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A NOTE ON GENERALIZED DERIVATIONS OF SEMIPRIME RINGS

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Abstract. In this paper we prove that generalized Jordan derivations and generalized Jordan triple derivation of $2-$torsion free semiprime rings are generalized derivations.

1. Introduction

This paper is motivated by the work of Jing and Lu [7].Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be $n-$torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let A be an algebra over the real or complex field and let B be a subalgebra of A. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [6] asserts that any Jordan derivation on a $2-$torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [1] . Cusack [5] generalized Herstein's result to $2-$torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping $D : R \to R$ is called Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. Bresar [3] has proved that any Jordan triple derivation on $2-$torsion free semiprime ring is a dedrivation. One can easily prove that any Jordan derivation of arbitrary ring is Jordan triple derivation (see for example [1] for the details) which means that the result we have just mentioned generalized Cusack's generalization of

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Herfstein's theorem. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. An additive mapping $T : R \to R$ is called left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of a right centralizer and a right Jordan centralizer

should be self-explanatory. Obviously, any left centralizer is a left Jordan cenrtralizer. Zalar [13] has proved that any left Jordan centralizer on a $2-$torsion free semiprime ring is a left centralizer. Molnar [8] has proved the following result. Let R be a $2-$torsion free prime ring and let $T : R \to R$ be an additive mapping. If $T(xyx) = T(x)yx$ holds for every $x, y \in R$, then T is a left centralizer. This result has been recently generalized to $2-$torsion free semiprime rings by Vukman and Kosi-Ulbl [12]. An additive mapping $F : R \to R$ is called generalized derivation in case $F(xy) = F(x)y + xD(y)$ holds for all pairs $x, y \in R$, where $D : R \to R$ is a derivation. The concept of generalized derivation has been introduced by Bresar [4]. It is easy to see that $F : R \to R$ is a generalized derivation iff F is of the form $F = D + T$, where D is a derivation and T a left centralizer. Recently, Jing and Lu [7] introduced a concept of generalized Jordan derivation and generalized Jordan triple derivation. An additive mapping $F : R \to R$ is generalized Jordan derivation if $F(x^2) = F(x)x + xD(x)$ holds for all $x \in R$ where $D : R \to R$ is a Jordan derivation. An additive mapping $F : R \to R$ is generalized Jordan triple derivation if $F(xyx) = F(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$ where $D : R \to R$ is a Jordan triple derivation. Jing and Lu [7] conjectured that in case $F : R \to R$, where R is $2-$torsion free semiprime ring, is either a generalized Jordan derivation or generalized Jordan triple derivation, is a generalized derivation. In is our aim in this note to prove both conjectures.

Theorem 1. *Let R be a $2-$torsion free semiprime ring and let $F : R \to R$ be a generalized Jordan derivation. In this case F is a generalized derivation.*

Proof. We have therefore the relation

$$F(x^2) = F(x)x + xD(x), (1)$$

for all $x \in R$, where D is a Jordan derivation of R. Since R is a semiprime ring one can conclude that D is a derivation. Let us denote $F - D$ by T. Then we have $T(x^2) = F(x^2) - D(x^2) = F(x)x + xD(x) - D(x)x - xD(x) = (F(x) - D(x))x = T(x)x$. We have therefore $T(x^2) = T(x)x$, for all $x \in R$. In other words, T is a left Jordan centralizer of R. Since R is a $2-$torsion free semiprime ring one can conclude that T is a left centralizer by Proposition 1.4 in [13]. Hence F is of the form $F = D + T$, where D is a derivation and T is a left centralizer of R, which means that F is a generalized derivation. The proof is complete.

Theorem 2. *Let R be a $2-$torsion free semiprime ring and let $F : R \to R$ be a generalized Jordan triple derivation. In this case F is a generalized derivation.*

Proof. We have therefore the relation

$$F(xyx) = F(x)yx + xD(y)x + xyD(x), (1)$$

for all pairs $x, y \in R$, where D is a Jordan triple derivation of R. Since R is a semiprime ring one can conclude that D is a derivation by Theorem A in $[3]$. Let us denote $F - D$ by T. We have $T(xyx) = F(xyx) - D(xyx) = F(x)yx + xD(y)x + xyD(x) - D(x)yx - xD(y)x - xyD(x) = (F(x) - D(x))yx = T(x)yx$. We have therefore $T(xyx) = T(x)yx$, for all pairs $x, y \in R$. By Theorem in $[12]$ one can conclude that T is a left centralizer. We have therefore proved that F can be written as $F = D + T$, where D is a derivation and T is a left centralizer, which means that F is a generalized derivation. The proof is complete.

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