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WEIGHTED ESTIMATE FOR A CLASS OF LITTLEWOOD-PALEY OPERATORS

Qingying Xue, Yong Ding and Kôzô Yabuta

Abstract. In this paper the authors give the weighted weak $(1,1)$ boundedness and the weighted L^p boundedness for the Littlewood-Paley operators with complex parameters.

1. Introduction

It is well-known that the Littlewood-Paley operators, such as the Littlewood-Paley g-function, g_λ^* function and the Lusin area integral S, play very important roles in harmonic analysis and PDE. The classical Littlewood-Paley g_λ^*-function and Lusin area integral S are defined by

$$g_\lambda^*(f)(x)=\left(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\big|\nabla(P_t*f)(y)\big|^2\frac{dydt}{t^{n-1}}\right)^{1/2}\quad \text{for}\quad \lambda>1$$

and

$$S(f)(x)=\left(\iint_{\Gamma(x)}\big|\nabla(P_t*f)(y)\big|^2\frac{dydt}{t^{n-1}}\right)^{1/2},$$

respectively, where $\Gamma(x)=\{(y,t)\in\mathbb{R}^{n+1}_+ : |x-y|<t\}$, $P_t(x)=t^{-n}P(x/t)$ denotes the Poisson kernel and $\nabla=(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\cdots,\frac{\partial}{\partial y_n},\frac{\partial}{\partial t})$.

The L^p boundedness and weak boundedness of g_λ^*-function and S have been studied by some authors. In 1961 and 1970, Stein [St1] and Fefferman [Fe] gave the following conclusions, respectively:

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Theorem A. ([St])

(a) *If* $\lambda > 2$, *then* g_λ^* *is of weak type* $(1,1)$*;*

(b) *If* $\lambda > 2$, *then* g_λ^* *is of type* (p, p) *for* $1 < p < \infty$*;*

(c) *If* $1 < \lambda \le 2$, *then* g_λ^* *is of type* (p, p) *for* $2/\lambda < p < \infty$.

Theorem B. ([Fe])

(a) *If* $1 < p < 2$ *and* $\lambda = 2/p$, *then* g_λ^* *is of weak type* (p, p)*;*

(b) *If* $1 < p < \infty$ *and* $\lambda > \max\{2/p, 1\}$, *then* g_λ^* *is of type* (p, p).

Remark 1. It is easy to check that the conclusion (b) of Theorem B is equivalent to the conclusions (b) and (c) of Theorem A.

In 1987, Chanillo and Wheeden [CW] proved the weighted boundedness of S.

Theorem C. ([CW]) *Suppose that* w *is a nonnegative locally integrable function on* $\mathbb{R}^n$, *then for any Schwartz function* f

(a) $\int_{\{x\in\mathbb{R}^n:\ S(f)(x)>\beta\}} w(x)dx \le (C_n/\beta)\int_{\mathbb{R}^n} |f(x)|w^*(x)dx$ *for any* $\beta > 0$;

(b) $\int_{\mathbb{R}^n} S(f)(x)^p w(x)dx \le C(n,p)\int_{\mathbb{R}^n} |f(x)|^p w^*(x)dx$ *for* $1 < p < 2$*;*

(c) $\int_{\mathbb{R}^n} S(f)(x)^p w(x)dx \le C(n,p)\int_{\mathbb{R}^n} |f(x)|^p w^*(x)^{p/2} w(x)^{-(p/2-1)}dx$ *for* $2 \le p < \infty$.

Remark 2. In [CW], the authors showed that the conclusion (b) in Theorem C doesn't hold for $p > 2$ by an example. Therefore, the conclusion (c) is a replacement of (b) since $w^* \le w^{*p/2} w^{-(p/2-1)}$ for $2 < p < \infty$.

On the other hand, in 1999, Sakamoto and Yabuta [SY] defined and studied the L^p boundedness of a class of Littlewood-Paley operators with complex parameter. Suppose that $\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero on $\mathbb{R}^n$ and satisfies

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where S^{n-1} denotes the unit sphere of $\mathbb{R}^n (n \ge 2)$ equipped with Lebesgue measure $d\sigma = d\sigma(x')$. Let $\varphi^\rho(x) = \Omega(x)|x|^{-n+\rho}\chi_B(x)$, where ρ is a complex number, $\rho = \gamma + i\tau$ with $\gamma > 0, \tau \in \mathbb{R}$ and B denotes the unit ball in $\mathbb{R}^n$. Then the parametrized Littlewood-Paley function $\mu_\lambda^{*,\rho}$ and the parametrized area integral μ_S^ρ are defined by

$$\mu_\lambda^{*,\rho}(f)(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|(\varphi_t^\rho * f)(y)\right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \quad \text{for} \quad \lambda > 1$$

and

$$\mu_S^\rho(f)(x) = \left(\iint_{\Gamma(x)} \left|(\varphi_t^\rho * f)(y)\right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

respectively, where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$ and $\varphi_t^\rho(x) = \frac{1}{t^n}\varphi^\rho(\frac{x}{t})$.

In [SY], the L^p-boundedness ($1 < p < \infty$) of $\mu_\lambda^{*,\rho}$ and μ_S^ρ were given when Ω satisfies $\mathrm{Lip}_\alpha(S^{n-1})$ condition for $0 < \alpha \le 1$. In 2002, Ding, Lu and Yabuta [DLY] obtained the following weighted L^2 boundedness of $\mu_\lambda^{*,\rho}$ and μ_S^ρ.

Theorem D. *Let $\Omega \in L\log^+ L(S^{n-1})$ satisfying (1.1). Then for $\rho = \gamma + i\tau$ with $\gamma > 0$ and $\lambda > 1$, there exists a constant $C > 0$ such that for any nonnegative locally integrable function $w(x)$,*

$$\begin{aligned} \int_{\mathbb{R}^n} \mu_S^\rho(f)(x)^2 w(x)dx &\le C_{n,\lambda} \int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^2 w(x)dx \\ &\le (C/\gamma) \int_{\mathbb{R}^n} |f(x)|^2 w^*(x)dx, \end{aligned} \tag{1.2}$$

where and in what follows, $w^*(x)$ denotes the Hardy-Littlewood maximal function of $w(x)$.

Remark 3. If we take $\rho \equiv 1$, then Theorem D is an improvement of the weighted L^2 boundedness of μ_S obtained by Chang, Wilson and Wolff [CWW] in 1985. In fact, in the result in [CWW], the condition assumed on the kernel function is $\varphi \in C_0^\infty$ with $\int \varphi(x) = 0$.

In comparison with the properties of g_λ^*-function and the parametrized Littlewood-Paley $\mu_\lambda^{*,\rho}$ function, the Lusin area integral S and the parametrized area integral μ_S^ρ, respectively, it is natural to ask the following problem: if the conclusions of Theorems A, B and C still hold when we replace the operators g_λ^* and S by the operators $\mu_\lambda^{*,\rho}$ and μ_S^ρ, respectively? The aim of this paper is to answer the interesting question above. However, we will get more in this paper. In fact, we will give the weighted weak type (1,1) and L^p estimates for the operators $\mu_\lambda^{*,\rho}$. As its corollary, we will obtain the weighted boundedness of μ_S^ρ. Before stating our results, let us recall some definitions.

For $\Omega(x') \in L^q(S^{n-1})$ $(1 \le q < \infty)$, the integral modulus $\omega_q(\delta)$ of continuity of Ω is defined by

$$\omega_q(\delta) = \sup_{\|\xi\| \le \delta} \left(\int_{S^{n-1}} |\Omega(\xi x') - \Omega(x')|^q d\sigma(x') \right)^{1/q},$$

where ξ is a rotation on S^{n-1} and $\|\xi\| = \sup_{x' \in S^{n-1}} |\xi x' - x'|$. A nonnegative locally integrable function w is said to satisfy the doubling property if there exists a constant $C > 0$ such that $w(2Q) \le Cw(Q)$ for any cube $Q \subset \mathbb{R}^n$.

The main result in this paper is as follows.

Theorem 1. *Suppose that $w(x)$ is a nonnegative locally integrable function to satisfy the doubling property. Let $\Omega \in L^2(S^{n-1})$ satisfy (1.1) and for some $\sigma > 1$*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta < \infty. \tag{1.3}$$

(i) *If $\gamma > n/2$ and $\lambda > 2$, then*

$$\int_{\{x\in\mathbb{R}^n:\mu_\lambda^{*,\rho}(f)(x)>\beta\}} w(x)dx \le [C(1+|\tau|)/\beta]\int_{\mathbb{R}^n}|f(x)|w^*(x)dx \text{ for any } \beta>0;$$

(ii) *If $\gamma > n/2$, $1 < p < 2$ and $\lambda > 2/p$, then*

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^p w(x)dx \le C(\gamma,|\tau|)\int_{\mathbb{R}^n}|f(x)|^p w^*(x)dx;$$

(iii) *If $0 < \gamma \le n/2$, $\frac{2n}{n+2\gamma} < p < 2$ and $\lambda > 2/p$, then*

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^p w(x)dx \le C(\gamma,|\tau|)\int_{\mathbb{R}^n}|f(x)|^p w^*(x)dx.$$

Note that the function Ω in Theorem 1 needs to satisfy the condition (1.3), although this is a very weak smoothness condition. However, below we will see that for the case $2 \le p < \infty$, in the results of weighted L^p-boundedness for the operators $\mu_\lambda^{*,\rho}$ and μ_S^ρ, the function Ω has not any smoothness on the unit sphere.

