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CHARACTERIZATIONS OF ALMOST CONVERGENT SEQUENCES IN A HILBERT SPACE OR IN $L^p(T)$

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Abstract. In [Acta Math. 80(1948),167-190], G. G. Lorentz characterized almost convergent sequences in $\mathbb{R}$ (or in $\mathbb{C}$) in terms of the concept of uniform convergence of the de la Vallée-Poussin means. In this paper, we give a further study on such kind of convergence for any Hilbert space or $L^p(T)$, where $1 \le p \le \infty$. Two new Cauchy forms for almost convergence are established. We prove that any of them is equivalent to the one established by Miller and Orhan. We use these forms to characterize almost convergent sequences in the aforementioned spaces in terms of coefficients.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a Hilbert space or $(L^p(T), \|\cdot\|_p)$, and $f_n \in X$, where $1 \le p \le \infty$ and $T = (-\pi, \pi]$. We say that $\{f_n\}_{n=0}^{\infty}$ is weakly almost convergent to $f \in X$, if the following statement is true:

$$(1.1)\qquad L(\{\lambda(f_n)\}) = \lambda(f) \text{ for all Banach limits } L \text{ and for all } \lambda \in X^*,$$

where X^* denotes the dual space of X. This concept was introduced in [7, p. 169] by G. G. Lorentz for the scalar fields $\mathbb{R}$ or $\mathbb{C}$, and extended to general X by Deeds [3] and Kurtz [5, p. 494]. In [7], Lorentz used "almost convergent" instead of "weakly almost convergent". It is known (see e.g. [7, Theorem 1] or [6, Theorem 3.2(d)]) that $(1.1) \Longleftrightarrow (1.2) \Longleftarrow (1.3)$, where

$$(1.2)\qquad \lim_{N\to\infty}\left\{\sup_{n\ge 0}\left|\lambda\left(\frac{1}{N}\sum_{k=n}^{n+N-1} f_k - f\right)\right|\right\} = 0 \quad \text{for all } \lambda \in X^*,$$

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$$\lim_{N\to\infty}\left\{\sup_{n\geq 0}\left\|\frac{1}{N}\sum_{k=n}^{n+N-1} f_k - f\right\|\right\} = 0. \tag{1.3}$$

In general, (1.1) and (1.2) are weaker than (1.3). But they are equivalent when $\{f_n\}_{n=0}^{\infty}$ is relatively compact (cf. [5, Theorem 2.1.3]). Condition (1.3) is analogous to the definition of the mean values which are used in the theory of almost periodic functions, (cf. [7, Eq. (11)] or [2, Eq. (1.31)]). This definition involves the de la Vallée-Poussin means $\frac{1}{N}\sum_{k=n}^{n+N-1} f_k \quad (n \geq 0; N \geq 1)$ and is related to the method $(C, 1)$ of the arithmetic means. It describes a uniform convergence property. When (1.3) is true, we shall say that $\{f_n\}_{n=0}^{\infty}$ is almost convergent to f.

In [7, §3], Lorentz used (1.3) to find several examples of almost convergent sequences for $X = \mathbb{R}$ or $\mathbb{C}$. These include almost periodic sequences. In this paper, we try to characterize almost convergent sequences in X from the following two directions. In §2, we use a weak form of (1.3) to derive two new Cauchy forms of (1.3). These forms allow us to examine the almost convergence property of $\{f_n\}_{n=0}^{\infty}$ directly from the behavior of the differences of its de la Vallée-Poussin means, without investigating the existence problem of the limit f. They are equivalent to the one appeared in [8, Theorem 2.5]. In §3-§4, we characterize the almost convergence property from the viewpoint of coefficients. More precisely, we shall use (1.3) and the aforementioned Cauchy forms to characterize those matrices $A = (c_{n,k})_{n,k\geq 0}$ for which the sequences

$$f_n = \sum_{k=0}^{\infty} c_{n,k}\phi_k \qquad (n \geq 0) \tag{1.4}$$

are almost convergent in X, where $\{\phi_n\}_{n=0}^{\infty}$ is a given set in X such that the associated Parseval formula or the Hausdorff-Young inequality holds. We state our results for the case that X is a Hilbert space or $L^p(T)$, where $1 \leq p \leq \infty$. For $p = 2$, these matrices have been completely characterized. As for $p \neq 2$, certain type of necessary conditions are found. However, it is unsolved for sufficient conditions. The details are given in §4.

