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STRONG CONVERGENCE THEOREMS OF REICH TYPE ITERATIVE SEQUENCE FOR NON-SELF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. The purpose of this paper is to give some necessary and sufficient conditions for the iterative sequence of Reich type to converging to a fixed point. The results presented in this paper extend and improve some recent results ([2-4]).

1. Introduction

Throughout this paper, we assume that E is a real Banach space, C is a nonempty closed convex subset of E, E^* is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = ||x|| \cdot ||f||,\ ||x|| = ||f||\},\ \ \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing.

In the sequel, we shall denote the single-valued normalized duality mapping J by j and denote the fixed point set of a mapping T by $F(T)$. If $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to a point $x \in E$.

Recall that a mapping $T : C \to C$ is said to be *asymptotically nonexpansive* ([1]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$||T^n x - T^n y|| \le k_n||x - y||,\ \ \forall x, y \in C,\ \ n \ge 0.$$

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A mapping T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||, \ \ \forall x, y \in C, \ \ n \geq 0.$$

It is clear that every non-expansive mapping is asymptotically non-expansive and every asymptotically non-expansive is uniformly L-Lipschitzian with a constant $L = \sup_{n \geq 0} k_n \geq 1$. The converses do not hold. The asymptotically non-expansive mappings are important generalization of non-expansive mappings.

Definition 1.1. Let E be a real Banach space and C be a nonempty subset of E.

(1) A mapping P from E onto C is said to be a *retraction* if $P^2 = P$.

(2) If there exists a continuous retraction $P : E \to C$ such that $Px = x$ for all $x \in C$, then the set C is said to be a *retract* of E.

(3) Especially , if there exists a non-expansive retraction $P : E \to C$ such that $Px = x$ for all $x \in C$, then the set C is said to be a *non-expansive retract* of E.

Definition 1.2. Let E be a real Banach space, C be a nonempty non-expansive retract of E with a non-expansive retraction $P : E \to C$. Let $T : C \to E$ be a non-self mappings.

(1) T is said to be *non-self asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq k_n||x - y||, \qquad \forall\, x, y \in C, \ \ n \geq 1.$$

(2) T is said to be *non-self uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq L||x - y||, \qquad \forall\, x, y \in C, \ \ n \geq 1.$$

Proposition 1.1. *If $T : C \to E$ is a non-self asymptotically non-expansive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ $(n \to \infty)$, then we have the following:*

(1) *The mapping $PT : C \to C$ is an asymptotically non-expansive mapping with the same sequence $\{k_n\}$.*

(2) *PT is a uniformly L-Lipschitzian mapping with $L = \sup_{n\geq 1} k_n \geq 1$.*

Proof. In fact, since $T : C \to E$ is a non-self asymptotically non-expansive mapping, for any $n \geq 1$ and $x, y \in C$, we have

$$||(PT)^n x - (PT)^n y|| \leq ||T(PT)^{n-1}x - T(PT)^{n-1}y|| \leq k_n||x - y||.$$

Thus the conclusion (1) is proved. The conclusion (2) is obvious. This completes the proof.

Definition 1.3. Let $S := \{x \in E : ||x|| = 1\}$ be the unit sphere of a Banach space E. The space E is said to have a *uniformly Gâteaux differentiable norm* if, for each $y \in S$, the limit

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}$$

is attained uniformly for all $x \in S$.

Remark 1.1. It is well-known that, if E has a uniformly Gâteaux differentiable norm, then the normalized duality mapping $J : E \to 2^{E^*}$ is uniformly continuous from the norm topology of E to the weak* topology of E^* on any bounded subsets of E.

Let K be a nonempty closed convex and bounded subset of E and the diameter of K be defined by $d(K) = \sup\{||x - y|| : x, y \in K\}$. For each $x \in K$, let $r(x, K) = \sup\{||x - y|| : y \in K\}$ and $r(K) = \inf\{r(x, K) : x \in K\}$ denote the *Chebyshev radius* of K relative to itself. The *normal structure coefficient* $N(E)$ of E is defined by

$$N(E) : \\ = \inf\{\frac{d(K)}{r(K)} : K \;\; is\ a\ closed\ convex\ bounded\ subset\ of E\ with \;\; d(K) > 0\}.$$

A Banach space E such that $N(E) > 1$ is said to have *uniformly normal structure*. It is known that every space with a uniformly normal structure is reflexive and all uniformly convex and uniformly smooth Banach spaces have uniformly normal structure (see [12]).

