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# APPROXIMATE DERIVATIONS MAPPING INTO THE RADICALS OF BANACH ALGEBRAS

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**Abstract.** In the present paper, we investigate the situations so that the generalized Hyers-Ulam-Rassias stability for functional equations $f(x^2) = f(x)x + xf(x)$ and $f(xy) = f(x)y + xf(y)$ is satisfied. As a result we obtain that every linear mapping on a commutative Banach algebra which is an $\varepsilon$-*approximate* derivation maps the algebra into its radical.

## 1. Introduction

A definition of stability in the case of homomorphisms between metric groups was suggested by a problem posed by S. M. Ulam [23] in 1940. Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In this case, the equation of homomorphism $h(x \cdot y) = h(x) * h(y)$ is called stable. In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. D. H. Hyers [9] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E_1 \to E_2$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

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for all $x, y \in E_1$ and for some $\delta \geq 0$. Then there exists a unique additive mapping $T : E_1 \to E_2$ satisfying

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then the mapping $T$ is linear. In 1951, D. G. Bourgin [2] was the second author to treat this problem for additive mappings. Th. M. Rassias [16] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference controlled by $\|x\|^p + \|y\|^p,\ p \in [0, 1)$ to be unbounded. A number of mathematicians were attracted to this result of Th.M. Rassias and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Th.M. Rassias in his 1978 paper is called the Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [6-8]. A Hyers-Ulam-Rassias stability theorem for the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

was proved by D.H. Hyers and Th. M. Rassias [10], Jun and Lee [11] and Th.M. Rassias [17], etc.

On the *34-th International Symposium on Functional Equations* G. Maksa [14] posed the Hyers-Ulam stability problem for the functional equation

$$f(xy) = xf(y) + yf(x) \tag{1.1}$$

on the unit interval. The first result concerning the superstability of this equation for functions between operator algebras was obtained by P. Semrl [18]. Z. Páles [15] proved that the functional equation (1.1) for real-valued functions on $[1, \infty)$ is stable in the sense of Hyers and Ulam. In 1997 C. Borelli [1] demonstrated the Hyers-Ulam-Rassias stability of the equation (1.1) on restricted domain of $\mathbb{R}$. Moreover the Hyers-Ulam-Rassias stability of $f(x^2) = 2xf(x)$ on restricted domain of $\mathbb{R}$ has been studied in [13]. J. Tabor [21] solved the corresponding Hyers-Ulam stability problem of (1.1) on the interval $(0, 1]$. Jung and Park [12] have solved the following functional equation $f(x + y + xy) = f(x) + f(y) + xf(y) + yf(x)$ motivated by the equation (1.1), and then investigated the Hyers-Ulam stability problem on the interval $(-1, 0]$ and the superstability on $[0, \infty)$ of the above equation, respectively.

On the other hand, Singer and Wermer [19] proved that every continuous derivation on a commutative Banach algebra maps the algebra into its radical. Thomas [22] proved that the Singer-Wermer theorem remains still true without assuming the continuity of the derivation. There are many papers extending the Singer-Wermer theorem [3-5, 20, 24]. It is well-known that any linear derivations on commutative semi-simple Banach algebras are zero by Thomas' result. Concerning these results, we show in this paper that every $\varepsilon$-*approximate* linear derivation on commutative Banach algebras maps still the algebra into its radical, and that any $\varepsilon$-*approximate* linear derivations on commutative semi-simple Banach algebras are also zero. For this purpose, we are going to investigate the generalized Hyers-Ulam-Rassias stability problem for functional equations of multiplicative derivations on Banach algebras.

## 2. Stability of Derivations on Banach Algebras

Concerning the above functional equation (1.1), we consider the stability of the following functional equations

$$\begin{aligned} f(x^2) &= f(x)x + xf(x) \\ \text{and} \quad f(xy) &= f(x)y + xf(y), \end{aligned}$$

which define additive Jordan derivations, and derivations on Banach algebras, respectively. In this section, using an idea from the direct method of Hyers, we shall give certain conditions so that the Hyers-Ulam-Rassias stability of the above functional equations works. Throughout this paper, let $\mathcal{B}$ be a Banach algebra with norm $\|\cdot\|$ and let $\mathbb{R}^+$ denote the set of all nonnegative real numbers.