Theorem 2. *Suppose that $2 \le p < \infty$ and $w(x) \ge 0$ is a locally integrable function on $\mathbb{R}^n$. If $\Omega \in L\log^+ L(S^{n-1})$ satisfying (1.1), then for $\gamma > 0$ and $\lambda > 1$,*

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^p w(x)dx \le (C/\gamma)\int_{\mathbb{R}^n}|f(x)|^p w^*(x)^{p/2} w(x)^{-(p/2-1)}dx,$$

where C is a constant independent of f and w.

Remark 4. Obviously, the conclusion (i) of Theorem 1 is the weighted extension of the conclusion (a) of Theorem A. The conclusions (ii) and (iii) of Theorem 1 extend the conclusion (b) of Theorem B for the case $1 < p < 2$, and Theorem 2 is an extension of the conclusion (b) of Theorem B for the case $2 \le p < \infty$. Moreover, it was shown in [DLX] that the condition (1.3) is weaker than the Lip_α condition for $0 < \alpha \le 1$, and it is well known that

$$\mathrm{Lip}_\alpha(S^{n-1})\,(0<\alpha\le 1) \subsetneqq L^q(S^{n-1})\,(q>1) \subsetneqq L\log^+ L(S^{n-1}).$$

Hence the conditions in Theorems 1 and 2 are weaker than that in Theorems A and

B. In particular, if we take $\rho \equiv 1$, then Theorem 2 is a substantial improvement of the conclusion (b) in Theorem B for the case $2 \leq p < \infty$.

Note the following fact: for any $x \in \mathbb{R}^n$

$$\mu_S^{\rho}(f)(x) \leq 2^{\lambda n} \mu_{\lambda}^{*,\rho}(f)(x). \tag{1.4}$$

(For example, see the proof of (19) in [St2, p. 89]). Hence the weighted weak (1,1) and the weighted L^p-boundedness of μ_S^{ρ} are direct results of Theorems 1 and 2.

Corollary 1. *Suppose that w and Ω satisfy the same conditions as in Theorem 1.*

(*i*) *If $\gamma > n/2$, then $\int_{\{x \in \mathbb{R}^n:\ \mu_S^{\rho}(f)(x) > \beta\}} w(x)dx \leq [C(1+|\tau|)/\beta] \int_{\mathbb{R}^n} |f(x)| w(x)^* dx$ for any $\beta > 0$;*

(*ii*) *If $\gamma > n/2$ and $1 < p < 2$, then $\int_{\mathbb{R}^n} \mu_S^{\rho}(f)(x)^p w(x)dx \leq C(\gamma, |\tau|) \int_{\mathbb{R}^n} |f(x)|^p w(x)^* dx$;*

(*iii*) *If $0 < \gamma \leq n/2$ and $\frac{2n}{n+2\gamma} < p < 2$, then $\int_{\mathbb{R}^n} \mu_S^{\rho}(f)(x)^p w(x)dx \leq C(\gamma, |\tau|) \int_{\mathbb{R}^n} |f(x)|^p w^*(x)dx$.*

Corollary 2. *Suppose that $w(x)$ and Ω satisfy the same conditions as in Theorem 2, then for $\gamma > 0$,*

$$\int_{\mathbb{R}^n} \mu_S^{\rho}(f)(x)^p w(x)dx \leq (C/\gamma) \int_{\mathbb{R}^n} |f(x)|^p w^*(x)^{p/2} w(x)^{-(p/2-1)}dx, \ \ 2 \leq p < \infty,$$

where C is a constant independent of f and w.

Remark 5. As shown in Remark 4, Corollaries 1 and 2 extend Theorem C, and the conditions in Corollaries 1 and 2 are weaker than that in Theorem C. Similarly, if we take $\rho \equiv 1$, then Corollary 2 is a substantial improvement of the conclusion (c) in Theorem C.

We only give the proofs of Theorems 1 and 2 in this paper, taking (1.4) into account. This paper is arranged as follows. In §2, we first prove the conclusions (ii) and (iii) of Theorem 1 by applying the conclusions (i) of Theorem 1 and a Banach space valued version of Stein's interpolation theorem of analytic families of linear operators. The proof of the conclusions (i) of Theorem 1 will be put in §3, because its proof is considerably long. Finally, in §4 we will give the proof of Theorem 2. In this paper, $C(\gamma, |\tau|)$ will express a constant which depends on $\gamma, |\tau|$ and n, σ, λ, but not on f. Moreover, $C(\gamma, |\tau|)$ may be different from line to line.

2. Weighted L^p $(1 < p \leq 2)$ Boundedness of $\mu_{\lambda}^{*,\rho}$

In this section we will give the proof of the conclusions (ii) and (iii) of Theorem

1 under the conclusion (i) of Theorem 1 holds. In the proof, we will apply the following Banach space valued version of the interpolation theorem of analytic families of linear operators.

Lemma 2.1. *Let B be a Banach space and B' be its dual. Let $S = \{z \in \mathcal{C}; 0 \leq \mathrm{Re}\, z \leq 1\}$, (X,μ), (Y,ν) be measure spaces. To each $z \in S$ there is assigned a linear operator T_z on the space of simple functions in $L^1(X)$ into B-valued measurable functions on Y such that $(T_z f)g$ is integrable on Y whenever f is a simple function in $L^1(X)$ and g is a B'-valued simple function in $L^1_{B'}(Y)$. Suppose further the mapping*

$$z \to \int_Y (T_z f) g \, d\nu$$

is analytic in the interior S^0 of S, continuous on S, and there exists a constant $a < \pi$ such that

$$e^{-a|y|} \log \left| \int_Y (T_z f) g \, d\nu \right|$$

is uniformly bounded above in the strip S. Finally suppose

$$\|T_{iy} f\|_{L^{q_0}_B(Y)} \leq M_0(y) \|f\|_{L^{p_0}(X)} \quad \textit{and} \quad \|T_{1+iy} f\|_{L^{q_1}_B(Y)} \leq M_1(y) \|f\|_{L^{p_1}(X)}$$

for all simple functions f in $L^1(X)$, where $1 \leq p_j,\, q_j \leq \infty$, $M_j(y)$, $j = 0, 1$, are independent of f and satisfy

$$\sup_{-\infty < y < \infty} e^{-b|y|} \log M_j(y) < \infty$$

for some $b < \pi$. Then, if $0 \leq t \leq 1$, there exists a constant M_t such that

$$\|T_t f\|_{L^{q_t}_B(Y)} \leq M_t \|f\|_{L^{p_t}(X)}$$

for all simple functions f provided $1/p_t = (1-t)/p_0 + t/p_1$ and $1/q_t = (1-t)/q_0 + t/q_1$.

The proof of Lemma 2.1 is quite similar to the proof of Theorem 4.1 of Chapter V in the book by Stein and Weiss [SW, p. 205]. We therefore omit the details here.

Let us now return to the proofs of the conclusions (ii) and (iii) of Theorem 1. First we consider the conclusion (ii). Note that for the case $\lambda > 2$, the conclusions (ii) is obviously. In fact, it is a direct corollary by using weighted interpolation Theorem between Theorem D and the conclusion (i) of Theorem 1. Hence, to prove the conclusion (ii), it remains to consider the case $2/p < \lambda \leq 2$. We will apply Lemma 2.1 to finish the proof.

For $1 < p < 2$ and $2/p < \lambda \le 2$, we take a number a to satisfy $0 < a < \lambda - 2/p$. Consider the $L^2(\mathbb{R}^{n+1}_+; dydt/t^{n+1})$-valued linear operator T^1_z, which is defined by

$$T^1_z f(x,y,t) = \Big(\frac{t}{t+|x-y|}\Big)^{[z+1+(\lambda-2/p)-a]n/2} t^{-\rho} \int_{|u|<t} \frac{\Omega(u)}{|u|^{n-\rho}} f(y-u)\, du. \tag{2.1}$$

It is easy to check that if we take $z = 2/p - 1 + a$ then

$$\|T^1_{2/p-1+a} f(x,\cdot,\cdot)\|_{L^2(dydt/t^{n+1})} = \mu^{*,\rho}_\lambda(f)(x). \tag{2.2}$$

On the other hand, if $z = iv$ and $1 + iv$ with $v \in \mathbb{R}$, respectively, we have

$$\|T^1_{iv} f(x,\cdot,\cdot)\|_{L^2(dydt/t^{n+1})} = \mu^{*,\rho}_{1+\lambda-2/p-a}(f)(x), \tag{2.3}$$

and

$$\|T^1_{1+iv} f(x,\cdot,\cdot)\|_{L^2(dydt/t^{n+1})} = \mu^{*,\rho}_{2+\lambda-2/p-a}(f)(x). \tag{2.4}$$

Since $\lambda - 2/p > a > 0$, we have $1 + \lambda - 2/p - a > 1$. By (2.3) and Theorem D we get

$$\int_{\mathbb{R}^n} \Big(\|T^1_{iv} f(x,\cdot,\cdot)\|_{L^2(dydt/t^{n+1})}\Big)^2 w(x)dx \le \frac{C}{\gamma} \int_{\mathbb{R}^n} |f(x)|^2 w^*(x)dx. \tag{2.5}$$

Note that $2 + \lambda - 2/p - a > 2$. Hence for any $1 < p_1 < 2$, applying the interpolation theorem for (2.4) between Theorem D and the conclusion (i) of Theorem 1 we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\|T^1_{1+iv} f(x,\cdot,\cdot)\|_{L^2(dydt/t^{n+1})}\Big)^{p_1} w(x)dx &\le C(\gamma, |\tau|) \\ \int_{\mathbb{R}^n} |f(x)|^{p_1} w^*(x)dx. \end{aligned} \tag{2.6}$$

Now take p_1 so that $1/p = (2 - 2/p - a)/2 + (2/p - 1 + a)/p_1$, i.e. $p_1 = (2/p - 1 + a)/(2/p - 1 + a/2)$. Note that $0 < 2/p - 1 + a < 1$ and $1 < p_1 < p < 2$. By (2.2), (2.5) and (2.6) and using Lemma 2.1 we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(\mu^{*,\rho}_\lambda(f)(x)\Big)^p w(x)dx &= \int_{\mathbb{R}^n} \Big(\|T^1_{2/p-1+a} f(x,\cdot,\cdot)\|_{L^2(dydt/t^{n+1})}\Big)^p w(x)dx \\ &\le C(\gamma, |\tau|) \int_{\mathbb{R}^n} |f(x)|^p w^*(x)dx. \end{aligned}$$

Thus we finish the proof of the conclusion (ii) of Theorem 1.

