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A NEW CONVOLUTION IDENTITY DEDUCIBLE FROM THE REMARKABLE FORMULA OF RAMANUJAN

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Abstract. In this paper we obtain a convolution identity for the coefficients $B_n(\alpha, \theta, q)$ defined by

$$\sum_{n=-\infty}^{\infty} B_n(\alpha, \theta, q)x^n = \frac{\prod_{n=1}^{\infty}(1 + 2xq^n \cos\theta + x^2q^{2n})}{\prod_{n=1}^{\infty}(1 + \alpha q^n xe^{i\theta})},$$

using the well-known Ramanujan's ${}_1\psi_1$-summation formula. The work presented here complements the works of K.-W. Yang, S. Bhargava, C. Adiga and D. D. Somashekara and of H. M. Srivastava.

1. Introduction

The famous ${}_1\psi_1$ summation formula of Ramanujan [5, Ch. 16] can be stated as

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (q/az)_\infty (q)_\infty (b/a)_\infty}{(z)_\infty (b/az)_\infty (b)_\infty (q/a)_\infty}, \tag{1.1}$$

where $|b/a| < |z| < 1$, $|q| < 1$,

$$\begin{aligned} (a)_\infty &= (a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n), \\ (a)_n &= (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \qquad n : \text{an integer.} \end{aligned}$$

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G. H. Hardy [3, pp. 222-223] has described (1.1) as a "remarkable formula with many parameters". There are several proofs of (1.1) in literature. For details one may refer the book [1] by B. C. Berndt. Setting $b = 0, a = -q/c$ and $z = cz$ in (1.1), we obtain

$$\sum_{n=-\infty}^{\infty} (-q/c)_n (cz)^n = \frac{(-qz)_\infty (-1/z)_\infty (q)_\infty}{(-c)_\infty (cz)_\infty}. \tag{1.2}$$

Changing q to q^2, z to z/q in (1.2) and then setting $c = 0$, we obtain the well-known Jacobi's triple product identity [4]

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (-qz; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty, \qquad z \neq 0. \tag{1.3}$$

The main purpose of the present note is to obtain an interesting convolution identity for the coefficients $B_n(\alpha, \theta, q)$ defined by

$$\sum_{n=-\infty}^{\infty} B_n(\alpha, \theta, q) x^n = \frac{\prod_{n=1}^{\infty} (1 + 2xq^n \cos\theta + x^2 q^{2n})}{\prod_{n=1}^{\infty} (1 + \alpha q^n x e^{i\theta})}. \tag{1.4}$$

Our work complements the works of S. Bhargava, C. Adiga, D. D. Somashekara [2], H. M. Srivastava [6], K.-W. Yang [7]. We prove our main theorem in Section 2. In Section 3 we deduce interesting special cases, which includes the convolution identities of Yang [7] and of Bhargava, Adiga and Somashekara [2].

2. Main Theorem

The following theorem contains the convolution identity for the coefficients $B_n(\alpha, \theta, q)$ given by (1.4).

Theorem. *If $B_n(\alpha, \theta, q)$ is as defined in (1.4), then*

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{-n} B_{n+m}(\alpha, \theta, q) B_n(\beta, \theta, q) &= \frac{(\alpha q)_\infty (\beta q)_\infty (1/\alpha)_m (-\alpha q e^{i\theta})^m}{(q)_\infty^2} \\ &\quad \times \sum_{n=-\infty}^{\infty} \left(\frac{q^m}{\alpha}\right)_n (1/\beta)_n (\alpha\beta q e^{2i\theta})^n. \end{aligned} \tag{2.1}$$

Proof. By (1.4), we have

$$\sum_{n=-\infty}^{\infty} B_n(\alpha,\theta,q)x^n \sum_{n=-\infty}^{\infty} B_n(\beta,\theta,q)(xq)^{-n}$$

$$= \frac{(-xqe^{i\theta})_\infty(-xqe^{-i\theta})_\infty}{(-\alpha qxe^{i\theta})_\infty} \cdot \frac{(-e^{i\theta}/x)_\infty(-e^{-i\theta}/x)_\infty}{(-\beta e^{i\theta}/x)_\infty}$$

$$= \left[\frac{(\alpha q)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\alpha}\right)_n (-\alpha xqe^{i\theta})^n\right] \left[\frac{(\beta q)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\beta}\right)_n \left(\frac{-\beta e^{i\theta}}{x}\right)^n\right],$$

on using (1.2). Comparing the coefficients of x^m we obtain,

$$\sum_{n=-\infty}^{\infty} q^{-n} B_{n+m}(\alpha,\theta,q)B_n(\beta,\theta,q)$$

$$= \frac{(\alpha q)_\infty(\beta q)_\infty}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\alpha}\right)_{n+m} (-\alpha qe^{i\theta})^{n+m} \left(\frac{1}{\beta}\right)_n (-\beta e^{i\theta})^n,$$

which on simplification yields (2.1).