Throughout this paper, $\ell^p(\Omega)$ denotes the space of all complex sequences $\{c_n\}_{n\in\Omega}$ with the property: $\left(\sum_{n\in\Omega}|c_n|^p\right)^{1/p} < \infty$. For the case that Ω is the set of all nonnegative integers, we write ℓ^p instead of $\ell^p(\Omega)$.

2. Cauchy forms of (1.3)

We know that $\sup_{n\geq 0}(*) \geq \sup_{n\geq N}(*)$. Hence, (2.1) is a weak form of (1.3), and so $(1.3) \Longrightarrow (2.1)$, where

$$\lim_{N\to\infty}\left\{\sup_{n\geq N}\left\|\frac{1}{N}\sum_{k=n}^{n+N-1} f_k - f\right\|\right\} = 0. \tag{2.1}$$

It is clear that (2.1) describes the behavior of the de la Vallée-Poussin means $V_{n,N}(\{f_k\})$ from the terms with indices $n \geq N$, where

$$V_{n,N}(\{f_k\}) = \frac{f_n + f_{n+1} + \cdots + f_{n+N-1}}{N} \qquad (n \geq 0; N \geq 1). \tag{2.2}$$

With no ambiguity, we write $V_{n,N}$ in the place of $V_{n,N}(\{f_k\})$. From

$$\sup_{n\geq N}\|f_n\| \leq N\left(\sup_{n\geq N}\|V_{n,N}\| + \frac{N-1}{N}\sup_{n\geq N}\|V_{n+1,N-1}\|\right),$$

we see that if (2.1) is satisfied, then $\sup_{n\geq 0}\|f_n\| < \infty$. With this help, we get the following equivalence.

Lemma 2.1. *We have* $(1.3) \Longleftrightarrow (2.1)$.

Proof. For completeness, we introduce the following proof, which is given in [4]. Let $N \geq 1$, $M \geq 1$, and ℓ be the non-negative integer such that $\ell M \leq N < (\ell+1)M$. We have

$$\begin{aligned}\|V_{n,N}\| &\leq \frac{1}{N}\sum_{k=n}^{M-1}\|f_k\| + \frac{M}{N}\sum_{u=1}^{\ell-1}\|V_{uM,M}\| + \frac{1}{N}\sum_{k=\ell M}^{n+N-1}\|f_k\| \\ &\leq \frac{3M}{N}\left(\sup_{k\geq 0}\|f_k\|\right) + \frac{(\ell-1)M}{N}\sup_{m\geq M}\|V_{m,M}\| \quad (n<M),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\|V_{n,N}\| &\leq \frac{M}{N}\sum_{u=0}^{\ell-1}\|V_{n+uM,M}\| + \frac{1}{N}\sum_{k=n+\ell M}^{n+N-1}\|f_k\| \\ &\leq \frac{\ell M}{N}\sup_{m\geq n}\|V_{m,M}\| + \frac{M}{N}\left(\sup_{k\geq n}\|f_k\|\right) \quad (n \geq M),\end{aligned} \tag{2.4}$$

so $(2.3) - (2.4)$ together yields

$$\sup_{n\geq 0}\|V_{n,N}\| \leq \frac{3M}{N}\left(\sup_{k\geq 0}\|f_k\|\right) + \sup_{m\geq M}\|V_{m,M}\| \quad (N \geq 1, M \geq 1). \tag{2.5}$$

Consider the replacement of f_n by $f_n - f$. Then the desired result follows. ■

Lemma 2.1 is important in the establishment of Cauchy criterions given below. Consider the following three Cauchy forms of (1.3):

$$\lim_{\min(M,N)\to\infty}\left\{\sup_{m\geq M,n\geq N}\left\|\frac{f_m+\cdots+f_{m+M-1}}{M}-\frac{f_n+\cdots+f_{n+N-1}}{N}\right\|\right\}=0, \tag{2.6}$$

$$\lim_{N\to\infty}\left\{\sup_{m,n\geq N}\left\|\frac{f_m+\cdots+f_{m+N-1}}{N}-\frac{f_n+\cdots+f_{n+N-1}}{N}\right\|\right\}=0, \tag{2.7}$$

$$\lim_{N\to\infty}\left\{\sup_{m,n\geq 0}\left\|\frac{f_m+\cdots+f_{m+N-1}}{N}-\frac{f_n+\cdots+f_{n+N-1}}{N}\right\|\right\}=0. \tag{2.8}$$

In [8, Theorem 2.5], Miller and Orhan have proved that $(2.6) \iff (1.3)$. In the following, we prove that any of (2.7) – (2.8) is equivalent to (1.3).