Recall that a linear continuous functional $\mu \in (l^\infty)^*$ is called a *Banach limit* if

$$||\mu|| = 1, \;\; \liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n, \;\; \mu_n(a_n) = \mu_n(a_{n+1})$$

for all $a = \{a_n\} \in l^\infty$, where $\mu(a),\ a = \{a_n\} \in l^\infty$ is denoted by $\mu_n(a_n)$.

Next, we introduce the concept of *Reich type iterative sequence* for non-self asymptotically nonexpansive mappings.

Definition 1.4. Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $u \in C$ be a given point and $T : C \to E$ be a non-self asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$, then the sequence $\{x_n\}$ in E defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n}(PT)^j y_n, \\ y_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n}(PT)^j x_n, \end{cases} \quad \forall\, n \geq 0, \tag{1.2}$$

is called *the first type iterative sequence* of Reich, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0, 1].

In (1.2), taking $\beta_n = 1$ for all $n \geq 0$, then the sequence $\{x_n\}$ in E defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n}(PT)^j x_n, \end{cases} \quad \forall\, n \geq 0, \tag{1.3}$$

is called *the second type iterative sequence* of Reich, where $\{\alpha_n\}$ is a sequence in [0, 1].

Next, we consider some special cases of (1.2) and (1.3).

In 1980 and 1983, Reich [4], [5] proved that if E is a uniformly smooth Banach space, C is a weakly compact convex subset of E with fixed point property for nonexpansive mappings, $T : C \to C$ is a nonexpansive mapping and $\alpha_n = n^{-a}$, $a \in (0, 1)$, then the sequence $\{x_n\}$ defined by

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n u + (1-\alpha_n)T x_n \end{cases} \quad \forall\, n \geq 0, \tag{1.4}$$

converges strongly to a fixed point of T.

In 1992, Wittmann [6] proved that, if E a Hilbert space and the sequence $\{\alpha_n\} \subset [0, 1]$ satisfies the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then the sequence (1.4) converges strongly to some fixed point of T.

In 1997, Shioji and Takahashi [7] extended Wittmann's result to Banach spaces with a uniformly Gâteaux differentiable norm.

Recently, Zhang [8], [9] and Zeng [10] studied the convergence problem of the following iterative sequence for asymptotically non-expansive mapping $T : C \to C$:

$$(1.5)\qquad \begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j y_n, \\ y_n = \beta_n x_n + (1-\beta_n)\dfrac{1}{n+1}\displaystyle\sum_{j=0}^{n} T^j x_n, \end{cases} \quad \forall\ n \geq 0,$$

which extended the corresponding results of Reich [4], [5], Shiojio and Takahashi [7], Wittmann [6].

In order to prove our main results, we need the following Lemmas.

Lemma 1.2. *Let E be a Banach space with uniformly normal structure, C be a nonempty bounded subset of E and $T : C \to C$ be a uniformly $L-$Lipschitzian mapping with $L < \sqrt{N(E)}$. Suppose further that there exists a nonempty bounded closed convex subset K of C with the following property (A):*

(A) $x \in K$ *implies* $\omega_w(x) \subset K$

where $\omega_w(x)$ is the weak $\omega-$limit set of T at x, i.e.,

$$\omega_w(x) = \{y \in E : y = weak - \lim_j T^{n_j} x \ for\ some\ n_j \to \infty\}.$$

Then T has a fixed point in K.

Lemma 1.3. ([7]) *Let E be a Banach space with a uniformly Gâteaux differentiable norm, C be a nonempty closed convex subset of E and $\{x_n\}$ be a bounded sequence of E. Let μ be the Banach limit and $z \in C$. Then*

$$\mu_n ||x_n - z||^2 = \min_{y \in C} \mu_n ||x_n - y||^2$$

if and only if

$$\mu_n \langle y - z, J(x_n - z)\rangle \leq 0, \ \ \forall y \in C.$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping.

