gamma}(t)-G^{\lambda-\gamma}(T)],$$

it is easy to show that Corollary 3.2 holds from Corollary 3.1. ■

By Theorem 2.2, we have

Corollary 3.3. *Let* $\lim_{t\to\infty}G(t)=\infty$. *Then (1.1) is oscillatory provided that one of the following conditions holds.*

(9). $0\le p(t)\le 1$, and there exists $\varphi\in C([t_0,\infty),\mathbb{R})$ such that for some $\lambda>\gamma$ and all $T\ge t_0$,

$$\limsup_{t\to\infty}\frac{1}{G^{\lambda}(t)}\int_T^t[G(t)-G(s)]^{\lambda}Q_3(s)ds\ge\varphi(T), \tag{3.7}$$

where φ satisfies

$$\liminf_{t\to\infty}\frac{1}{G^{\lambda}(t)}\int_T^t[G(t)-G(s)]^{\lambda}g^{-1/\gamma}(s)\varphi^{(\gamma+1)/\gamma}(s)ds=\infty. \tag{3.8}$$

(10). $-1<p_0\le p(t)\le 0$, and there exists $\varphi\in C([t_0,\infty),\mathbb{R})$ such that (3.8) holds and for some $\lambda>\gamma$ and all $T\ge t_0$,

$$\limsup_{t\to\infty}\frac{1}{G^{\lambda}(t)}\int_T^t[G(t)-G(s)]^{\lambda}Q_4(s)ds\ge\varphi(T), \tag{3.9}$$

where $Q_3(t)$, $Q_4(t)$ and $G(t)$ are defined in Corollary 3.2.

Remark 3.1. Following the classical ideas of Kamenev [13], we define $H(t,s)$ as

$$H(t,s)=(t-s)^{\lambda},\quad (t,s)\in D,$$

where $\lambda>\gamma$. Therefore, as a consequence of Theorems 2.1 and 2.2, we can obtain two oscillation criteria. Here, we omit the details.

The following two examples illustrate our main results

Example 3.1. Consider the following neutral differential equation

$$\begin{aligned}&(|x'(t)|x'(t))'+q_1(t)|y(t-3)|^{1/(2n+1)}y(t-3)\\&+q_2(t)|y(t-2)|^{(4n+1)/(2n+1)}y(t-2)=0,\quad t\ge 3,\end{aligned} \tag{3.10}$$

where $r(t) = 1$, $x(t) = y(t) + p(t)x(t-1)$, $p(t) = p/\sqrt{t+3}$, $p = \pm 1$, $\gamma = 2$, $\alpha = (2n+2)/(2n+1)$, $\beta = (6n+2)/(2n+1)$, n is positive integer, and $q_1,\ q_2 \in C([3,\infty), \mathbb{R}^+)$ with $q_1(t)q_2(t) \geq \lambda_1^2/t^2$, $(\lambda_1 > 0)$.

Now, we consider the following two cases.

Case 1. $p = 1$. Noting that $p(t-3) > p(t-2)$, we have

$$\mu = 2, \quad G(t) = t-3, \quad \text{and} \quad Q_3(t) \geq \frac{2\lambda_1}{t}(1 - \frac{1}{\sqrt{t}})^2.$$

Then, by Theorem 41 in [12],

$$(t-s)^\lambda \geq t^\lambda - \lambda s t^{\lambda-1} \quad \text{for } \lambda > 1,\ t \geq s \geq 1,$$

we have, for $\lambda > \gamma$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{G^\lambda(t)} \int_{t_0}^{t} [G(t) - G(s)]^\lambda Q_3(s)ds \\
&\quad \geq \limsup_{t\to\infty} \frac{2\lambda_1}{t^\lambda} \int_3^t \frac{(t-s)^\lambda}{s} \left(1 - \frac{1}{\sqrt{s}}\right)^2 ds \\
&\quad \geq 2\lambda_1 \left(1 - \frac{1}{\sqrt{3}}\right)^2 \limsup_{t\to\infty} \frac{1}{t^\lambda} \int_3^t \frac{t^\lambda - \lambda t^{\lambda-1} s}{s} ds = \infty.
\end{aligned}$$

Hence, by Corollary 3.2 (7), (3.10) is oscillatory.

Case 2. $p = -1$. We have

$$\mu = 2, \quad G(t) = t-3, \quad \text{and} \quad Q_4(t) \geq \frac{2\lambda_1}{t}.$$

The rest of proof is similar to that of Case 1. Hence (3.10) is oscillatory by Corollary 3.2 (8).