Let us now turn to the proof of the conclusion (iii) of Theorem 1. For $0 < \gamma \le n/2$, $\frac{2n}{n+2\gamma} < p < 2$ and $\lambda > 2/p$, take a number a satisfying $0 < a <$

$\min(\lambda - \frac{2}{p}, \frac{2\gamma}{n} + 1 - \frac{2}{p})$. Consider the $L^2(\mathbb{R}^{n+1}_+; \frac{dydt}{t^{n+1}})$-valued linear operator T^2_z, which is defined by

$$T^2_z f(x,y,t) = \Big(\frac{t}{t+|x-y|}\Big)^{n(z+\lambda-\frac{2}{p}+1-a)/2} t^{-n(z+\frac{2\rho}{n}+1-\frac{2}{p}-a)/2}$$
$$\int_{|u|<t} \frac{\Omega(u)}{|u|^{n-n(z+\frac{2\rho}{n}+1-\frac{2}{p}-a)/2}} f(y-u)\,du.$$

Then we have

$$\|T^2_z f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})} = \mu^{*,n(z+\frac{2\rho}{n}+1-\frac{2}{p}-a)/2}_{\lambda-\frac{2}{p}+1-a+\mathrm{Re}\,z}(f)(x), \tag{2.7}$$

$$\|T^2_{\frac{2}{p}-1+a} f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})} = \mu^{*,\rho}_{\lambda}(f)(x), \tag{2.8}$$

$$\|T^2_{iv} f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})} = \mu^{*,n(iv+\frac{2\rho}{n}+1-\frac{2}{p}-a)/2}_{\lambda-\frac{2}{p}+1-a}(f)(x) \tag{2.9}$$

and

$$\|T^2_{1+iv} f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})} = \mu^{*,n(iv+\frac{2\rho}{n}+2-\frac{2}{p}-a)/2}_{\lambda-\frac{2}{p}+2-a}(f)(x). \tag{2.10}$$

Let $z_0 = \frac{2\rho}{n} - \{2(\frac{2\rho+n}{2n} - \frac{1}{p}) - a\} = \frac{2}{p} - 1 + a$. Note that $0 < z_0 < \frac{2\gamma}{n} \le 1$. Take p_1 so that

$$\frac{1}{p} = \frac{1-z_0}{2} + \frac{z_0}{p_1},$$

i.e. $p_1 = (2/p - 1 + a)/(2/p - 1 + a/2)$. Then we see that $1 < p_1 < p < 2$. For $0 \le \mathrm{Re\,z} \le 1$, we have $\mathrm{Re}\,\{\mathrm{n}(\mathrm{z}+\frac{2\rho}{\mathrm{n}}+1-\frac{2}{\mathrm{p}}-\mathrm{a})/2\} = \mathrm{n}(\mathrm{Re\,z}+\frac{2\gamma}{\mathrm{n}}+1-\frac{2}{\mathrm{p}}-\mathrm{a})/2 > 0$. We see also that $\lambda - \frac{2}{p} + 1 - a + \mathrm{Re}\,z > 1 + \mathrm{Re}\,z > 1$. Hence, by (2.7) and Theorem D, we get

$$\begin{aligned}\big\|\|T^2_z f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})}\big\|_{L^2(w)} &= \big\|\mu^{*,n(z+\frac{2\rho}{n}+1-\frac{2}{p}-a)/2}_{\lambda-\frac{2}{p}+1-a+\mathrm{Re}\,z}(f)(x)\big\|_{L^2(w)} \\ &\le \frac{C}{\sqrt{n(\mathrm{Re\,z}+\frac{2\gamma}{\mathrm{n}}+1-\frac{2}{\mathrm{p}}-\mathrm{a})/2}}\|f\|_{L^2(w^*)}.\end{aligned}$$

Naturally, for $z = iv$ we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \Big(\|T^2_{iv} f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})}\Big)^2 w(x)dx \\ &\le \frac{C}{\gamma + \frac{n}{2} - \frac{n}{p} - \frac{na}{2}} \int_{\mathbb{R}^n} |f(x)|^2 w^*(x)dx.\end{aligned} \tag{2.11}$$

Note that $\mathrm{Re}\,\{\mathrm{n}(\mathrm{iv}+\frac{2\rho}{\mathrm{n}}-\frac{2}{\mathrm{p}}+1-\mathrm{a}+1)/2\} = \gamma+\mathrm{n}-\frac{\mathrm{n}}{\mathrm{p}}-\frac{\mathrm{na}}{2} > \frac{\mathrm{n}}{2}$, $1 < p_1 < 2$ and $\lambda - \frac{2}{p} + 2 - a > 2 > 2/p_1$. Hence, by (2.10) and the conclusion (ii) of Theorem 1, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \Big(\|T^2_{1+iv} f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})} \Big)^{p_1} w(x)dx \\ (2.12)\qquad &= \int_{\mathbb{R}^n} \Big(\mu^{*,n(iv+\frac{2\rho}{n}+2-\frac{2}{p}-a)/2}_{\lambda-\frac{2}{p}+2-a}(f)(x) \Big)^{p_1} w(x)dx \\ &\le C(\gamma, |nv+2\tau|) \int_{\mathbb{R}^n} |f(x)|^{p_1} w^*(x)dx. \end{aligned}$$

By (2.8), (2.11), (2.12) and Lemma 2.1 we obtain

$$\begin{aligned} (2.13)\qquad \int_{\mathbb{R}^n} \Big(\mu^{*,\rho}_{\lambda}(f)(x) \Big)^p w(x)dx &= \int_{\mathbb{R}^n} \Big(\|T^2_{z_0} f(x,\cdot,\cdot)\|_{L^2(\frac{dydt}{t^{n+1}})} \Big)^p w(x)dx \\ &\le C(\gamma, |\tau|) \int_{\mathbb{R}^n} |f(x)|^p w^*(x)dx. \end{aligned}$$

Thus, we complete the proof of the conclusion (iii) of Theorem 1.

3. Weighted Weak (1,1) Boundedness of $\mu^{*,\rho}_{\lambda}$

In this section we will prove the conclusion (i) of Theorem 1, that is, we will give the proof of weighted weak (1,1) boundedness of $\mu^{*,\rho}_{\lambda}$. We need the following lemma, which is an extension of the result obtained by Kurtz and Wheeden in 1979 [KW].

Lemma 3.1. *Let $1 \le q < \infty$ and $\rho = \gamma + i\tau$ $(\sigma, \tau \in \mathbb{R})$ with $\gamma > 0$. Suppose that Ω is homobeneous of degree zero and satisfies the L^q-Dini condition. Then, there exists $C > 0$ such that for any $R > 0$ and $|y| < \frac{1}{2}R$,*

$$\begin{aligned} &\left(\int_{R<|x|<2R} \left| \frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x)}{|x|^{n-\rho}} \right|^q dx \right)^{\frac{1}{q}} \\ &\quad \le C(1+|\tau|)\, R^{n/q-(n-\gamma)} \left\{ \|\Omega\|_{L^q(S^{n-1})} \frac{|y|}{R} + \int_{|y|/2R}^{|y|/R} \frac{\omega_q(\delta)}{\delta} d\delta \right\}, \end{aligned}$$

where C is independent of R and y.

Proof. First we note that if $\rho = \gamma + i\tau$ and $|y| < |x|/2$, then

$$(3.1)\qquad \left| \frac{1}{|x-y|^{n-\rho}} - \frac{1}{|x|^{n-\rho}} \right| \le C(1+|\tau|) \frac{|y|}{|x|^{n-\gamma+1}}.$$

In fact, for $|y| < |x|/2$ we have

$$\frac{1}{2} < 1 - \frac{|y|}{|x|} \leq \frac{|x-y|}{|x|} \leq 1 + \frac{|y|}{|x|} < \frac{3}{2}. \tag{3.2}$$

On the other hand, it is easy to see that

$$0 > \log(1-t) > -2(\log 2)t \quad \text{for} \quad 0 < t < 1/2, \tag{3.3}$$

and

$$0 < \log(1+t) < t \quad \text{for} \quad t > 0. \tag{3.4}$$

Thus, if $|x-y| \leq |x|$ then by (3.2) and (3.3) we have

$$\left|\log \frac{|x-y|}{|x|}\right| \leq \left|\log \frac{|x|-|y|}{|x|}\right| < 2(\log 2)\frac{|y|}{|x|}.$$

if $|x-y| > |x|$ then by (3.2) and (3.4) we have

$$0 < \log \frac{|x-y|}{|x|} \leq \log\left(1 + \frac{|y|}{|x|}\right) < \frac{|y|}{|x|} < 2(\log 2)\frac{|y|}{|x|}.$$

Hence

$$\left|\log \frac{|x-y|}{|x|}\right| < 2(\log 2)\frac{|y|}{|x|} \quad \text{for} \quad |y| < |x|/2. \tag{3.5}$$