Lemma 1.4. ([13]) *Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n + c_n, \qquad \forall\ \ n \geq n_0,$$

where n_0 is a nonnegative integer, $\{\lambda_n\}$ is a sequence in [0,1] with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5. *Let E be a real Banach space, E^* be the dual space of E and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,*

(1) $||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall\, j(x+y) \in J(x+y).$

(2) $||x+y||^2 \geq ||x||^2 + 2\langle y, j(x)\rangle, \quad \forall\, j(x) \in J(x).$

The purpose of this paper is to prove the following main results for non-self asymptotically non-expansive which improve and extend the corresponding results of Reich [4], [5], Shiojio and Takahashi [7], Wittmann [6], Zhang [8], [9] and Zeng [10].

2. The Main Results

Now, we give the main results in this paper.

Theorem 2.1. *Let E be a real Banach space with a uniformly normal structure and whose norm is uniformly $G\hat{a}teaux$ differentiable. Let C be a nonempty bounded closed convex subset of E and $T : C \to E$ be a non-self asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ satisfying the folowing conditions:*

(*i*) $k_n \to 1$, $1 \leq \sup_{n\geq 1} k_n < \sqrt{N(E)}$;

(*ii*) $\sum_{n=0}^{\infty}(e_n - 1) < \infty$, *where*

$$e_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j \geq 1, \ \ \forall n \geq 0.$$

Let $P : E \to C$ be the nonexpansive retraction, $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in [0, 1] satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. For any given $u \in C$ and $n \geq 1$ define a contractive mapping $S_n : C \to C$ by

$$S_n(z) = (1-d_n)u + d_n(PT)^n z, \ \ n \geq 1, \tag{1.6}$$

where $d_n = \frac{t_n}{k_n}$, $\forall n \geq 1$, $t_n \in (0,1)$, $t_n \to 1$ $(n \to \infty)$ and $k_n^2 - 1 \leq (1-d_n)^2$, $\forall\, n \geq n_0$, for some nonnegative integer n_0. If $z_n \in C$ is the unique fixed point of S_n, i.e.,

$$z_n = S_n z_n = (1-d_n)u + d_n(PT)^n z_n, \tag{1.7}$$

then the sequence $\{z_n\}$ defined by (1.7) and the sequence $\{x_n\}$ defined by (1.2) converge strongly to the same fixed point $z \in F(PT)$ if and only if

$$||x_n - PT(x_n)|| \to 0, \quad ||z_n - PT(z_n)|| \to 0 \ \ (n \to \infty). \tag{1.8}$$

Proof.

Necessity.

If the sequences $\{x_n\}$ and $\{z_n\}$ defined by (1.2) and (1.7), respectively, converge strongly to the same point $z \in F(PT)$, then, by Proposition 1.1, we have

$$\begin{aligned}||x_n - PTx_n|| &\leq ||x_n - z|| + ||PTz - PTx_n|| \\ &\leq (1 + k_1)||x_n - z|| \to 0 \quad (n \to \infty).\end{aligned}$$

Similarly, we can prove that $||z_n - PTz_n|| \to 0$ $(as\ n \to \infty)$.

Sufficiency.

For any $v \in F(PT)$, from (1.7), we have

$$\begin{aligned}&||z_n - [(1 - d_n)u + d_n v]|| \\ &= ||(1 - d_n)u + d_n(PT)^n z_n - [(1 - d_n)u + d_n v]|| \\ &= d_n||(PT)^n z_n - v|| \leq d_n k_n||z_n - v||.\end{aligned} \tag{2.1}$$

Again, from Lemma 1.5, we have

$$\begin{aligned}||z_n - [(1 - d_n)u + d_n v]||^2 &= ||d_n(z_n - v) + (1 - d_n)(z_n - u)||^2 \\ &\geq d_n^2||z_n - v||^2 + 2d_n(1 - d_n)\langle z_n - u, J(z_n - v)\rangle.\end{aligned}$$

From (2.1), we have

$$\begin{aligned}&2d_n(1 - d_n)\langle z_n - u, J(z_n - v)\rangle \\ &\leq ||z_n - [(1 - d_n)u + d_n v]||^2 - d_n^2||z_n - v||^2 \\ &\leq d_n^2(k_n^2 - 1)||z_n - v||^2.\end{aligned} \tag{2.2}$$