**Theorem 2.1.** *Let $f : \mathcal{B} \to \mathcal{B}$ be a given mapping and let $\varphi_1 : \mathcal{B} \to \mathbb{R}^+$, $\varphi_2 : \mathcal{B} \times \mathcal{B} \to \mathbb{R}^+$ be mappings for which there exist nonzero rational numbers $a, b$ with $|a + b| \neq 0, 1$ such that*

$$\|f(x^2) - f(x)x - xf(x)\| \leq \varphi_1(x), \tag{2.1}$$

$$\|f(ax + by) - af(x) - bf(y)\| \leq \varphi_2(x, y) \tag{2.2}$$

*for all $x, y \in \mathcal{B}$. Assume that the series*

$$\begin{aligned} &\Phi_2(x, y) := \sum_{i=0}^{\infty} \frac{\varphi_2(\lambda^i x, \lambda^i y)}{|\lambda|^i} < \infty \\ &\left( \widehat{\Phi}_2(x, y) := \sum_{i=1}^{\infty} |\lambda|^i \varphi_2\left(\frac{x}{\lambda^i}, \frac{y}{\lambda^i}\right) < \infty, \ respectively, \right) \end{aligned} \tag{2.3}$$

*converges and the limit*

$$\begin{aligned} &\lim_{n\to\infty} \frac{\varphi_1(\lambda^n x)}{|\lambda|^{2n}} = 0, \\ &\left(\lim_{n\to\infty} |\lambda|^{2n} \varphi_1\left(\frac{x}{\lambda^n}\right) = 0\right) \end{aligned}$$

*for all $x, y \in \mathcal{B}$, where $\lambda := a + b$. Then there exists a unique additive mapping $g : \mathcal{B} \to \mathcal{B}$ such that*

$$g(ax + by) - ag(x) - bg(y) = 0, g(x^2) - g(x)x - xg(x) = 0 \tag{2.4}$$

*for all $x, y \in \mathcal{B}$, that is, $g$ is an additive Jordan derivation on $\mathcal{B}$, and the inequality*

$$\begin{aligned} &\|f(x) - g(x)\| \leq \frac{1}{|\lambda|}\Phi_2(x, x) \\ &\left(\|f(x) - g(x)\| \leq \frac{1}{|\lambda|}\widehat{\Phi}_2(x, x)\right) \end{aligned} \tag{2.5}$$

*holds for all $x \in \mathcal{B}$.*

*Moreover, if $f$ is measurable or the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a real Banach algebra $\mathcal{B}$, then the mapping $g$ is a $\mathbb{R}$-linear Jordan derivation on $\mathcal{B}$. Alternatively, if $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a complex Banach algebra $\mathcal{B}$ and there exists a mapping $\varphi_3 : \mathcal{B} \to \mathbb{R}^+$ for which*

$$\begin{aligned} &\|f(iu) - if(u)\| \leq \varphi_3(u) \quad \textit{and} \\ &\lim_{n\to\infty} \frac{\varphi_3(\lambda^n u)}{|\lambda|^n} = 0, \ \left(\lim_{n\to\infty} |\lambda|^n \varphi_3\left(\frac{u}{\lambda^n}\right) = 0\right) \end{aligned} \tag{2.6}$$

*hold for all $u \in \mathcal{B}(\|u\| = 1)$, then the mapping $g$ is a $\mathbb{C}$-linear Jordan derivation on $\mathcal{B}$.*

*Proof.* Setting $y = x$ in (2.2) yields

$$\left\|\frac{f(\lambda x)}{\lambda} - f(x)\right\| \leq \frac{1}{|\lambda|}\varphi_2(x, x) \tag{2.7}$$

for all $x \in \mathcal{B}$. Thus given integers $m, n (m > n \geq 0)$,

$$\begin{aligned} &\left\|\frac{f(\lambda^i x)}{\lambda^i} - \frac{f(\lambda^{i+1} x)}{\lambda^{i+1}}\right\| \leq \frac{1}{|\lambda|^{i+1}}\varphi_2(\lambda^i x, \lambda^i x), \\ &\left\|\frac{f(\lambda^m x)}{\lambda^m} - \frac{f(\lambda^n x)}{\lambda^n}\right\| \leq \frac{1}{|\lambda|}\sum_{i=n}^{m-1} \frac{1}{|\lambda|^i}\varphi_2(\lambda^i x, \lambda^i x) \end{aligned} \tag{2.8}$$