Note that

$$\begin{aligned}\left|\frac{1}{|x-y|^{n-\rho}} - \frac{1}{|x|^{n-\rho}}\right| &\leq \left|\frac{1}{|x-y|^{n-\rho}} - \frac{1}{|x-y|^{-i\tau}|x|^{n-\gamma}}\right| \\ &\quad + \left|\frac{1}{|x-y|^{-i\tau}} - \frac{1}{|x|^{-i\tau}}\right| \frac{1}{|x|^{n-\gamma}} \\ &= \left|\frac{1}{|x-y|^{n-\gamma}} - \frac{1}{|x|^{n-\gamma}}\right| + \left||x-y|^{i\tau} - |x|^{i\tau}\right| \frac{1}{|x|^{n-\gamma}}.\end{aligned}$$

By (3.2) we have

$$\left|\frac{1}{|x-y|^{n-\gamma}} - \frac{1}{|x|^{n-\gamma}}\right| \leq C\frac{|y|}{|x|^{n-\gamma+1}}.$$

On the other hand, by (3.5)

$$\begin{aligned}\left||x|^{i\tau} - |x-y|^{i\tau}\right| &= |e^{i\tau \log|x|} - e^{i\tau \log|x-y|}| = |1 - e^{i\tau \log(|x-y|/|x|)}| \\ &\leq \left|\tau \log \frac{|x-y|}{|x|}\right| < 2(\log 2)|\tau|\frac{|y|}{|x|}.\end{aligned}$$

Thus we obtain (3.1). It is easy to see that $|y| < |x|/2$ under the conditions of Lemma 3.1. Therefore, by (3.1)

$$\begin{aligned}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x)}{|x|^{n-\rho}}\right| &= \left|\Omega(x)\left(\frac{1}{|x-y|^{n-\rho}} - \frac{1}{|x|^{n-\rho}}\right) + \frac{\Omega(x-y)-\Omega(x)}{|x-y|^{n-\rho}}\right| \\ &\le C(1+|\tau|)|\Omega(x)|\frac{|y|}{|x|^{n-\gamma+1}} + \frac{|\Omega(x-y)-\Omega(x)|}{|x|^{n-\gamma}} \\ &\le C(1+|\tau|)\left(|\Omega(x)|\frac{|y|}{|x|^{n-\gamma+1}} + \frac{|\Omega(x-y)-\Omega(x)|}{|x|^{n-\gamma}}\right).\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\int_{R<|x|<2R}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x)}{|x|^{n-\rho}}\right|^q dx\right)^{1/q} \\ &\le C(1+|\tau|)\Bigg[\left(\int_{R<|x|<2R}|\Omega(x)|^q\frac{|y|^q}{|x|^{(n-\gamma+1)q}}dx\right)^{1/q} \\ &\quad+\left(\int_{R<|x|<2R}\frac{|\Omega(x-y)-\Omega(x)|^q}{|x|^{(n-\gamma)q}}dx\right)^{1/q}\Bigg].\end{aligned}$$

From [KW] we know that

$$\left(\int_{R<|x|<2R}|\Omega(x)|^q\frac{|y|^q}{|x|^{(n-\gamma+1)q}}dx\right)^{1/q} \le C\|\Omega\|_{L^q(S^{n-1})}R^{n/q-(n-\gamma)}\left(\frac{|y|}{R}\right)$$

and

$$\left(\int_{R<|x|<2R}\frac{|\Omega(x-y)-\Omega(x)|^q}{|x|^{(n-\gamma)q}}dx\right)^{1/q} \le CR^{n/q-(n-\alpha)}\left(\int_{|y|/2R}^{|y|/R}\omega_q(\delta)\frac{d\sigma}{\delta}\right).$$

Thus we prove Lemma 3.1.

Now let us return to the proof of the conclusion (i) in Theorem 1. Note that $\lambda > 2$ and $\rho = \gamma + i\tau$ with $\gamma > n/2$. Denote $E = \{x \in \mathbb{R}^n : f^*(x) > \beta\}$, by [FS], we have

$$w(E) := \int_E w(x)dx \le \frac{C}{\beta}\int_{\mathbb{R}^n}|f(x)|w^*(x)dx. \tag{3.6}$$

Let $E = \cup Q_k$ be the Whitney decomposition, $\{Q_k\}$ is a sequence of the cube with interiors are disjoint. For $f \in L^1_{loc}(\mathbb{R}^n)$ and any $\beta > 0$, denote

$$h(x) = \begin{matrix} f(x), & \text{for} \quad x \in E^c \\ \frac{1}{|Q_k|}\int_{Q_k} f(y)dy, & \text{for} \quad x \in Q_k. \end{matrix}$$

By [St2], $h(x) \leq C\beta \quad a.e.\ x \in \mathbb{R}^n$. Now set $b(x) = f(x) - h(x)$. Then $b(x) = 0$ for $x \in E^c$, $\int_{Q_k} b(x)dx = 0$ for each k. Denote

$$b_k(x) = \begin{matrix} b(x), & \text{for} & x \in Q_k \\ 0, & \text{for} & x \notin Q_k. \end{matrix}$$

We only need to prove (i) holds for $h(x)$ and $b(x)$, respectively. By (3.6) and Chebychev's inequality

$$\begin{aligned} \int_{\{x:\mu_\lambda^{*,\rho}(h)(x)>\beta\}} w(x)dx &\leq \int_{x\in E} w(x)dx + \int_{\{x:\mu_\lambda^{*,\rho}(h)(x)>\beta\}} w(x)\chi_{\mathbb{R}^n\setminus E}(x)dx \\ &\leq \frac{C}{\beta}\int_{\mathbb{R}^n} |f(x)|w^*(x)dx \\ &+\frac{1}{\beta^2}\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(h)(x)^2 w(x)\chi_{\mathbb{R}^n\setminus E}(x)dx. \end{aligned}$$

By Theorem D and noting that $h(x) \leq C\beta \quad a.e.\ x \in \mathbb{R}^n$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(h)(x)^2 w(x)\chi_{\mathbb{R}^n\setminus E}(x)dx &\leq C\int_{\mathbb{R}^n} |h(x)|^2 (w\chi_{\mathbb{R}^n\setminus E})^*(x)dx \\ &\leq C\beta\int_{\mathbb{R}^n} |h(x)|(w\chi_{\mathbb{R}^n\setminus E})^*(x)dx. \end{aligned}$$

By [CW, pp.282-283], we have

$$\beta\int_{\mathbb{R}^n} |h(x)|(w\chi_{\mathbb{R}^n\setminus E})^*(x)dx \leq C\beta\int_{\mathbb{R}^n} |f(x)|w^*(x)dx.$$

Thus

$$\int_{\{x:\mu_\lambda^{*,\rho}(h)(x)>\beta\}} w(x)dx \leq \frac{C}{\beta}\int_{\mathbb{R}^n} |f(x)|w^*(x)dx. \tag{3.7}$$

To prove the conclusion (i) of Theorem 1, it remains to show that

$$\int_{\{x:\mu_\lambda^{*,\rho}(b)(x)>\beta\}} w(x)dx \leq \frac{C}{\beta}\int_{\mathbb{R}^n} |f(x)|w^*(x)dx. \tag{3.8}$$

Denote by x_k and a_k the center and side length of Q_k, respectively. Let B_k be the ball with center at x_k and radius $r_k = \frac{\sqrt{n}}{2}a_k$ for each k. Moreover, we denote $E^* = \cup_k 8B_k$. Since w satisfies the doubling property and (3.6), we have

$$\begin{aligned} \int_{\{x:\mu_\lambda^{*,\rho}(b)(x)>\beta\}} w(x)dx &\leq w(E^*) + \int_{\{x:\mu_\lambda^{*,\rho}(b)(x)>\beta\}} w(x)\chi_{\{\mathbb{R}^n\setminus E^*\}}(x)dx \\ &\leq Cw(E) + \frac{c}{\beta}\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)w(x)dx \\ &\leq \frac{C}{\beta}\int_{\mathbb{R}^n} |f(x)|w^*(x)dx + \frac{C}{\beta}\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)w(x)dx. \end{aligned}$$

Hence, to obtain (3.8) it suffices to prove

$$\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)w(x)dx \le C(1+|\tau|)\int_{\mathbb{R}^n} |f(x)|w^*(x)dx. \tag{3.9}$$

By the Minkowski inequality

$$\begin{aligned}
&\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)w(x)dx = \int_{(\cup 8B_j)^c} \mu_\lambda^{*,\rho}(b)(x)w(x)dx \\
&= \int_{(\cup 8B_j)^c} \Bigg[\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \\
&\quad \left| \sum_k \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\gamma+1}} \Bigg]^{1/2} w(x)dx \\
&\le \int_{(\cup 8B_j)^c} \sum_k \Bigg[\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \\
&\quad \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\gamma+1}} \Bigg]^{1/2} w(x)dx.
\end{aligned}$$

Let

$$\begin{aligned}
J_1 &= \int_{(\cup 8B_j)^c} \sum_k \Bigg[\iint_{|y-x|<t} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \\
&\quad \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\gamma+1}} \Bigg]^{1/2} w(x)dx, \\
J_2 &= \int_{(\cup 8B_j)^c} \sum_k \Bigg[\iint_{|y-x|\ge t y\in 4B_k} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \\
&\quad \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\gamma+1}} \Bigg]^{1/2} w(x)dx
\end{aligned}$$

and

$$\begin{aligned}
J_3 &= \int_{(\cup 8B_j)^c} \sum_k \Bigg[\iint_{|y-x|\ge t y\in (4B_k)^c} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \\
&\quad \left| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \right|^2 \frac{dydt}{t^{n+2\gamma+1}} \Bigg]^{1/2} w(x)dx.
\end{aligned}$$