Theorem 2.2. *Let $f_n \in X$. Then (1.3) holds for some $f \in X$ if and only if (2.7) (or (2.8)) is true.*

Proof. First, we prove $(2.6) \iff (2.7)$. Obviously, $(2.6) \implies (2.7)$. For $m \geq M \geq 1$, we have

$$\begin{aligned}\|V_{m,M}-V_{MN,MN}\| &\leq \frac{1}{N}\sum_{\ell=N}^{2N-1}\|V_{m,M}-V_{M\ell,M}\| \\ &\leq \sup_{m^*,m'\geq M}\|V_{m^*,M}-V_{m',M}\|.\end{aligned} \tag{2.9}$$

Analogously, for $n \geq N \geq 1$,

$$\|V_{n,N}-V_{MN,MN}\| \leq \sup_{n^*,n'\geq N}\|V_{n^*,N}-V_{n',N}\|. \tag{2.10}$$

Putting (2.9) – (2.10) together yields

$$\begin{aligned}\sup_{m\geq M,n\geq N}\|V_{m,M}-V_{n,N}\| \leq& \sup_{m^*,m'\geq M}\|V_{m^*,M}-V_{m',M}\| \\ &+\sup_{n^*,n'\geq N}\|V_{n^*,N}-V_{n',N}\|,\end{aligned}$$

and so $(2.7) \implies (2.6)$. Therefore, $(2.6) \iff (2.7)$. From

$$\sup_{m,n\geq 0}\|V_{m,N}-V_{n,N}\| \leq \sup_{m\geq 0}\|V_{m,N}-f\|+\sup_{n\geq 0}\|V_{n,N}-f\|,$$

we obtain the implication: $(1.3) \Longrightarrow (2.8)$. It is clear that $(2.8) \Longrightarrow (2.7)$. By [8, Theorem 2.5], we know that $(2.7) \Longleftrightarrow (2.6) \Longleftrightarrow (1.3)$. Hence, $(2.8) \Longleftrightarrow (1.3)$. This completes the proof. ■

Theorem 2.2 enables us to examine the almost convergence property of the sequence $\{f_n\}_{n=0}^\infty$ directly from the behavior of the differences of its de la Vallée-Poussin means. This method does not involve the existence of the limit f.

3. Characterization from the Viewpoint of Coefficients

Let $\{\phi_n\}_{n=0}^\infty$ be an orthonormal set in a Hilbert space $(X, \|\cdot\|)$, and $A = (c_{n,k})_{n,k\geq 0}$, f_n be related by (1.4). We know that $f_n \in X$ if and only if $\sum_{k=0}^\infty |c_{n,k}|^2 < \infty$. Moreover, it follows from the Parseval formula that

$$\begin{aligned}\left\|\frac{1}{N}\sum_{\ell=n}^{n+N-1} f_\ell - f\right\|^2 &= \left\|\sum_{k=0}^\infty\left(\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k\right)\phi_k\right\|^2 \\ &= \sum_{k=0}^\infty\left|\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k\right|^2,\end{aligned}$$

where $f = \sum_{k=0}^\infty c_k\phi_k$. Putting these with (1.3) together yields the following result.

Theorem 3.1. *Let $X, \phi_n, c_{n,k}$, and f_n be given as above. Then $\{f_n\}_{n=0}^\infty$ is an almost convergent sequence in X if and only if the following two assertions hold:*

(i) *$\sum_{k=0}^\infty |c_{n,k}|^2 < \infty$ for each $n \geq 0$,*

(ii) *there exists a sequence $\{c_n\}_{n=0}^\infty \in \ell^2$ such that*

$$\lim_{N\to\infty}\left\{\sup_{n\geq 0}\left(\sum_{k=0}^\infty\left|\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k\right|^2\right)\right\} = 0. \tag{3.1}$$