By the assumption that $k_n^2 - 1 \leq (1 - d_n)^2$, we have

$$\langle z_n - u, J(z_n - v)\rangle \leq s_n D^2,$$

where $D = diamC$ (the diameter of C), $s_n := \frac{d_n(1-d_n)}{2} \to 0$ as $n \to \infty$. Therefore, we have

$$\limsup_{n\to\infty}\langle z_n - u, J(z_n - v)\rangle \leq 0. \tag{2.3}$$

Let μ be the Banach limit. We define a mapping $\phi : C \to [0, \infty)$ by

$$\phi(x) := \mu_n||z_n - x||^2, \ \ \forall x \in C. \tag{2.4}$$

Since E has uniformly normal structure, it is reflexive. Again, since $\phi(x) \to \infty \ \ (as\ ||x|| \to \infty)$, ϕ is continuous and convex, by Mazur and Schauder's Theorem [19], there exists $x^* \in K$ such that $\phi(x^*) = \inf_{x \in C} \phi(x)$. This implies that the set

$$K = \{y \in C : \phi(y) = \inf_{x \in C} \phi(x)\}$$

is a nonempty bounded closed convex subset of C.

Now, we prove that

$$\bigcup_{y \in K} \omega_w(y) \subset K, \tag{2.5}$$

where $\omega_w(y)$ is the weak $\omega-$limit set of mapping PT at y.

Indeed, for any $y \in K$ and any $p \in \omega_w(y)$, there exists a subsequence $\{m_j\} \subset \{m\}$ such that $p = weak - \lim_{j \to \infty} (PT)^{m_j} y$. By the weakly lower semi-continuity of ϕ, we have

$$\begin{aligned} \phi(p) &\le \liminf_{j \to \infty} \phi((PT)^{m_j} y) \le \limsup_{m \to \infty} \phi((PT)^m y) \\ &= \limsup_{m \to \infty} \mu_n ||z_n - (PT)^m y||^2 \\ &= \limsup_{m \to \infty} \mu_n ||z_n - (PT)^m z_n + (PT)^m z_n - (PT)^m y||^2. \end{aligned} \tag{2.6}$$

By the condition (1.8), $||z_n - (PT) z_n|| \to 0$ as $n \to \infty$, we can prove (by using induction) that

$$||z_n - (PT)^m z_n|| \to 0, \ (as\ n \to \infty), \ \forall\, m \ge 1. \tag{2.7}$$

Therefore, from (2.6)and (2.7), it follows that

$$\begin{aligned} \phi(p) &\le \limsup_{m \to \infty} \mu_n ||z_n - (PT)^m z_n + (PT)^m z_n - (PT)^m y||^2 \\ &= \limsup_{m \to \infty} \mu_n ||(PT)^m z_n - (PT)^m y||^2 \\ &\le \limsup_{m \to \infty} k_m^2 \mu_n ||z_n - y||^2 \\ &= \mu_n ||z_n - y||^2 = \min_{x \in C} \phi(x), \end{aligned} \tag{2.8}$$

i.e., $p \in K$. The conclusion (2.5) is proved. Therefore, the set K satisfies the condition (A). By Lemma 1.2, the mapping PT has a fixed point z in K and z is also a minimal point of ϕ in C. By Lemma 1.3, we have

$$\mu_n \langle y - z, \ J(z_n - z) \rangle \le 0, \ \ \forall y \in C. \tag{2.9}$$

Especially, taking $y = u$, we have

$$\mu_n\langle u - z,\ J(z_n - z)\rangle \leq 0. \tag{2.10}$$

Combining (2.3) and (2.10), we have

$$\begin{aligned}\mu_n||z_n - z||^2 &= \mu_n\langle z_n - z,\ J(z_n - z)\rangle \\ &= \mu_n\langle z_n - u + u - z,\ J(z_n - z)\rangle \\ &\leq \mu_n\langle z_n - u,\ J(z_n - z)\rangle \leq 0.\end{aligned}$$

Therefore, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $\{z_{n_i}\}$ converges strongly to z.

Next, we prove that every subsequence of $\{z_n\}$ converges strongly to the same z. Suppose the contrary, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \to p$. It follows from $||z_n - PTz_n|| \to 0$ as $n \to \infty$ that p is a fixed point of PT. By the assumption that the norm of E is uniformly Gâteaux differentiable, the normalized duality mapping J is uniformly continuous on every bounded subset of E from the strong topology of E to the weak* topology of E^*.