Then

$$\int_{(\cup 8B_j)^c} \mu_\lambda^{*,\rho}(b)(x)wdx \le J_1 + J_2 + J_3. \tag{3.10}$$

Below we will give the estimates of J_1, J_2 and J_3, respectively. First let us consider J_1. We have

$$
\begin{aligned}
J_1 \le & \int_{(\cup 8B_j)^c} \sum_k \Bigg[\iint_{|y-x|<t} \\
& \bigg| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \bigg|^2 \frac{dydt}{t^{n+2\gamma+1}} \Bigg]^{1/2} w(x)dx \\
\le & \int_{(\cup 8B_j)^c} \sum_k \Bigg[\bigg(\iint_{\substack{|y-x|<t \\ y\in 4B_k}} + \iint_{\substack{|y-x|<t \\ y\in (4B_k)^c}} \bigg) \\
& \bigg| \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} b_k(z)dz \bigg|^2 \frac{dydt}{t^{n+2\gamma+1}} \Bigg]^{1/2} w(x)dx \\
\le & J_{11} + J_{12}.
\end{aligned}
$$

Note that for fixed k, $x \in (8B_k)^c$ by $x \in (\cup 8B_j)^c$. Thus for $y \in 4B_k$ and $z \in Q_k$, we have

$$
|x - x_k| - 4r_k \le |x - x_k| - |y - x_k| \le |x - y| < t,
$$

and $|x-y| \sim |x-x_k| \sim |x-z| \sim |x-x_k| - 4r_k,\ |y-z| < 4r_k + |z-x_k| \le 5r_k$. Since $\int_{Q_k} |b(x)|dx \le 2\int_{Q_k} |f(x)|dx$, applying the Minkowski inequality again, we get

$$
\begin{aligned}
J_{11} \le & \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \\
& \bigg(\iint_{\substack{y-x|<t \\ |y-z|<t \\ y\in 4B_k}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} \frac{dydt}{t^{n+2\gamma+1}} \bigg)^{1/2} dz\, w(x)dx \\
\le & \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\int_{|y-z|<5r_k} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} \\
& \bigg(\int_{|x-x_k|-4r_k}^{\infty} \frac{dt}{t^{n+2\gamma+1}} \bigg) dy \Bigg]^{1/2} dz\, w(x)dx \\
\le & C \sum_k \int_{Q_k} |b(z)| \bigg(\int_{|y-z|<5r_k} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} dy \bigg)^{1/2} \\
& \int_{(\cup 8B_j)^c} \frac{w(x)}{(|x-x_k| - 4r_k)^{n/2+\gamma}} dxdz \\
\le & C \sum_k (r_k)^{\gamma - n/2} \int_{Q_k} |b(z)|dz \int_{(\cup 8B_j)^c} \frac{w(x)}{(|x-x_k| - 4r_k)^{n/2+\gamma}} dx \\
\le & C \sum_k (r_k)^{\gamma - n/2} \int_{Q_k} |f(z)|dz \int_{(\cup 8B_j)^c} \frac{w(x)}{(|x-x_k| - 4r_k)^{n/2+\gamma}} dx
\end{aligned} \tag{3.11}
$$

$$\le C\sum_k\int_{Q_k}(r_k)^{\gamma-n/2}|f(z)|\sum_{l=2}^{\infty}\int_{2^l r_k<|x-z|\le 2^{l+1}r_k}\frac{w(x)}{(|x-z|)^{n/2+\gamma}}dxdz$$
$$\le C\sum_k\int_{Q_k}|f(z)|\sum_{l=2}^{\infty}\frac{1}{(2^l)^{\gamma-n/2}}$$
$$\int_{|x-z|\le 2^{l+1}r_k}\frac{w(x)}{(2^{l+1}r_k)^n}dxdz \quad (\text{since } \gamma>n/2)$$
$$\le C\sum_k\int_{Q_k}|f(z)|w^*(z)dz$$
$$\le C\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.$$

As for J_{12}, we have

$$\begin{aligned} J_{12} &= \int_{(\cup 8B_j)^c}\sum_k\Bigg[\Bigg(\iint_{\substack{|y-x|<t\\ t\le|y-x_k|+2r_k\\ y\in(4B_k)^c}}\iint_{\substack{|y-x|<t\\ >|y-x_k|+2r_k\\ y\in(4B_k)^c}}\Bigg)\\ &\quad +\left|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}b_k(z)dz\right|^2\frac{dydt}{t^{n+2\gamma+1}}\Bigg]^{1/2}w(x)dx\\ &\le J_{12}^1+J_{12}^2. \end{aligned}$$

If $z\in Q_k$, $x\in(\cup 8B_j)^c$, $y\in(4B_k)^c$ and $|x-y|<t$, it is easy to see that $|y-x_k|>4r_k$, $|y-z|\sim|y-x_k|$ and $|x-x_k|\le|x-y|+|y-x_k|\le t+|y-x_k|$. Thus for $t\le|y-x_k|+2r_k$, we have $|x-x_k|\le 2|y-x_k|+2r_k\le 3|y-x_k|$. Noting that for $\alpha>0$,

$$\int_{|y-z|}^{|y-x_k|+2r_k}\frac{dt}{t^{\alpha+1}}\le\left|\frac{1}{|y-z|^{\alpha}}-\frac{1}{(|y-x_k|+2r_k)|^{\alpha}}\right|\le\frac{Cr_k}{|y-z|^{\alpha+1}}, \tag{3.12}$$

and taking $\alpha=n+2\gamma$ in (3.12), we have

$$\begin{aligned} J_{12}^1 &\le \int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Bigg(\iint_{\substack{|y-x|<t\\ t\le|y-x_k|+2r_k\\ |y-z|<t\\ y\in(4B_k)^c}}\\ &\quad \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}}\frac{dydt}{t^{n+2\gamma+1}}\Bigg)^{1/2}dz\,w(x)dx\\ &\le \int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Bigg[\int_{\substack{y\in(4B_k)^c\\ |x-x_k|\le 3|y-x_k|}}\\ &\quad \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}}\Bigg(\int_{|y-z|}^{|y-x_k|+2r_k}\frac{dt}{t^{n+2\gamma+1}}\Bigg)dy\Bigg]^{1/2}dz\,w(x)dx \qquad (3.13)\\ &\le C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Bigg(\int_{\substack{y\in(4B_k)^c\\ |x-x_k|\le 3|y-x_k|}} \end{aligned}$$

$$\begin{aligned}
&\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}}\frac{r_k}{|y-z|^{n+2\gamma+1}}dy\Big)^{1/2}dz\,w(x)dx\\
&\le C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Big(\int_{\substack{y\in(4B_k)^c\\|x-x_k|\le 3|y-x_k|}}\\
&\frac{|\Omega(y-z)|^2}{|y-z|^{n+1/2}}\frac{r_k}{|y-x_k|^{2n+1/2}}dy\Big)^{1/2}dz\,w(x)dx\\
&\le C\sum_k\int_{Q_k}|b(z)|\Big(\int_{y\in(4B_k)^c}\frac{|\Omega(y-z)|^2r_k^{1/2}}{|y-z|^{n+1/2}}dy\Big)^{1/2}dz\\
&\int_{(\cup 8B_j)^c}\frac{w(x)r_k^{1/4}}{|x-x_k|^{n+1/4}}dx\\
&\le C\sum_k\int_{Q_k}|b(z)|dz\int_{(\cup 8B_j)^c}\frac{w(x)r_k^{1/4}}{|x-x_k|^{n+1/4}}dx\\
&\le C\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.
\end{aligned}$$

The last inequality is obtained in the same way as in the estimate (3.11).

Now we give the estimate of J_{12}^2. Note that $Q_k\subset\{z:|y-z|<t\}$ since $y\in(4B_k)^c$ and $t>|y-x_k|+2r_k$. In addition, $|x-x_k|<|x-y|+|y-x_k|<2t$. Hence by the cancellation property of b on Q_k, we have

$$\begin{aligned}
J_{12}^2 &= \int_{(\cup 8B_j)^c}\sum_k\Bigg(\iint_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ y\in(4B_k)^c}}\Bigg|\int_{|y-z|<t}\Big(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Big)\\
&\qquad b_k(z)dz\Bigg|^2\frac{dydt}{t^{n+2\gamma+1}}\Bigg)^{1/2}w(x)dx\\
&\le \int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Bigg(\int_{y\in(4B_k)^c}\Bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\Bigg|^2\\
&\qquad\times\Big(\int_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ |y-z|<t}}\frac{dt}{t^{n+2\gamma+1}}\Big)dy\Bigg)^{1/2}dz\,w(x)dx.
\end{aligned}$$

Take an ε such that $0<\varepsilon<\min\{1/2,(\lambda-2)n/2,\gamma-n/2,\sigma-1\}$. Then we have

$$\begin{aligned}
\int_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ |y-z|<t}}\frac{dt}{t^{n+2\gamma+1}} &= \int_{\substack{|y-x|<t\\ t>|y-x_k|+2r_k\\ |y-z|<t}}\frac{(\log\frac{t}{r_k})^{2+2\varepsilon}dt}{t^{2\gamma-n+1}t^{2n}(\log\frac{t}{r_k})^{2+2\varepsilon}}\\
&\le \int_{t>|y-x_k|+2r_k}\frac{(\log\frac{t}{r_k})^{2+2\varepsilon}dt}{t^{2\gamma-n+1}(|x-x_k|/2)^{2n}(\log\frac{|x-x_k|}{2r_k})^{2+2\varepsilon}}.
\end{aligned}$$

Thus

$$J_{12}^2 \le \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \Big[\int_{y\in(4B_k)^c} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}} \Big|^2 \times \Big(\int_{>|y-x_k|+2r_k} \frac{(\log \frac{t}{r_k})^{2+2\varepsilon} dt}{t^{2\gamma-n+1}(|x-x_k|/2)^{2n}(\log \frac{|x-x_k|}{2r_k})^{2+2\varepsilon}} \Big) dy \Big]^{1/2} dz\, w(x) dx. \tag{3.14}$$

To complete the estimate of J_{12}^2, we give an inequality as follows.