It is clear that (3.1) reduces to (1.3), if $c_{n,k^*} = f_n$, $c_{k^*} = f$, and $c_{n,k} = c_k = 0$ for $k \neq k^*$, where $k^* \geq 0$ is given. Hence, (3.1) is an infinite dimensional version of (1.3). In particular, for each $k^* \geq 0$, $\{c_{n,k^*}\}_{n=0}^\infty$ is almost convergent to c_{k^*}. This can be derived from the inequality:

$$\left(\sup_{n\geq 0}\left|\frac{1}{N}\sum_{\ell=n}^{n+N-1} c_{\ell,k^*} - c_{k^*}\right|\right)^2 \leq \sup_{n\geq 0}\left(\sum_{k=0}^\infty\left|\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k\right|^2\right).$$

It should be noticed that the reverse implication is false, in general. Consider the matrix: $c_{n,k} = (-1)^{n-k}$ for $n \geq k \geq 0$, and 0 otherwise. For this matrix, Theorem

3.1(i) is true and $\{c_{n,k^*}\}_{n=0}^{\infty}$ is almost convergent to 0 for each $k^* \geq 0$. However, for $n \geq 0$ and odd N, we have $\displaystyle\sum_{k=0}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k}\right|^2 = \frac{(n+1)+(N-1)/2}{N^2}$, and so (3.1) fails for the case $c_k = 0$.

Set $\rho_{k,N} = \sup_{n\geq 0}\left|\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k\right|$. It is clear that $(3.2) \Longrightarrow (3.1)$:

$$\sum_{k=0}^{\infty} \rho_{k,N}^2 = o(1) \qquad \text{as} \quad N \to \infty. \tag{3.2}$$

Hence, Theorem 3.1 can apply to those matrices which satisfy (3.2). For the case that $c_{n,k} = c_k$ for $k \leq n$ and 0 otherwise, we have $\rho_{k,N} = \frac{\min(k,N)}{N}|c_k|$, and so (3.2) holds if and only if $\sum_{k=0}^{\infty}(\frac{\min(k,N)}{N})^2|c_k|^2 = o(1)$ as $N \to \infty$, or equivalently, $\{c_k\}_{k=0}^{\infty} \in \ell^2$. In this case, the sequence defined by $f_n = \sum_{k=0}^{n} c_k\phi_k \quad (n \geq 0)$ is almost convergent in X. Another example is given as follows: $c_{n,k} = 0$ for all $k > k_0$, and $\{c_{n,k}\}_{n=0}^{\infty}$ is almost convergent in $\mathbb{C}$ for each $k \leq k_0$, where k_0 is a fixed integer. In this case, the sequence defined by $f_n = \sum_{k=0}^{k_0} c_{n,k}\phi_k \quad (n \geq 0)$ is almost convergent in X.

We know that $(L^2(T), \|\cdot\|_2)$ is a Hilbert space and $\{e^{int}\}_{n=-\infty}^{\infty}$ forms an orthonormal basis for $L^2(T)$. Rewrite $\{e^{ikt}\}_{k=-\infty}^{\infty}$ as $\{\phi_n\}_{n=0}^{\infty}$ in the order: $k = 0, 1, -1, 2, -2, \cdots$. Let $\{f_n\}_{n=0}^{\infty}$ be the corresponding sequence defined by (1.4), in symbol,

$$f_n(t) = \sum_{k=-\infty}^{\infty} c_{n,k}e^{ikt} \qquad (n \geq 0; t \in T). \tag{3.3}$$

Applying Theorem 3.1 to this case, we get the following characterization of almost convergent sequences in $L^2(T)$.

Corollary 3.2. *Let $c_{n,k}$ and f_n be related by (3.3). Then $\{f_n\}_{n=0}^{\infty}$ is an almost convergent sequence in $L^2(T)$ if and only if the following two assertions hold:*

(i) $\sum_{k=-\infty}^{\infty}|c_{n,k}|^2 < \infty$ *for each* $n \geq 0$,

(ii) *there exists a sequence* $\{c_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$ *such that*

$$\lim_{N\to\infty}\left\{\sup_{n\geq 0}\left(\sum_{k=-\infty}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k\right|^2\right)\right\} = 0. \tag{3.4}$$