Observe that

$$\begin{aligned}&|\langle z_{n_i} - u,\ J(z_{n_i} - p)\rangle - \langle z - u,\ J(z - p)\rangle| \\ &\quad = |\langle z_{n_i} - z,\ J(z_{n_i} - p)\rangle + \langle z - u,\ J(z_{n_i} - p) - J(z - p)\rangle| \\ &\quad \leq ||z_{n_i} - z|| \cdot ||z_{n_i} - p|| + |\langle z - u,\ J(z_{n_i} - p) - J(z - p)\rangle|.\end{aligned}$$

Since $z_{n_i} \to z$ (as $i \to \infty$), we have

$$\lim_{i\to\infty}\langle z_{n_i} - u,\ J(z_{n_i} - p)\rangle = \langle z - u,\ J(z - p)\rangle. \tag{2.11}$$

Similarly, we can prove that

$$\lim_{j\to\infty}\langle z_{n_j} - u,\ J(z_{n_j} - z)\rangle = \langle p - u,\ J(p - z)\rangle. \tag{2.12}$$

It follows from (2.3), (2.11) and (2.12) that $\langle z-u,\ J(z-p)\rangle \leq 0$ and $\langle p-u,\ J(p-z)\rangle \leq 0$. Adding up these two inequalities, we have

$$\langle z - p,\ J(z - p)\rangle = ||z - p||^2 = 0,$$

i.e., $z = p$. This implies that $z_n \to z \in F(PT)$ as $n \to \infty$.

Next, we prove that the sequence $\{x_n\}$ defined by (1.2) converges strongly to $z \in F(PT)$ too. In fact, it follows from (1.2), Lemma 1.5(i) and Proposition 1.1

that, for any $n \geq 1$,

$$
\begin{aligned}
&||x_{n+1} - z||^2 \\
&= ||x_{n+1} - [\alpha_n u + (1-\alpha_n)z] + \alpha_n(u-z)||^2 \\
&\leq ||x_{n+1} - [\alpha_n u + (1-\alpha_n)z]||^2 + 2\alpha_n \langle u-z,\ j(x_{n+1}-z)\rangle \\
&= ||\alpha_n u + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - \alpha_n u - (1-\alpha_n)z||^2 \\
&\quad + 2\alpha_n\langle u-z,\ j(x_{n+1}-z)\rangle \\
&\leq ||(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - z)||^2 + 2\alpha_n\langle u-z,\ j(x_{n+1}-z)\rangle \\
&\leq (1-\alpha_n)^2\{\frac{1}{n+1}\sum_{j=1}^{n+1}k_j\}^2||y_n - z||^2 + 2\alpha_n\langle u-z,\ j(x_{n+1}-z)\rangle \\
&= (1-\alpha_n)^2 e_n^2 ||y_n - z||^2 + 2\alpha_n\langle u-z,\ j(x_{n+1}-z)\rangle.
\end{aligned}
\tag{2.13}
$$

Now, we consider the first term on the right side of (2.13). From (1.2), we have

$$
\begin{aligned}
&(1-\alpha_n)^2 e_n^2 ||y_n - z||^2 \\
&= (1-\alpha_n)^2 e_n^2 ||\beta_n x_n + (1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j x_n - z||^2 \\
&\leq (1-\alpha_n)^2 e_n^2 ||\beta_n(x_n - z) + (1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}((PT)^j x_n - z)||^2 \\
&\leq (1-\alpha_n)^2 e_n^2 \{\beta_n||x_n - x^*|| + (1-\beta_n)\frac{1}{n+1}\sum_{j=0}^{n}k_j||x_n - z||\}^2 \\
&\leq (1-\alpha_n)e_n^2\{\beta_n||x_n - z|| + (1-\beta_n)e_n||x_n - z||\}^2 \\
&\leq (1-\alpha_n)e_n^4||x_n - z||^2 \\
&\leq (1-\alpha_n)||x_n - z||^2 + (1-\alpha_n)(e_n^4 - 1)||x_n - z||^2 \\
&\leq (1-\alpha_n)||x_n - z||^2 + (e_n - 1)M.
\end{aligned}
\tag{2.14}
$$

where $M = D^2 \cdot \sup_{n\geq 0}(e_n^3 + e_n^2 + e_n + 1) < \infty$ and $D = diam(C)$. Substituting (2.14) into (2.13), we have

$$
||x_{n+1} - z||^2 \leq (1-\alpha_n)||x_n - z||^2 + (e_n - 1)M + 2\alpha_n\langle u-z,\ j(x_{n+1}-z)\rangle. \tag{2.15}
$$