Lemma 3.2. *Let $A > e$, $\eta > 2\varepsilon$. Then*

$$\int_A^\infty \frac{(\log s)^{2+2\varepsilon}}{s^{\eta+1}} ds \le C \frac{(\log A)^{2+2\varepsilon}}{A^\eta}. \tag{3.15}$$

Proof. Integrating by parts twice, we get

$$\int_A^\infty \frac{(\log s)^{2+2\varepsilon}}{s^{\eta+1}} ds = \frac{(\log A)^{2+2\varepsilon}}{\eta A^\eta} + \frac{2+2\varepsilon}{\eta^2} \frac{(\log A)^{1+2\varepsilon}}{A^\eta} + \frac{(2+2\varepsilon)(1+2\varepsilon)}{\eta^2} \int_A^\infty \frac{(\log s)^{2\varepsilon}}{s^{\eta+1}} ds. \tag{3.16}$$

By the decreasing property of $\frac{\log s}{s}$ for $s > e$, we have

$$\int_A^\infty \frac{(\log s)^{2\varepsilon}}{s^{\eta+1}} ds \le \frac{(\log A)^{2\varepsilon}}{A^{2\varepsilon}} \int_A^\infty \frac{1}{s^{\eta+1-2\varepsilon}} ds \le C \frac{(\log A)^{2\varepsilon}}{A^\eta}.$$

Thus, (3.15) follows from (3.16).

Let us now continue to estimate J_{12}^2. Let $\frac{t}{r_k} = s$. Then

$$\int_{|y-x_k|+2r_k}^\infty \frac{(\log \frac{t}{r_k})^{2+2\varepsilon}}{t^{2\gamma-n+1}} dt = \frac{1}{r_k^{2\gamma-n}} \int_{\frac{|y-x_k|}{r_k}+2}^\infty \frac{(\log s)^{2+2\varepsilon}}{s^{2\gamma-n+1}} ds.$$

Since $\frac{|y-x_k|}{r_k} + 2 > e$ by $y \in (4B_k)^c$, taking $A = \frac{|y-x_k|}{r_k} + 2$, $\eta = 2\gamma - n$ and applying (3.15), we have

$$\int_{|y-x_k|+2r_k}^\infty \frac{(\log \frac{t}{r_k})^{2+2\varepsilon}}{t^{2\gamma-n+1}} dt \le C \frac{[\log(\frac{|y-x_k|}{r_k}+2)]^{2+2\varepsilon}}{(|y-x_k|+2r_k)^{2\gamma-n}}. \tag{3.17}$$

By (3.14) and (3.17), we get

$$\begin{aligned} J_{12}^2 \quad &\le C \sum_k \int_{(\cup 8B_j)^c} \frac{1}{(|x-x_k|/2)^n (\log \frac{|x-x_k|}{2r_k})^{1+\varepsilon}} \int_{Q_k} |b(z)| \\ &\times \Big(\int_{y \in (4B_k)^c} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}} \Big|^2 \\ &\frac{[\log(\frac{|y-x_k|}{r_k}+2)]^{2+2\varepsilon}}{(|y-x_k|+2r_k)^{2\gamma-n}} dy \Big)^{1/2} dz\, w(x) dx. \end{aligned}$$

For $y \in (4B_k)^c$ and $z \in Q_k$, we have $|z-x_k| \le r_k \le \frac{|y-x_k|}{4}$. Applying Lemma 3.1 we get

$$\begin{aligned} J_{12}^2 &\le C \sum_k \int_{(\cup 8B_j)^c} \frac{1}{(|x-x_k|/2)^n (\log \frac{|x-x_k|}{2r_k})^{1+\varepsilon}} \int_{Q_k} |b(z)| \\ &\sum_{l=2}^{\infty} \Big(\int_{2^l r_k \le |y-x_k| < 2^{l+1} r_k} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\ &- \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}} \Big|^2 \frac{[\log(\frac{2^{l+1} r_k}{r_k}+2)]^{2+2\varepsilon}}{(2^l r_k + 2r_k)^{2\gamma-n}} dy \Big)^{1/2} dz\, w(x) dx \\ &\le C \sum_k \int_{(\cup 8B_j)^c} \frac{1}{(|x-x_k|/2)^n (\log \frac{|x-x_k|}{2r_k})^{1+\varepsilon}} \int_{Q_k} |b(z)| \\ &\sum_{l=2}^{\infty} \frac{(l+2)^{1+\varepsilon}}{(2^l r_k + 2r_k)^{\gamma-n/2}} (1+|\tau|)(2^l r_k)^{n/2-(n-\gamma)} \\ &\times \Big\{ \frac{|z-x_k|}{2^l r_k} + \int_{\frac{|z-x_k|}{2^{l+1} r_k}}^{\frac{|z-x_k|}{2^l r_k}} \frac{w_2(\delta)}{\delta} d\delta \Big\} dz\, w(x) dx. \end{aligned} \tag{3.18}$$

Note that

$$\begin{aligned} \int_{\frac{|z-x_k|}{2^{l+1} r_k}}^{\frac{|z-x_k|}{2^l r_k}} \frac{w_2(\delta)}{\delta} d\delta &= \int_{\frac{|z-x_k|}{2^{l+1} r_k}}^{\frac{|z-x_k|}{2^l r_k}} \frac{\omega_2(\delta)(1+|\log\delta|)^\sigma}{\delta(1+|\log\delta|)^\sigma} d\delta \\ &\le \frac{C}{l^\sigma} \int_{\frac{|z-x_k|}{2^{l+1} r_k}}^{\frac{|z-x_k|}{2^l r_k}} \frac{\omega_2(\delta)(1+|\log\delta|)^\sigma}{\delta} d\delta. \end{aligned} \tag{3.19}$$

By (3.18), (3.19) and the condition (1.3), we get

$$J_{12}^2 \leq C(1+|\tau|)\sum_k \int_{(\cup 8B_j)^c} \frac{1}{|x-x_k|^n(\log\frac{|x-x_k|}{2r_k})^{1+\varepsilon}} \int_{Q_k} |b(z)|$$
$$\times \sum_{l=2}^{\infty}(l+2)^{1+\varepsilon}\left\{\frac{1}{2^l}+\frac{C}{l^\sigma}\int_{\frac{|z-x_k|}{2^{l+1}r_k}}^{\frac{|z-x_k|}{2^l r_k}} \frac{\omega_2(\delta)(1+|\log\delta|)^\sigma}{\delta}d\delta\right\}dz\, w(x)dx$$
$$\leq C(1+|\tau|)\sum_k \int_{(\cup 8B_j)^c} \frac{1}{|x-x_k|^n(\log\frac{|x-x_k|}{2r_k})^{1+\varepsilon}} \int_{Q_k} |b(z)|$$
$$(3.20)\qquad \times\left(1+\int_0^1 \frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta\right)dz\, w(x)dx$$
$$\leq C(1+|\tau|)\sum_k \int_{Q_k}|f(z)|\int_{(\cup 8B_j)^c} \frac{w(x)}{|x-x_k|^n(\log\frac{|x-x_k|}{2r_k})^{1+\varepsilon}}dxdz$$
$$\leq C(1+|\tau|)\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.$$

The last inequality is also obtained in a way similar to the estimate (3.11). Thus, by (3.11), (3.13) and (3.20) we obtain

$$(3.21)\qquad J_1 \leq C(1+|\tau|)\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.$$

Now let us estimate J_2. From $y\in 4B_k, x\in(\cup 8B_j)^c$ and $z\in Q_k$, we see that $|y-z|<5r_k$, $|x-y|\sim|x-x_k|$ and

$$|y-x|>|x-x_k|-|y-x_k|>|x-x_k|-4r_k\geq|x-x_k|/2.$$

By the Minkowski inequality we have

$$J_2 \leq \int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Bigg[\iint_{\substack{|y-x|\geq t\\ |y-x|>|x-x_k|/2\\ |y-z|<t\\ |y-z|<5r_k\\ y\in 4B_k}}\left(\frac{t}{t+|x-y|}\right)^{2n+2\varepsilon}$$
$$\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}}\frac{dydt}{t^{n+2\gamma+1}}\Bigg]^{1/2}dz\, w(x)dx$$
$$(3.22)\qquad \leq C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Bigg[\int_{|y-z|<5r_k}\frac{|\Omega(y-z)|^2}{(|x-x_k|/2)^{2n+2\varepsilon}|y-z|^{2n-2\gamma}}$$
$$\times\left(\int_0^{|y-x|}\frac{t^{2n+2\varepsilon}}{t^{2n+\varepsilon+1}|y-z|^{2\gamma-n-\varepsilon}}dt\right)dy\Bigg]^{1/2}dzwdx$$
$$\leq C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Bigg(\int_{|y-z|<5r_k}$$

$$\frac{|\Omega(y-z)|^2|y-x|^\varepsilon}{(|x-x_k|/2)^{2n+2\varepsilon}|y-z|^{n-\varepsilon}}dy\Big)^{1/2} dz\, w(x)dx$$

$$\le C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\Big(\int_{|y-z|<5r_k}$$

$$\frac{|\Omega(y-z)|^2}{(|x-x_k|/2)^{2n+\varepsilon}|y-z|^{n-\varepsilon}}dy\Big)^{1/2} dz\, w(x)dx$$

$$\le C\sum_k\int_{Q_k}|b(z)|\Big(\int_{|y-z|<5r_k}\frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}}dy\Big)^{1/2}$$

$$\int_{(\cup 8B_j)^c}\frac{w(x)}{|x-x_k|^{n+\varepsilon/2}}dxdz$$

$$\le C\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.$$

The last inequality is obtained in the same way as in the estimate (3.11).