Setting $\alpha = 0 = \beta$, in (2.1) we obtain the following corollary.

Corollary.

$$\text{(2.2)} \quad \sum_{n=-\infty}^{\infty} q^{-n} B_{n+m}(0,\theta,q)B_n(0,\theta,q) = \frac{q^{m(m+1)/2}e^{mi\theta}}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{n^2+nm}e^{2ni\theta}.$$

The above corollary can also be obtained from a known result [2, p. 157, Theorem 2.1] (see also [6, p. 434, Theorem 1]).

3. Some Special Cases

In this Section we obtain as special cases of (2.2) the convolution identities of Yang [7], Bhargava, Adiga and Somashekara [2] and some more which seem new.

Theorem 3.1. [Yang]. *If the coefficients A_n are defined by*

$$\prod_{n=1}^{\infty}(1 + xq^n + x^2q^{2n}) = \sum_{n=-\infty}^{\infty} A_n x^n,$$

then

$$\sum_{n=-\infty}^{\infty} q^{-n} A_n A_{2m+n} = \frac{q^{m(m+1)}(-q^3;q^6)_\infty (q^2;q^2)_\infty}{(q;q)_\infty}, \tag{3.1}$$

$$\sum_{n=-\infty}^{\infty} q^{-n} A_n A_{2m+n-1} = \frac{q^{m^2}(-q^6;q^6)_\infty (-q;q^2)_\infty}{(q;q)_\infty}. \tag{3.2}$$

Proof. Changing m to $2m$ in (2.2), setting $\theta = \pi/3$ and noting from (1.4) that

$$A_n = A_n(q) = B_n(0, \pi/3, q) \tag{3.3}$$

we obtain on some simplification

$$\sum_{n=-\infty}^{\infty} q^{-n} A_n A_{2m+n} = \frac{q^{m(m+1)}}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(m+n)^2} \omega^{m+n}, \tag{3.4}$$

where ω is a cube root of unity. Using the Jacobi's triple product identity (1.3) on the right side of (3.4) we obtain (3.1) after some simplification.

Similarly on changing m to $(2m-1)$ in (2.2) and then proceeding as above we obtain on some simplification

$$\sum_{n=-\infty}^{\infty} q^{-n} A_n A_{2m+n-1} = \frac{q^{m^2}(-\omega)}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(m+n)^2} (\omega/q)^{m+n}. \tag{3.5}$$

Using the Jacobi's triple product identity (1.3) on the right side of (3.5) we obtain (3.2) after some simplification.

Theorem 3.2. [Bhargava, Adiga and Somashekara]. *If the coefficients D_n are defined by*

$$\prod_{n=1}^{\infty} (1 + 2xq^n + x^2 q^{2n}) = \sum_{n=-\infty}^{\infty} D_n x^n,$$

then

$$\sum_{n=-\infty}^{\infty} q^{-n} D_n D_{2m+n} = \frac{q^{m(m+1)}(-q;q^2)_\infty^2}{(q;q)_\infty (q;q^2)_\infty}, \tag{3.6.}$$

$$\sum_{n=-\infty}^{\infty} q^{-n} D_n D_{2m+n-1} = \frac{2q^{m^2}(-q^2;q^2)_\infty^2}{(q;q)_\infty (q;q^2)_\infty}. \tag{3.7}$$

Proof. Changing m to $2m$ in (2.2), setting $\theta = 0$ and noting from (1.4) that

$$D_n = D_n(q) = B_n(0,0,q) \tag{3.8}$$

we obtain on some simplification

$$\sum_{n=-\infty}^{\infty} q^{-n} D_n D_{2m+n} = \frac{q^{m(m+1)}}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(m+n)^2}\,. \tag{3.9}$$

Using the Jacobi's triple product identity (1.3) on the right side of (3.9) we obtain (3.6) after some simplification.

Similarly, on changing m to $(2m-1)$ in (2.2) and then proceeding as above we obtain on some simplification

$$\sum_{n=-\infty}^{\infty} q^{-n} D_n D_{2m+n-1} = \frac{q^{m^2}}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(m+n)^2} q^{-\,(m+n)}. \tag{3.10}$$

Using the Jacobi's triple product identity (1.3) on the right side of (3.10) we obtain (3.7) after some simplification.