Example 3.2. Consider the following neutral differential equation

$$\begin{aligned}
&(|x'(t)|^{\gamma-1}x'(t))' + q_1(t)|y(t-3)|^{\alpha-1}y(t-3) \\
&\qquad + q_2(t)|y(t-2)|^{\beta-1}y(t-2) = 0, \quad t \geq 3,
\end{aligned} \tag{3.11}$$

where $x(t) = y(t) + py(t-1)$, $-1 < p < 1$, $r(t) = 1$, $0 < \alpha < \gamma < \beta$ with $\gamma = (\alpha+\beta)/2$, and $q_1,\ q_2 \in C([3,\infty), \mathbb{R}^+)$ with $q_1(t)q_2(t) \geq (\lambda_2 t^{\gamma-1})^2$, $(\lambda_2 > 0)$.

For Theorem 2.2, let $\phi(t) = t^{-(\gamma+1)}$. Now, we consider the following two cases.

Case 1. $0 \le p < 1$. A direct computation yields

$$\mu = 2, \quad g(t) = t^{-(\gamma+1)}, \quad \text{and} \quad Q_1(t) \ge \frac{2\lambda_2(1-p)^\gamma}{t^2}.$$

Let $H(t,s) = (t-s)^{\gamma+1}$, then $h(t,s) = (\gamma+1)/(t-s)$. Thus, for all $t > T \ge 3$,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{H(t,T)} A(Q_1;T,t) &= \limsup_{t\to\infty} \frac{1}{(t-T)^{\gamma+1}} \int_T^t (t-s)^{\gamma+1} Q_1(s)ds \\
&\ge 2\lambda_2(1-p)^\gamma \limsup_{t\to\infty} \frac{1}{(t-T)^{\gamma+1}} \int_T^t \frac{(t-s)^{\gamma+1}}{s^2} ds \\
&= 2\lambda_2(1-p)^\gamma \limsup_{t\to\infty} \frac{1}{(t-T)^{\gamma+1}} \int_T^t \frac{t^{\gamma+1} - (\lambda+1)t^\gamma s}{s^2} ds \\
&= \frac{2\lambda_2(1-p)^\gamma}{T},
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} A(g|h - \phi'\phi^{-1}|^{\gamma+1};T,t) \\
&\quad = \limsup_{t\to\infty} \frac{(\gamma+1)^{\gamma+1}}{(t-T)^{\gamma+1}} \int_T^t \frac{t^{\gamma+1}}{s^{2(\gamma+1)}}\, ds = \frac{(\gamma+1)^{\gamma+1}}{(2\gamma+1)T^{2\gamma+1}}.
\end{aligned}$$

Set,

$$\varphi_1(T) = \frac{2\lambda_2(1-p)^\gamma}{T} \text{ and } \varphi_2(T) = \frac{(\gamma+1)^{\gamma+1}}{2\gamma+1} \frac{1}{T^{2\gamma+1}}.$$

Thus, for $\lambda_2 > (9(1-p))^{-\gamma}/(4\gamma+2)$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} A(g^{-1/\gamma}(\varphi_1 - \theta\varphi_2)_+^{\gamma+1};T,t) \\
&\quad = \liminf_{t\to\infty} \frac{1}{(t-T)^{\gamma+1}} \int_T^t (t-s)^{\gamma+1} \left(2\lambda_2(1-p)^\gamma - \frac{1}{(2\gamma+1)s^{2\gamma}}\right)^{(\gamma+1)/\gamma} ds \\
&\quad \ge \left(2\lambda_2(1-p)^\gamma - \frac{1}{9^\gamma(2\gamma+1)}\right)^{(\gamma+1)/\gamma} \liminf_{t\to\infty} \frac{1}{(t-T)^{\gamma+1}} \int_T^t (t-s)^{\gamma+1} ds = \infty.
\end{aligned}$$

It follows from Theorem 2.2 (3) that (2.11) is oscillatory for $\lambda_2 > (9(1-p))^{-\gamma}/(4\gamma+2)$.

Case 2. $-1 \leq p \leq 0$. Then

$$\mu = 2, \quad g(t) = t^{-(\gamma+1)}, \quad Q_2(t) \geq \frac{2\lambda_2}{t^2}.$$

The rest of proof is similar to that of Case 1. Hence, by Theorem 2.2 (4), (3.11) is oscillatory for $\lambda_2 > 9^{-\gamma}/(4\gamma + 2)$.

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