for all $x \in \mathcal{B}$. Using conditions (2.3) and (2.8), we obtain by direct method [7, 9, 16] that the sequence $\left\{\frac{f(\lambda^n x)}{\lambda^n}\right\}$ is a Cauchy sequence and the mapping $g : \mathcal{B} \to \mathcal{B}$ given by

$$g(x) := \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n}$$

is well defined. Replacing $x$ by $\lambda^n x$ and $y$ by $\lambda^n y$ in (2.2), and dividing the result by $|\lambda|^n$, and then taking $n \to \infty$, we see from (2.3) that the mapping $g$ satisfies the equation (2.4), which is equivalent to $g(u+v) - g(u) - g(v) = 0$ for all $u, v \in \mathcal{B}$, that is, $g$ is additive. Letting $n = 0$ and taking $m \to \infty$ in (2.8), one can obtain the inequality (2.5). Substituting $\lambda^n x$ for $x$ in (2.1) and dividing the result by $|\lambda|^{2n}$, we obtain

$$\left\| \frac{f(\lambda^{2n}x^2)}{\lambda^{2n}} - \frac{f(\lambda^n x)}{\lambda^n}x - x\frac{f(\lambda^n x)}{\lambda^n} \right\| \le \frac{\varphi_1(\lambda^n x)}{|\lambda|^{2n}} \tag{2.9}$$

for all $x \in \mathcal{B}$. Taking the limit as $n \to \infty$ in the inequality (2.9), one obtains that

$$g(x^2) - g(x)x - xg(x) = 0$$

for all $x \in \mathcal{B}$ since $\lim_{n\to\infty} \dfrac{\varphi_1(\lambda^n x)}{|\lambda|^{2n}} = 0$ and

$$\lim_{n\to\infty} \frac{f(\lambda^{2n}x^2)}{\lambda^{2n}} = \frac{g(\lambda^n x^2)}{\lambda^n} = g(x^2).$$

Thus $g$ is an additive Jordan derivation on $\mathcal{B}$.

Furthermore, under the assumption that $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a real or complex Banach algebra $\mathcal{B}$, the additive mapping $g : \mathcal{B} \to \mathcal{B}$ satisfies

$$g(tx) = tg(x)$$

for all $t \in \mathbb{R}$ and all $x \in \mathcal{B}$ [9, 16]. That is, $g$ is a $\mathbb{R}$-linear Jordan derivation on $\mathcal{B}$. Additionally using the condition (2.6), we figure out

$$\|g(iu) - ig(u)\| = \lim_{n\to\infty} \frac{\|f(\lambda^n iu) - if(\lambda^n u)\|}{\lambda^n} \le \frac{\varphi_3(\lambda^n u)}{|\lambda|^n}.$$

for all $u \in \mathcal{B}(\|u\| = 1)$. Taking the limit in the last inequality, one gets $g(iu) = ig(u)$ for all $u \in \mathcal{B}(\|u\| = 1)$. Since $g$ is a $\mathbb{R}$-linear, for any nonzero $x \in \mathcal{B}$

$$g(ix) = g(i\|x\|\frac{x}{\|x\|}) = \|x\|\frac{i}{\|x\|}g(x) = ig(x)$$

and thus $g(ix) = ig(x)$ for all $x \in \mathcal{B}$. This fact implies $g(cx) = cg(x)$ for all $c \in \mathbb{C}$. Therefore $g$ is a $\mathbb{C}$-linear Jordan derivation on $\mathcal{B}$.

The proof of assertion indicated by parentheses in the theorem is similarly verified by the following inequalities due to (2.7)

$$\left\| f(x) - \lambda f\left(\frac{x}{\lambda}\right) \right\| \le \varphi_2\left(\frac{x}{\lambda}, \frac{x}{\lambda}\right),$$

$$\left\| \lambda^{n-1} f\left(\frac{x}{\lambda^{n-1}}\right) - \lambda^m f\left(\frac{x}{\lambda^m}\right) \right\| \le \frac{1}{|\lambda|} \sum_{i=n}^{m} |\lambda|^i \varphi_2\left(\frac{x}{\lambda^i}, \frac{x}{\lambda^i}\right)$$

for all $x \in \mathcal{B}$. This completes the proof. ■

It is well-known that an additive Jordan derivation $g$ on $\mathcal{B}$ becomes an additive derivation if $\mathcal{B}$ is a semi-prime Banach algebra [3].