The above corollary indicates that under (i), (3.4) characterizes the almost convergent sequences in $L^2(T)$. We point out that this condition can not be used as a tool to determine whether a sequence is norm convergent. Let us illustrate this by the example: $f_n(t) = e^{int} = \sum_{k=-\infty}^{\infty} c_{n,k}e^{ikt}$, where $c_{n,k} = 1$ for $k = n$ and 0 otherwise. Obviously, $\sum_{k=-\infty}^{\infty} |c_{n,k}|^2 < \infty$ for all $n \geq 0$. Moreover,

$$\sup_{n\geq 0}\left(\sum_{k=-\infty}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1} c_{n+\ell,k} - 0\right|^2\right) = \frac{1}{N} \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

This shows that (3.4) holds for the case $c_k = 0$. By Corollary 3.2, we conclude that $\{f_n\}_{n=0}^{\infty}$ is almost convergent in $L^2(T)$. However, it is clear that this sequence is not norm convergent in $L^2(T)$.

In Theorem 3.1(ii) and Corollary 3.2(ii), we assume the existence of the sequence $\{c_n\}$. This difficulty can be removed by applying the Cauchy criterions established in Theorem 2.2. The details are given below. Let f_n and ϕ_n be related by (1.4). It follows from the Parseval formula that

$$\begin{aligned}&\left\|\frac{f_m + \cdots + f_{m+N-1}}{N} - \frac{f_n + \cdots + f_{n+N-1}}{N}\right\| \\ =&\left\|\sum_{k=0}^{\infty}\left(\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k} - c_{n+\ell,k})\right)\phi_k\right\| = \left\{\sum_{k=0}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k} - c_{n+\ell,k})\right|^2\right\}^{1/2}.\end{aligned}$$

Applying Theorem 2.2 to the case (2.8), we get the following characterization of almost convergent sequences in a Hilbert space $(X, \|\cdot\|)$.

Theorem 3.3. *Let $X, \phi_n, c_{n,k}$, and f_n be defined as in Theorem 3.1. Then $\{f_n\}_{n=0}^{\infty}$ is an almost convergent sequence in X if and only if $\sum_{k=0}^{\infty} |c_{n,k}|^2 < \infty$ for all $n \geq 0$ and (3.5) holds:*

$$\lim_{N\to\infty}\left\{\sup_{m,n\geq 0}\left(\sum_{k=0}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k} - c_{n+\ell,k})\right|^2\right)\right\} = 0. \tag{3.5}$$

Take the particular case: $X = L^2(T)$ and $\{\phi_n\}_{n=0}^{\infty} = \{e^{ikt}\}_{k=-\infty}^{\infty}$. Then Theorem 3.3 can be rewritten in the following form.

Corollary 3.4. *Let $c_{n,k}$ and f_n be related by (3.3). Then $\{f_n\}_{n=0}^{\infty}$ is an almost convergent sequence in $L^2(T)$ if and only if $\sum_{k=-\infty}^{\infty} |c_{n,k}|^2 < \infty$ for all $n \geq 0$ and (3.6) holds:*

$$\lim_{N\to\infty}\left\{\sup_{m,n\geq 0}\left(\sum_{k=-\infty}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k} - c_{n+\ell,k})\right|^2\right)\right\} = 0. \tag{3.6}$$

Corollary 3.4 is different from Corollary 3.2 at the point of the existence of the sequence $\{c_n\}_{n=-\infty}^{\infty}$. Condition (3.6) allows us to check the almost convergence property of a given sequence in $L^2(T)$ directly from the behavior of the differences of their coefficients.

4. Extensions of §3 to the Case $L^p(T)$

In Theorem 3.1 through Corollary 3.4, X is assumed to be a Hilbert space. A natural question arises: can they be extended to Banach spaces X? For $X = L^p(T)$, $1 \le p \le 2$, the Hausdorff-Young inequality, (see [9, Vol. II, p. 101]), implies that if $f_n \in X$, then $\sum_{k=-\infty}^{\infty} |c_{n,k}|^q < \infty$, where $1/p + 1/q = 1$ and $c_{n,k}$, f_n are related by (3.3). Moreover,

$$\begin{aligned} \left\| \frac{1}{N} \sum_{k=n}^{n+N-1} f_k - f \right\|_p &= \left\| \sum_{k=-\infty}^{\infty} \left(\frac{1}{N} \sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k \right) e^{ikt} \right\|_p \\ &\ge \left\{ \sum_{k=-\infty}^{\infty} \left| \frac{1}{N} \sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k \right|^q \right\}^{1/q}, \end{aligned} \tag{4.1}$$

where $f = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$. With the help of (1.3), we get the following result, which extends the "only if" part of Corollary 3.2.