Theorem 3.3. [Bhargava, Adiga and Somashekara]. *If the coefficients C_n are defined by*

$$\prod_{n=1}^{\infty}(1+xq^n) = \sum_{n=-\infty}^{\infty} c_n x^n,$$

then

$$\sum_{n=-\infty}^{\infty} q^{-n} C_n C_{m+n} = \frac{q^{m(m+1)/2}}{(q;q)_\infty}\,. \tag{3.11}$$

Proof. Changing m to $2m$ in (2.2), setting $\theta = \pi/2$ and noting from (1.4) that

$$C_n = C_n(q) = B_n(0, \pi/2, \sqrt{q}) \tag{3.12}$$

we obtain on some simplification

$$\sum_{n=-\infty}^{\infty} q^{-n} C_n C_{2m+n} = \frac{q^{m(m+1)}}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(m+n)^2}(-1)^{m+n}\,. \tag{3.13}$$

Using the Jacobi's triple product identity (1.3) on the right side of (3.13) and then changing q to $q^{\frac{1}{2}}$ we get (3.11) after some simplification.

Theorem 3.4. *If the coefficients E_n are defined by*

$$\prod_{n=1}^{\infty}(1+\sqrt{3}xq^n+x^2q^{2n})=\sum_{n=-\infty}^{\infty}E_nx^n,$$

then

$$\sum_{n=-\infty}^{\infty}q^{-n}E_nE_{2m+n}=\frac{q^{m(m+1)}(q^3;q^6)_\infty(-q;q)_\infty}{(q)_\infty(q;q^2)_\infty}\,, \tag{3.14}$$

$$\sum_{n=-\infty}^{\infty}q^{-n}E_nE_{2m+n-1}=\frac{q^{m^2}\omega(\omega-1)i(q^6;q^6)_\infty}{(q)_\infty^2}\,. \tag{3.15}$$

Proof. Changing m to $2m$ in (2.2), setting $\theta=\pi/6$ and noting from (1.4) that

$$E_n=E_n(q)=B_n(0,\pi/6,q) \tag{3.16}$$

we obtain on some simplification

$$\sum_{n=-\infty}^{\infty}q^{-n}E_nE_{2m+n}=\frac{q^{m(m+1)}}{(q)_\infty^2}\sum_{n=-\infty}^{\infty}q^{(m+n)^2}(-\omega^2)^{m+n}\,. \tag{3.17}$$

Using the Jacobi's triple product identity (1.3) on the right side of (3.17) we obtain (3.14) after some simplification.

Similarly on changing m to $(2m-1)$ in (2.2) and then proceeding as above we obtain on some simplification

$$\sum_{n=-\infty}^{\infty}q^{-n}E_nE_{2m+n-1}=\frac{q^{m^2}\omega^2 i}{(q)_\infty^2}\sum_{n=-\infty}^{\infty}q^{(m+n)^2}\left(\frac{-\omega^2}{q}\right)^{m+n}\,. \tag{3.18}$$

Using the Jacobi's triple product identity (1.3) on the right side of (3.18) we obtain (3.15) after some simplification.

Theorem 3.5. *If the coefficients G_n are defined by*

$$\prod_{n=1}^{\infty}\left(1+2xq^n\cos\left(\frac{\pi}{12}\right)+x^2q^{2n}\right)=\sum_{n=-\infty}^{\infty}G_nx^n,$$

then

$$\sum_{n=-\infty}^{\infty}q^{-n}G_nG_{2m+n}=\frac{q^{m(m+1)}\prod_{n=0}^{\infty}(1+\sqrt{3}q^{2n+1}+q^{4n+2})}{(q)_\infty(q;q^2)_\infty}\,. \tag{3.19}$$

Proof. Changing m to $2m$ in (2.2), setting $\theta = \pi/12$ and noting from (1.4) that

$$G_n = G_n(q) = B_n(0, \pi/12, q) \tag{3.20}$$

we obtain on some simplification

$$\sum_{n=-\infty}^{\infty} q^{-n} G_n G_{2m+n} = \frac{q^{m(m+1)}}{(q)_\infty^2} \sum_{n=-\infty}^{\infty} q^{(m+n)^2} (-\omega i)^{m+n} . \tag{3.21}$$

Using the Jacobi's triple product identity (1.3) on the right side of (3.21) we obtain (3.19) after some simplification.

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