**Theorem 2.2.** *Let $f : \mathcal{B} \to \mathcal{B}$ be a given mapping and let $\psi_1 : \mathcal{B} \times \mathcal{B} \to \mathbb{R}^+$, $\psi_2 : \mathcal{B} \times \mathcal{B} \to \mathbb{R}^+$ be mappings for which there exist nonzero rational numbers $a, b$ with $|a+b| \neq 0, 1$ such that*

$$\|f(xy) - f(x)y - xf(y)\| \leq \psi_1(x,y), \tag{2.10}$$

$$\|f(ax+by) - af(x) - bf(y)\| \leq \psi_2(x,y) \tag{2.11}$$

*for all $x, y \in \mathcal{B}$. Assume that the series*

$$\begin{gathered} \Psi_2(x,y) := \sum_{i=0}^{\infty} \frac{\psi_2(\lambda^i x, \lambda^i y)}{\lambda^i} < \infty \\ \left( \widehat{\Psi}_2(x,y) := \sum_{i=1}^{\infty} |\lambda|^i \psi_2\left(\frac{x}{\lambda^i}, \frac{y}{\lambda^i}\right) < \infty,\ respectively, \right) \end{gathered} \tag{2.12}$$

*converges and the limit*

$$\lim_{n\to\infty} \frac{\psi_1(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0 \quad \left( \lim_{n\to\infty} |\lambda|^{2n} \psi_1\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right) = 0 \right)$$

*for all $x, y \in \mathcal{B}$, where $\lambda := a+b$. Then there exists a unique additive mapping $g : \mathcal{B} \to \mathcal{B}$ such that*

$$\begin{gathered} g(ax+by) - ag(x) - bg(y) = 0, \\ g(xy) - g(x)y - xg(y) = 0 \end{gathered}$$

*for all $x, y \in \mathcal{B}$, that is, $g$ is an additive derivation on $\mathcal{B}$ and the following inequality*

$$\begin{gathered} \|f(x) - g(x)\| \leq \frac{1}{|\lambda|} \Psi_2(x,x) \\ \left( \|f(x) - g(x)\| \leq \frac{1}{|\lambda|} \widehat{\Psi}_2(x,x) \right) \end{gathered} \tag{2.13}$$

*holds for all $x \in \mathcal{B}$.*

*Moreover, if $f$ is measurable or the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a real Banach algebra $\mathcal{B}$, then the mapping $g$ is a $\mathbb{R}$-linear derivation on*

*$\mathcal{B}$. Alternatively, if $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a complex Banach algebra $\mathcal{B}$ and there exists a mapping $\psi_3 : \mathcal{B} \to \mathbb{R}^+$ such that*

$$\|f(ix) - if(x)\| \le \psi_3(x) \quad \textit{and}$$

$$\lim_{n\to\infty} \frac{\psi_3(\lambda^n x)}{|\lambda|^n} = 0 \quad \left(\lim_{n\to\infty} |\lambda|^n \psi_3\left(\frac{x}{\lambda^n}\right) = 0\right) \tag{2.14}$$

*are fulfilled for all $x \in \mathcal{B}$, then the mapping $g$ is a $\mathbb{C}$-linear derivation on $\mathcal{B}$.*

*Proof.* Setting $x = y$ in (2.10) yields $\|f(x^2) - f(x)x - xf(x)\| \le \psi_1(x, x)$. Taking $\varphi_1(x) := \psi_1(x, x)$ and applying Theorem 2.1, we obtain from conditions (2.11) and (2.12) that there exists a unique additive mapping $g : \mathcal{B} \to \mathcal{B}$, defined by $g(x) = \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n}$, satisfying the inequality (2.13). Replacing $x$ and $y$ in (2.10) with $\lambda^n x$ and $\lambda^n y$, respectively, and dividing the result by $|\lambda|^{2n}$, we obtain

$$\left\|\frac{f(\lambda^{2n}xy)}{\lambda^{2n}} - \frac{f(\lambda^n x)}{\lambda^n} y - x\frac{f(\lambda^n y)}{\lambda^n}\right\| \le \frac{\psi_1(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}}$$

for all $x, y \in \mathcal{B}$. Taking the limit in the last inequality, one obtains that

$$g(xy) - g(x)y - xg(y) = 0$$

for all $x, y \in \mathcal{B}$ since $\lim_{n\to\infty} \frac{\phi_1(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = 0$ and

$$\lim_{n\to\infty} \frac{f(\lambda^{2n}xy)}{\lambda^{2n}} = \frac{g(\lambda^n xy)}{\lambda^n} = g(xy).$$