Theorem 4.1. *Let* $1 \le p \le 2$, $1/p + 1/q = 1$, *and* $c_{n,k}$, f_n *be related by* (3.3). *If* $\{f_n\}_{n=0}^{\infty}$ *is an almost convergent sequence in* $L^p(T)$, *then* $\sum_{k=-\infty}^{\infty} |c_{n,k}|^q < \infty$ *for all* $n \ge 0$, *and there exists a sequence* $\{c_n\}_{n=-\infty}^{\infty} \in \ell^q(\mathbb{Z})$ *such that*

$$\lim_{N\to\infty} \left\{ \sup_{n\ge 0} \left(\sum_{k=-\infty}^{\infty} \left| \frac{1}{N} \sum_{\ell=0}^{N-1} c_{n+\ell,k} - c_k \right|^q \right) \right\} = 0. \tag{4.2}$$

The converse of Theorem 4.1 is false for the case $1 \le p < 2$. Consider the following example:

$$c_{n,k} = \begin{cases} \dfrac{1}{2\sqrt{\ell}} & (|k| = 2^\ell, 1 \le \ell \le n), \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Obviously, $\sum_{k=-\infty}^{\infty} |c_{n,k}|^q < \infty$ for all $n \ge 0$, where $1/p + 1/q = 1$. Set $c_k = \frac{1}{2\sqrt{\ell}}$ for $|k| = 2^\ell$ with $\ell = 1, 2, \cdots$, and $c_k = 0$ otherwise. We have

$$\sup_{n\geq 0}\Bigg(\sum_{k=-\infty}^{\infty}\bigg|\frac{1}{N}\sum_{\ell=0}^{N-1}c_{n+\ell,k}-c_k\bigg|^q\Bigg)=2^{1-q}\Bigg(\sum_{\ell=1}^{\infty}\bigg|\frac{\min(\ell,N)}{N\sqrt{\ell}}\bigg|^q\Bigg)$$
$$\leq 2^{1-q}\Bigg(N^{-q}\int_0^N x^{q/2}dx+\sum_{\ell\geq N}^{\infty}\ell^{-q/2}\Bigg)$$
$$\longrightarrow 0 \qquad \text{as} \qquad N\to\infty,$$

because $q>2$. Hence, (4.2) is satisfied. However, by [1, Vol. I, p.223], we know that the sequence $\{f_n\}_{n=0}^{\infty}$, defined by $f_n=\sum_{k=-\infty}^{\infty}c_{n,k}e^{ikt}=\sum_{\ell=1}^{n}\frac{\cos 2^{\ell}t}{\sqrt{\ell}}$ is not an almost convergent sequence in $L^p(T)$, otherwise, $\sum_{n=1}^{\infty}\frac{\cos 2^{n}t}{\sqrt{n}}$ is a Fourier series, which gives us a contradiction.

Let $2<p\leq\infty$ and $1/p+1/p^*=1$. Then $1\leq p^*<2<p$. By the Hölder inequality, we get $\left\|\frac{1}{N}\sum_{k=n}^{n+N-1}f_k-f\right\|_{p^*}\leq\left\|\frac{1}{N}\sum_{k=n}^{n+N-1}f_k-f\right\|_{p}$. Putting this with (1.3) together, we infer that if $\{f_n\}_{n=0}^{\infty}$ is almost convergent to f in $L^p(T)$, then it is also almost convergent to f in $L^{p^*}(T)$. Applying Theorem 4.1 to p^* instead of p, we obtain the following result.