Since

$$
z_m - ((1-d_m)u + d_m x_n) = (z_m - x_n) - (1-d_m)(u - x_n), \quad \forall\, n \geq 0,\ m \geq 1,
$$

it follows from Lemma 1.4 and (1.7) that

$$\begin{aligned}||z_m - x_n||^2 &= ||z_m - ((1-d_m)u + d_m x_n) + (1-d_m)(u - x_n)||^2 \\ &\le ||z_m - ((1-d_m)u + d_m x_n)||^2 + 2(1-d_m)\langle u - x_n, j(z_m - x_n)\rangle \\ &= ||(1-d_m)u + d_m(PT)^m z_m) - ((1-d_m)u + d_m x_n)||^2 \\ &\quad + 2(1-d_m)\langle u - z_m + z_m - x_n, j(z_m - x_n)\rangle \\ &\le d_m^2 ||(PT)^m z_m - x_n)||^2 + 2(1-d_m)||z_m - x_n||^2 \\ &\quad + 2(1-d_m)\langle u - z_m, j(z_m - x_n)\rangle.\end{aligned}$$

Since the normalized duality mapping J is odd, i.e., $J(-x) = -J(x),\ x \in E$, we have

$$\begin{aligned}&\langle u - z_m, j(x_n - z_m)\rangle \\ &\le \frac{1-2d_m}{2(1-d_m)}||z_m - x_n||^2 + \frac{d_m^2}{2(1-d_m)}||(PT)^m z_m - x_n||^2 \\ &= \frac{2d_m - 1}{2(1-d_m)}\{||(PT)^m z_m - x_n||^2 - ||z_m - x_n||^2\} + \frac{1-d_m}{2}||(PT)^m z_m - x_n||^2 \\ &\le \frac{2d_m - 1}{2(1-d_m)}\{(||(PT)^m z_m - (PT)^m x_n|| + ||(PT)^m x_n - x_n||)^2 \\ &\quad - ||x_n - z_m||^2\} + \frac{1-d_m}{2}||(PT)^m z_m - x_n||^2 \\ &\le \frac{2d_m - 1}{2(1-d_m)}\{(k_m||z_m - x_n|| + ||(PT)^m x_n - x_n||)^2 - ||x_n - z_m||^2\} \\ &\quad + \frac{1-d_m}{2}||(PT)^m z_m - x_n||^2 \\ &\le \frac{2d_m - 1}{2(1-d_m)}\{(k_m^2 - 1)||z_m - x_n||^2 + 2k_m||z_m - x_n|| \cdot ||(PT)^m x_n - x_n|| \\ &\quad + ||(PT)^m x_n - x_n||^2\} + \frac{1-d_m}{2}||(PT)^m z_m - x_n||^2.\end{aligned}$$

Since C is a bounded subset of E and

$$\begin{aligned}||(PT)^m z_m - x_n|| &\le ||(PT)^m z_m - z|| + ||x_n - z|| \\ &\le k_m||z_m - z|| + ||x_n - z||\end{aligned}$$

and hence $\{||(PT)^m z_m - x_n||\}$ is also bounded. Letting

$$S = \sup_{m\ge 1, n\ge 0}\{||(PT)^m z_m - x_n||,\ D\} < \infty,$$

we have

$$\begin{aligned}\langle u - z_m, j(x_n - z_m)\rangle \leq \frac{2d_m - 1}{2(1-d_m)}\{(k_m^2 - 1)S^2 \\ + 2k_m S||(PT)^m x_n - x_n|| \\ + ||(PT)^m x_n - x_n||^2\} + \frac{1-d_m}{2}S^2.\end{aligned} \tag{2.16}$$