The rest of the proof is similar to that of Theorem 2.1. This completes the proof. ■

The following two corollaries are immediate consequences of Theorem 2.2.

**Corollary 2.3.** *Let $\theta_1, \theta_2$ and $\theta_3$ be nonnegative reals, and let $p, q, r$ be real numbers such that either $p < 1$, $q + r < 2$ or $p > 1$, $q + r > 2$. Suppose that there exist nonzero rational numbers $a, b$ with $|a + b| \neq 0, 1$ for which a mapping $f : \mathcal{B} \to \mathcal{B}$ satisfies*

$$\|f(xy) - f(x)y - xf(y)\| \le \theta_1 \|x\|^q \|y\|^r,$$

$$\|f(ax + by) - af(x) - bf(y)\| \le \theta_2(\|x\|^p + \|y\|^p)$$

*for all $x, y \in \mathcal{B}$. Then there exists a unique additive mapping $g : \mathcal{B} \to \mathcal{B}$ on $\mathcal{B}$ such that*

$$g(xy) - g(x)y - xg(y) = 0,$$

$$g(ax + by) - ag(x) - bg(y) = 0$$

*for all $x, y \in \mathcal{B}$, and the inequality*

$$\|f(x) - g(x)\| \le \frac{2\theta_2 \|x\|^p}{|a+b| - |a+b|^p}, \text{ if } p < 1 \text{ and } q + r < 2$$

$$\left( \|f(x) - g(x)\| \le \frac{2\theta_2 \|x\|^p}{|a+b|^p - |a+b|} \text{ if } p > 1 \text{ and } q + r > 2 \quad , respectively \right)$$

*holds for all $x \in \mathcal{B}$.*

*Moreover, if $f$ is measurable or the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a real Banach algebra $\mathcal{B}$, then the mapping $g$ is a $\mathbb{R}$-linear Jordan derivation on $\mathcal{B}$. If, alternatively, $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a complex Banach algebra $\mathcal{B}$ and in addition the following inequality*

$$\|f(ix) - if(x)\| \le \theta_3 \|x\|^s$$

*holds for all $x \in \mathcal{B}$ and for some $s \neq 1$, where $s < 1$ if $p < 1$, $(s > 1$ if $p > 1,\ respectively,)$ then the mapping $g$ is a $\mathbb{C}$-linear derivation on $\mathcal{B}$.*

*Proof.* Setting $\psi_1(x,y) := \theta_1 \|x\|^q \|y\|^r$, $\psi_2(x,y) := \theta_2(\|x\|^p + \|y\|^p)$ and $\psi_3(x,y) := \theta_3 \|x\|^s$ in the previous Theorem 2.2, we can obtain the desired result. In fact, the series

$$\Psi_2(x,y) = \sum_{i=0}^{\infty} \frac{\theta_2(\|x\|^p + \|y\|^p)|\lambda|^{ip}}{|\lambda|^i} < \infty \quad \text{if } p < 1$$

$$\left( \widehat{\Psi}_2(x,y) = \sum_{i=1}^{\infty} \frac{\theta_2(\|x\|^p + \|y\|^p)|\lambda|^i}{|\lambda|^{ip}} < \infty \quad \text{if } p > 1,\ respectively, \right)$$

converges and the limit

$$\lim_{n\to\infty} \frac{\psi_1(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} = \lim_{n\to\infty} \frac{\theta_1 \|x\|^q \|y\|^r |\lambda|^{(q+r)n}}{|\lambda|^{2n}} = 0 \quad \text{if } q + r < 2$$

$$\left( \lim_{n\to\infty} |\lambda|^{2n} \psi_1(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}) = \lim_{n\to\infty} \frac{\theta_1 \|x\|^q \|y\|^r |\lambda|^{2n}}{|\lambda|^{(q+r)n}} = 0 \quad \text{if } q + r > 2 \right)$$

for all $x, y \in \mathcal{B}$.