Finally, we give the estimate of J_3. Denote

$$J_{31} = \int_{(\cup 8B_j)^c}\sum_k\Big[\iint_{\substack{y\in(4B_k)^c\\ t\le|y-x_k|+\beta(\varepsilon)r_k\\ |y-x|\ge t}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}$$

$$\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\chi_{Q_k}(z)b(z)dz\Big|^2\times\frac{dydt}{t^{n+2\gamma+1}}\Big]^{1/2}w(x)dx$$

and

$$J_{32} = \int_{(\cup 8B_j)^c}\sum_k\Big[\iint_{\substack{y\in(4B_k)^c\\ |y-x_k|+\beta(\varepsilon)r_k<t\\ |y-x|\ge t}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}$$

$$\Big|\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\chi_{Q_k}(z)b(z)dz\Big|^2\times\frac{dydt}{t^{n+2\gamma+1}}\Big]^{1/2}w(x)dx,$$

where $\beta(\varepsilon)=8e^{(2+2\varepsilon)/\varepsilon}$. Then $J_3\le J_{31}+J_{32}$. By $y\in(4B_k)^c$ and $z\in Q_k$, we see that $|y-z|\sim|y-x_k|$ and $|y-x_k|\le|y-z|+|z-x_k|\le t+r_k$. For $\alpha>0$, by (3.12) we get

$$\int_{|y-z|}^{|y-x_k|+\beta(\varepsilon)r_k}\frac{1}{t^{\alpha+1}}dt\le\frac{Cr_k}{|y-x_k|^{\alpha+1}}. \tag{3.23}$$

Since

$$J_{31} \le \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\Bigg(\iint_{\substack{y\in(4B_k)^c\\ |y-x|\ge t\\ t\le|y-x_k|+\beta(\varepsilon)r_k\\ |y-z|<t\\ |x-x_k|\le 2|y-x_k|}} + \iint_{\substack{y\in(4B_k)^c\\ |y-x|\ge t\\ t\le|y-x_k|+\beta(\varepsilon)r_k\\ |y-z|<t\\ |x-x_k|>2|y-x_k|}} \Bigg) \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} \frac{dtdy}{t^{n+2\gamma+1}} \Bigg]^{1/2} dz\, w(x)dx$$

$$\le J_{31}^1 + J_{31}^2,$$

by (3.23) we get

$$J_{31}^1 \le \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\int_{\substack{y\in(4B_k)^c\\ |x-x_k|\le 2|y-x_k|}} \int_{|y-z|}^{|y-x_k|+\beta(\varepsilon)r_k} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} \frac{dtdy}{t^{n+2\gamma+1}} \Bigg]^{1/2} dz\, w(x)dx$$

$$\le C \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \Bigg(\int_{\substack{y\in(4B_k)^c\\ |x-x_k|\le 2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} \frac{r_k}{|y-x_k|^{n+2\gamma+1}} dy \Bigg)^{1/2} dz\, w(x)dx.$$

Using the same method of estimating J_{12}^1 in (3.13), we may get

$$J_{31}^1 \le C \int_{\mathbb{R}^n} |f(z)| w^*(z)dz. \tag{3.24}$$

Now we consider J_{31}^2. Take $0 < \varepsilon < \min\{1/2, \gamma - n/2, \sigma - 1\}$. Then, since $|y-x| \ge |x-x_k| - |y-x_k| \ge |x-x_k|/2$ and $|y-z| \sim |y-x_k|$, by (3.23) we have

$$J_{31}^2 \le \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\int_{\substack{y\in(4B_k)^c\\ |x-x_k|>2|y-x_k|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} \times \Bigg(\int_{|y-z|}^{|y-x_k|+\beta(\varepsilon)r_k} \left(\frac{t}{t+|x-y|}\right)^{2n+2\varepsilon} \frac{dt}{t^{n+2\gamma+1}} \Bigg) dy \Bigg]^{1/2} dz\, w(x)dx$$

$$\le C \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} |b(z)| \Bigg[\int_{\substack{y\in(4B_k)^c\\ |x-x_k|>2|y-x_k|}} \Bigg(\int_{|y-z|}^{|y-x_k|+\beta(\varepsilon)r_k} \frac{t^{2n+2\varepsilon-n-2\gamma-1}}{|x-y|^{2n+2\varepsilon}} dt \Bigg) \times \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}} dy \Bigg]^{1/2} dz\, w(x)dx \tag{3.25}$$

$$\leq C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\bigg(\int_{\substack{y\in(4B_k)^c\\|x-x_k|>2|y-x_k|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\gamma}}\frac{1}{|x-x_k|^{2n+2\varepsilon}}$$
$$\times\frac{r_k}{|y-x_k|^{2\gamma-n-2\varepsilon+1}}dy\bigg)^{1/2}dz\,w(x)dx$$
$$\leq C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\bigg(\int_{\substack{y\in(4B_k)^c\\|x-x_k|>2|y-x_k|}}$$
$$\frac{|\Omega(y-z)|^2}{|y-z|^{n-2\varepsilon+1}}\frac{r_k}{|x-x_k|^{2n+2\varepsilon}}dy\bigg)^{1/2}dz\,w(x)dx$$
$$\leq C\sum_k\int_{Q_k}|b(z)|\bigg(\int_{(4B_k)^c}\frac{|\Omega(y-z)|^2r_k^{1-2\varepsilon}}{|y-z|^{n-2\varepsilon+1}}dy\bigg)^{1/2}$$
$$\int_{(\cup 8B_j)^c}\frac{r_k^{\varepsilon}}{|x-x_k|^{n+\varepsilon}}w(x)dxdz$$
$$\leq C\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.$$

Finally let us consider J_{32}. By $y\in(4B_k)^c$ and $t>|y-x_k|+\beta(\varepsilon)r_k$, we have $Q_k\subset\{z:|y-z|<t\}$. On the other hand, it is easy to see that

$$t+|x-y|\geq t+|x-x_k|-|y-x_k|\geq|y-x_k|+\beta(\varepsilon)r_k+|x-x_k|-|y-x_k|\geq|x-x_k|+\beta(\varepsilon)r_k.$$

Thus, by the cancellation property of b on Q_k and the Minkowski inequality, we have

$$J_{32}\leq\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\bigg(\iint_{\substack{y\in(4B_k)^c\\|y-x_k|+\beta(\varepsilon)r_k<t\\|y-z|<t\leq|y-x|}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}$$
$$\times\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2\frac{dtdy}{t^{n+2\gamma+1}}\bigg)^{1/2}dz\,w(x)dx$$
$$=\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}|b(z)|\bigg(\iint_{\substack{y\in(4B_k)^c\\|y-x_k|+\beta(\varepsilon)r_k<t\\|y-z|<t\leq|y-x|}}$$
$$\frac{t^{\lambda n}}{(t+|x-y|)^{2n}[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}\frac{[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n}}$$
$$\times\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2\frac{dtdy}{t^{n+2\gamma+1}}\bigg)^{1/2}dz\,w(x)dx$$

$$\leq C \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} \frac{|b(z)|}{(|x-x_k|+\beta(\varepsilon)r_k)^n [\log(\frac{|x-x_k|+\beta(\varepsilon)r_k}{r_k})]^{1+\varepsilon}}$$

$$\times \Bigg(\iint_{\substack{y\in(4B_k)^c \\ |y-x_k|+\beta(\varepsilon)r_k<t \\ |y-z|<t\leq|y-x|}} \frac{t^{\lambda n}[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n}}$$

$$\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\right|^2 \frac{dtdy}{t^{n+2\gamma+1}}\Bigg)^{1/2} dz\, w(x)dx$$

$$\leq C \int_{(\cup 8B_j)^c} \sum_k \int_{Q_k} \frac{|b(z)|}{(|x-x_k|+\beta(\varepsilon)r_k)^n [\log(\frac{|x-x_k|+\beta(\varepsilon)r_k}{r_k})]^{1+\varepsilon}}$$

$$\Bigg(\int_{\substack{y\in(4B_k)^c \\ |y-x|\geq|y-x_k|+\beta(\varepsilon)r_k}} \left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\right|^2 \int_{|y-x_k|+\beta(\varepsilon)r_k}^{|y-x|}$$

$$\frac{t^{\lambda n}[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n} t^{n+2\gamma+1}} dtdy\Bigg)^{1/2} dz\, w(x)dx.$$

Notice that the function $g(s) = \frac{(\log s)^{2+2\varepsilon}}{s^\varepsilon}$ is decreasing when $s > e^{(2+2\varepsilon)/\varepsilon}$ and

$$\frac{t+|x-y|}{r_k} \geq \frac{|y-x_k|+\beta(\varepsilon)r_k+|x-y|}{r_k} > \frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k} > \beta(\varepsilon) > e^{(2+2\varepsilon)/\varepsilon},$$

and hence

$$\frac{[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}{(\frac{t+|x-y|}{r_k})^\varepsilon} = g(\frac{t+|x-y|}{r_k}) < g(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k}) = \frac{[\log(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})]^{2+2\varepsilon}}{(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})^\varepsilon}.$$