From the condition (2.8): $||x_n - PTx_n|| \to 0$ $(n \to \infty)$, by induction, we can prove that

$$||(PT)^m x_n - x_n|| \to 0, \ \ (n \to \infty) \quad \forall\, m \geq 1. \tag{2.17}$$

Again, since $k_m^2 - 1 \leq (1-d_m)^2$, it follows from (2.16) and (2.17) that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z_m, j(x_n - z_m)\rangle &\leq \frac{2d_m - 1}{2(1-d_m)}(k_m^2 - 1)S^2 + \frac{1-d_m}{2}S^2 \\ &\leq (2d_m - 1)\frac{1-d_m}{2}S^2 + \frac{1-d_m}{2}S^2 \\ &= (1-d_m)d_m S^2, \quad \forall\, m \geq 1.\end{aligned} \tag{2.18}$$

Hence, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\langle u - z_m, j(x_n - z_m)\rangle \leq (1-d_m)d_m S^2 + \varepsilon, \quad \forall\, n \geq n_0, \ m \geq 1. \tag{2.19}$$

Since $d_m \to 1, \ z_m \to z\,(m \to \infty)$ and J is uniformly continuous on any bounded subset of E from the norm topology of E to the weak* topology of E^* and hence we have

$$\limsup_{m\to\infty}\langle u - z_m, \ j(x_n - z_m)\rangle = \langle u - z, \ j(x_n - z)\rangle \leq \varepsilon, \quad \forall\, n \geq n_0,$$

i.e.,

$$\langle x - z, \ j(x_n - z)\rangle \leq \varepsilon, \quad \forall\, n \geq n_0,$$

and so

$$\limsup_{n\to\infty}\langle u - z, \ j(x_n - z)\rangle \leq \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we have

$$\limsup_{n\to\infty}\langle u - z, \ j(x_n - z)\rangle \leq 0. \tag{2.20}$$

Letting $\xi_n = max\{\langle u - z, \ j(x_n - z)\rangle, 0\}$ for all $n \geq 0$, we know that $\xi_n \geq 0$. Now, we prove that

$$\lim_{n\to\infty} \xi_n = 0. \tag{2.21}$$

In fact, it follows from (2.20) that, for any give $\varepsilon > 0$, there exists a positive integer n_1 such that

$$\langle u - z,\ j(x_n - z)\rangle \leq \varepsilon, \quad \forall\, n \geq n_1,$$

and so we have $0 \leq \xi_n < \varepsilon$ for all $n \geq n_1$. By the arbitrariness of $\varepsilon > 0$, this implies that

$$\lim_{n\to\infty} \xi_n = 0.$$

From (2.15), we have

$$\begin{aligned} ||x_{n+1} - z||^2 &\leq (1-\alpha_n)||x_n - z||^2 + (e_n - 1)M + 2\alpha_n\langle u - z,\ j(x_{n+1} - z)\rangle \\ &\leq (1-\alpha_n)||x_n - z||^2 + 2\alpha_n\xi_{n+1} + (e_n - 1)M. \end{aligned}$$

In Lemma 1.4, take

$$a_n = ||x_n - z||^2, \ \ \lambda_n = \alpha_n, \ \ b_n = 2\alpha_n\xi_{n+1}, \ \ c_n = (e_n - 1)M.$$

Since $\sum_{n=0}^{\infty}(e_n - 1) < \infty$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have

$$\sum_{n=0}^{\infty} \lambda_n = \infty, \quad b_n = o(\lambda_n), \quad \sum_{n=0}^{\infty} c_n < \infty.$$

This shows that all the conditions in Lemma 1.4 are satisfied. Therefore, we have

$$\lim_{n\to\infty} ||x_n - z|| = 0.$$

i.e., $\{x_n\}$ converges strongly to $z \in F(PT)$. This completes the proof.

Taking $\beta_n = 1$ for all $n \geq 0$ in Theorem 2.1, we have the following theorem:

Theorem 2.1. *Let* $\{z_n\}$ *and* $\{x_n\}$ *be the sequences defined by* (1.7) *and* (1.3)*, respectively. If the conditions in Theorem* 2.1 *are satisfied. Then the sequences* $\{z_n\}$ *and* $\{x_n\}$ *converge strongly to the same fixed point of* PT *in* C *if and only if the condition* (1.8) *is satisfied.*

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