Theorem 4.2. *Let $2<p\leq\infty$ and $c_{n,k},f_n$ be related by (3.3). If $\{f_n\}_{n=0}^{\infty}$ is an almost convergent sequence in $L^p(T)$, then $\sum_{k=-\infty}^{\infty}|c_{n,k}|^p<\infty$ for all $n\geq 0$, and there exists a sequence $\{c_n\}_{n=-\infty}^{\infty}\in\ell^p(\mathbb{Z})$ such that*

$$\lim_{N\to\infty}\Bigg\{\sup_{n\geq 0}\Bigg(\sum_{k=-\infty}^{\infty}\bigg|\frac{1}{N}\sum_{\ell=0}^{N-1}c_{n+\ell,k}-c_k\bigg|^p\Bigg)\Bigg\}=0. \tag{4.4}$$

The matrix defined by (4.3) shows that the converse of Theorem 4.2 is false, in general. Replace (4.1) by

$$\left\|\frac{f_m+\cdots+f_{m+N-1}}{N}-\frac{f_n+\cdots+f_{n+N-1}}{N}\right\|_p$$
$$\geq\Bigg(\sum_{k=-\infty}^{\infty}\bigg|\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k}-c_{n+\ell,k})\bigg|^q\Bigg)^{1/q}.$$

Then the Hausdorff-Young inequality and Theorem 2.2 lead us to the following substitute for Theorem 4.1.

Theorem 4.3. *Let $1\leq p\leq 2$, $1/p+1/q=1$, and $c_{n,k},f_n$ be related by (3.3). If $\{f_n\}_{n=0}^{\infty}$ is an almost convergent sequence in $L^p(T)$, then $\sum_{k=-\infty}^{\infty}|c_{n,k}|^q<\infty$ for all $n\geq 0$, and*

$$\lim_{N\to\infty}\left\{\sup_{m,n\geq 0}\left(\sum_{k=-\infty}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k}-c_{n+\ell,k})\right|^q\right)\right\}=0. \tag{4.5}$$

Theorem 4.3 extends the "only if" part of Corollary 3.4 from $p = 2$ to $1 \leq p \leq 2$. Let $\{c_{n,k}\}_{n,k\geq 0}$ be the matrix defined by (4.3). By Minkowski's inequality and the definition of $c_{n,k}$, we know that

$$\begin{aligned}&\sup_{m,n\geq 0}\left(\sum_{k=-\infty}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k}-c_{n+\ell,k})\right|^q\right)^{1/q}\\ &\leq \sup_{m>n\geq 0}\frac{1}{N}\sum_{\ell=0}^{N-1}\left(\sum_{k=-\infty}^{\infty}|c_{m+\ell,k}-c_{n+\ell,k}|^q\right)^{1/q}\\ &\leq \frac{1}{N}\sum_{\ell=0}^{N-1}\left(2\sum_{\ell^*=\ell+1}^{\infty}(\frac{1}{2\sqrt{\ell^*}})^q\right)^{1/q}\longrightarrow 0 \qquad \text{as} \qquad N\to\infty,\end{aligned}$$

where $q > 2$. This shows that (4.5) is satisfied. As the argument behind Theorem 4.1 indicates, the corresponding sequence $\{f_n\}_{n=0}^{\infty}$ is not an almost convergent sequence in $L^p(T)$, so the converse of Theorem 4.3 is false for the case $1 \leq p < 2$.

Following the argument before Theorem 4.2, we see that Theorem 4.3 has the following consequence .

Theorem 4.4. *Let $2<p\leq\infty$ and $c_{n,k}, f_n$ be related by (3.3). If $\{f_n\}_{n=0}^{\infty}$ is an almost convergent sequence in $L^p(T)$, then $\sum_{k=-\infty}^{\infty}|c_{n,k}|^p<\infty$ for all $n\geq 0$, and*

$$\lim_{N\to\infty}\left\{\sup_{m,n\geq 0}\left(\sum_{k=-\infty}^{\infty}\left|\frac{1}{N}\sum_{\ell=0}^{N-1}(c_{m+\ell,k}-c_{n+\ell,k})\right|^p\right)\right\}=0. \tag{4.6}$$

The converse of Theorem 4.4 is false, in general, (see (4.3) for a counterexample). As Theorems 4.1 through 4.4 show, (4.2) and $(4.4) - (4.6)$ are necessary conditions for the almost convergence of the sequence $\{f_n\}_{n=0}^{\infty}$ defined by (3.3). In Corollaries 3.2 and 3.4, we have proved that these are sufficient conditions for the case $p = 2$. However, the preceding arguments indicate that they are not the case for $p \neq 2$. Hence, we might look for replacements of (4.2) and $(4.4) - (4.6)$ for $p \neq 2$. This problem is open.

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