That is,

$$\frac{[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}{(t+|x-y|)^\varepsilon} < \frac{[\log(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})]^{2+2\varepsilon}}{(|y-x_k|+\beta(\varepsilon)r_k)^\varepsilon}. \tag{3.26}$$

Since $t+|x-y| \geq 2t$, we have $\frac{1}{t+|x-y|} \leq \frac{1}{2t}$. Together with this and (3.26), we get

$$\int_{|y-x_k|+\beta(\varepsilon)r_k}^{|x-y|} \frac{[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}{(t+|x-y|)^{\lambda n-2n}t^{n+2\gamma+1-\lambda n}}dt$$

$$\leq \int_{|y-x_k|+\beta(\varepsilon)r_k}^{|x-y|} \frac{[\log(\frac{t+|x-y|}{r_k})]^{2+2\varepsilon}}{(t+|x-y|)^{\varepsilon}} \frac{1}{(t+|x-y|)^{\lambda n-2n-\varepsilon}t^{n+2\gamma+1-\lambda n}}dt$$

(3.27)
$$\leq \int_{|y-x_k|+\beta(\varepsilon)r_k}^{\infty} \frac{[\log(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})]^{2+2\varepsilon}}{(|y-x_k|+\beta(\varepsilon)r_k)^{\varepsilon}} \cdot \frac{C}{t^{2\gamma-n+1-\varepsilon}}dt$$

$$\leq C\frac{[\log(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})]^{2+2\varepsilon}}{(|y-x_k|+\beta(\varepsilon)r_k)^{\varepsilon}} \cdot \frac{1}{(|y-x_k|+\beta(\varepsilon)r_k)^{2\gamma-n-\varepsilon}}$$

$$= C\frac{[\log(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})]^{2+2\varepsilon}}{(|y-x_k|+\beta(\varepsilon)r_k)^{2\gamma-n}}.$$

By (3.27), we have

$$J_{32} \leq C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}\frac{|b(z)|}{(|x-x_k|+\beta(\varepsilon)r_k)^n\big[\log(\frac{|x-x_k|+\beta(\varepsilon)r_k}{r_k})\big]^{1+\varepsilon}} \times\Bigg(\int_{\substack{y\in(4B_k)^c\\ |y-x|\geq|y-x_k|+\beta(\varepsilon)r_k}}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2 \frac{\big[\log(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})\big]^{2+2\varepsilon}}{(|y-x_k|+\beta(\varepsilon)r_k)^{2\gamma-n}}dy\Bigg)^{1/2}dz\,w(x)dx$$

$$\leq C\int_{(\cup 8B_j)^c}\sum_k\int_{Q_k}\frac{|b(z)|}{(|x-x_k|+\beta(\varepsilon)r_k)^n\big[\log(\frac{|x-x_k|+\beta(\varepsilon)r_k}{r_k})\big]^{1+\varepsilon}} \times\Bigg(\int_{y\in(4B_k)^c}\bigg|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(y-x_k)}{|y-x_k|^{n-\rho}}\bigg|^2 \frac{\big[\log(\frac{|y-x_k|+\beta(\varepsilon)r_k}{r_k})\big]^{2+2\varepsilon}}{(|y-x_k|+\beta(\varepsilon)r_k)^{2\gamma-n}}dy\Bigg)^{1/2}dz\,w(x)dx.$$

Applying the same method of estimating J_{12}^2, we may get

$$J_{32} \leq C(1+|\tau|)\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.$$

From this and (3.24), (3.25), we see that

(3.28)
$$J_3 \leq C(1+|\tau|)\int_{\mathbb{R}^n}|f(z)|w^*(z)dz.$$

Thus, from (3.10), (3.21), (3.22) and (3.28) we get (3.9)

$$\int_{(E^*)^c} \mu_\lambda^{*,\rho}(b)(x)w(x)dx \le C(1+|\tau|)\int_{\mathbb{R}^n} |f(z)|w^*(z)dz.$$

Thus, we obtain the conclusion (i) of Theorem 1 for $\gamma > n/2$ and $\lambda > 2$.

4. Weighted L^p $(2 \le p < \infty)$ Boundedness of $\mu_\lambda^{*,\rho}$

In this section we will prove Theorem 2. The basic idea of proving Theorem 2 is taken from [CW] and [TW]. When $p = 2$, Theorem 2 is just Theorem D, so we only need to give the proof of Theorem 2 for $2 < p < \infty$. By duality, we have

$$\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^p w(x)dx = \sup_h \left| \int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^2 h(x)w(x)dx \right|, \tag{4.1}$$

where $h(x) \ge 0$ and $\|h\|_{L^{(p/2)'}(w)} \le 1$. Since

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^2 h(x)w(x)dx \\
&= \int_{\mathbb{R}^n} \iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|y-x|}\right)^{n\lambda} |\varphi_t^\rho * f(y)|^2 \frac{dydt}{t^{n+1}} h(x)w(x)dx \\
&= \iint_{\mathbb{R}^{n+1}_+} |\varphi_t^\rho * f(y)|^2 \frac{1}{t}\left(\frac{1}{t^n}\int_{\mathbb{R}^n} h(x)w(x)\left(\frac{t}{t+|y-x|}\right)^{n\lambda} dx\right) dydt.
\end{aligned} \tag{4.2}$$

Set

$$E_k = \left\{(y,t): \quad \frac{1}{t^n}\int_{\mathbb{R}^n} h(x)w(x)\left(\frac{t}{t+|y-x|}\right)^{n\lambda} dx \quad \sim \; 2^k\right\}.$$

By [CW], for $(y,t) \in E_k$ and $|y-z| < t$

$$\begin{aligned}
2^k &< \tfrac{1}{t^n}\textstyle\int_{\mathbb{R}^n} h(x)w(x)\left(\tfrac{t}{t+|y-x|}\right)^{n\lambda} dx \sim \tfrac{1}{t^n}\textstyle\int_{\mathbb{R}^n} h(x)w(x)\left(\tfrac{t}{t+|x-z|}\right)^{n\lambda} dx \\
&\le Cw^*(z)M_w(h)(z),
\end{aligned}$$

where

$$M_w(h)(z) = \sup_{t>0}\left(\frac{1}{w(B_t(z))}\int_{B_t(z)} h(x)w(x)dx\right),$$

and $B_t(z)$ denotes the ball in $\mathbb{R}^n$ with center at z and radius t. It is easy to check that $C2^k \le w^*(z)M_w(h)(z)$ for $(y,t) \in E_k$ and $|y-z| < t$. So, it also holds that

$\varphi_t^\rho * f(y) = \varphi_t^\rho * (f\chi_{\{w^* M_w(h) > C2^k\}})(y)$ for $(y,t) \in E_k$. Thus, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^{n+1}_+} \left|\varphi_t^\rho * f(y)\right|^2 \tfrac{1}{t}\left(\tfrac{1}{t^n}\int_{\mathbb{R}^n} h(x)w(x)\left(\tfrac{t}{t+|y-x|}\right)^{n\lambda} dx\right) dydt \\
&\quad\le C\sum_k 2^{k+1}\int_{E_k}\left|\varphi_t^\rho * (f\chi_{\{w^* M_w(h)>C2^k\}})(y)\right|^2 \frac{dydt}{t} \\
&\quad\le C\sum_k 2^{k+1}\int_{\mathbb{R}^{n+1}_+}\left|\varphi_t^\rho * (f\chi_{\{w^* M_w(h)>C2^k\}})(y)\right|^2 \frac{dydt}{t} \qquad (4.3)\\
&\quad\le C\sum_k 2^{k+1}\int_{\mathbb{R}^n}\left(\int_0^\infty \left|\varphi_t^\rho * (f\chi_{\{w^* M_w(h)>C2^k\}})(y)\right|^2 \frac{dt}{t}\right) dy \\
&\quad= C\sum_k 2^{k+1}\int_{\mathbb{R}^n} \mu^\rho(f\chi_{\{w^* M_w(h)>C2^k\}})(y)^2 dy,
\end{aligned}$$

where μ^ρ denotes the parametrized Littlewood-Paley g-function (i.e. Marcinkiewicz integral), which is defined by

$$\mu^\rho(f)(x) = \left(\int_0^\infty \left|(\varphi_t^\rho * f)(y)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

By the L^2-boundedness of μ^ρ (see Theorem 1 in [DLY, p.15])

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f)(x)^2 h(x)w(x)dx \\
&\quad\le (C/\gamma)\sum_k 2^{k+1}\int_{\mathbb{R}^n}\left|f(y)\chi_{\{w^* M_w(h)>C2^k\}}(y)\right|^2 dy. \qquad (4.4)
\end{aligned}$$

On the other hand, by [CW, p.293] there exists a constant C, independent of h and f, such that

$$\begin{aligned}
&\sum_k 2^{k+1}\int_{\mathbb{R}^n}\left|f(y)\chi_{\{w^*(y) M_w(h)>C2^k\}}(y)\right|^2 dy \\
&\quad\le C\int_{\mathbb{R}^n}|f(y)|^p w^{*p/2}(y) w^{-(p/2-1)}(y)dy. \qquad (4.5)
\end{aligned}$$

Hence Theorem 2 follows from (4.1), (4.4) and (4.5).

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Qingying Xue and Yong Ding
Department of Mathematics,
Beijing Normal University,
Beijing 100875,
China
E-mail: qyxue@bnu.edu.cn and dingy@bnu.edu.cn

Kôzô Yabuta
School of Science,
Kwansei Gakuin University,
Gakuen 2-1, Sanda 669-1337,
Japan
E-mail: yabuta@kwansei.ac.jp