Moreover, we get

$$\lim_{n\to\infty} \frac{\psi_3(\lambda^n x)}{|\lambda|^n} = \lim_{n\to\infty} \frac{\theta_3 \|x\|^s |\lambda|^{ns}}{|\lambda|^n} = 0 \text{ if } s < 1$$

$$\left( \lim_{n\to\infty} |\lambda|^n \psi_3(\frac{x}{\lambda^n}) = \lim_{n\to\infty} \frac{\theta_3 \|x\|^s |\lambda|^n}{|\lambda|^{ns}} = 0, \text{ if } s > 1 \right)$$

for all $x \in \mathcal{B}$. ■

**Corollary 2.4.** *Let $\theta_1, \theta_2$ be nonnegative real numbers. Suppose that there exist nonzero rational numbers $a, b$ with $|a+b| \neq 0, 1$ for which a mapping $f : \mathcal{B} \to \mathcal{B}$ satisfies*

$$\begin{aligned} \|f(xy) - f(x)y - xf(y)\| &\leq \theta_1, \\ \|f(ax+by) - af(x) - bf(y)\| &\leq \theta_2 \end{aligned}$$

*for all $x, y \in \mathcal{B}$. Then there exists a unique mapping $g : \mathcal{B} \to \mathcal{B}$ such that*

$$g(xy) - g(x)y - xg(y) = 0,$$

$$g(ax+by) - ag(x) - bg(y) = 0$$

*for all $x, y \in \mathcal{B}$, that is, $g$ is an additive derivation on $\mathcal{B}$ and*

$$\|f(x) - g(x)\| \leq \theta_2$$

*for all $x \in \mathcal{B}$.*

*Moreover, if $f$ is measurable or the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a real Banach algebra $\mathcal{B}$, then the mapping $g$ is a $\mathbb{R}$-linear Jordan derivation on $\mathcal{B}$. If, alternatively, $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x$ in a complex Banach algebra $\mathcal{B}$ and in addition the following inequality*

$$\|f(ix) - if(x)\| \leq \theta_3$$

*holds for all $x \in \mathcal{B}$ and for some $\theta_3 \geq 0$, then the mapping $g$ is a $\mathbb{C}$-linear derivation on $\mathcal{B}$.*

*Proof.* Applying Theorem 2.2, one obtains the desired conclusion. ■

**Example 2.5.** Consider the Banach algebra $M_2(\mathbb{C})$ with norm

$$\|x\| = \sqrt{\sum_{1 \leq i,j \leq 2} |x_{ij}|^2}\ , \text{ where } x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

For given $a, b \in M_2(\mathbb{C})$ a mapping $f : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by $f(x) = [a, x] + b$, satisfies

$$\psi_1(x, y) := \|f(xy) - f(x)y - xf(y)\| = \|b - xb - by\|,$$

and

$$\psi_2(x, y) := \|f(x+y) - f(x) - f(y)\| = \|b\|.$$

Then it is easy to verify that

$$\sum_{i=0}^{\infty} \frac{\psi_2(2^i x, 2^i y)}{2^i} = 2\|b\|$$

and

$$\lim_{n\to\infty} \frac{\psi_1(2^n x, 2^n y)}{4^n} = 0.$$

Thus the limit $g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = [a, x]$ exists, and it is an additive derivation which satisfies the equation $\|f(x) - g(x)\| = \|b\|$.

Furthermore, the fact that $f(tx) = t[a, x] + b$ is continuous in $t \in \mathbb{R}$ and $\psi_3(x) := \|f(ix) - if(x)\| = |1 - i|\|b\|$ satisfies

$$\lim_{n\to\infty} \frac{\psi_3(2^n x)}{2^n} = \lim_{n\to\infty} \frac{|1-i|\|b\|}{2^n} = 0$$

implies that the above derivation $g$ is $\mathbb{C}$-linear.

As an application of the main theorems, we obtain the following superstability of approximate multiplicative derivations on commutative Banach algebras.

**Theorem 2.5.** *Let $\mathcal{B}$ be a commutative Banach algebra. Let $f : \mathcal{B} \to \mathcal{B}$ be a given linear mapping and an approximate derivation with perturbation Df bounded by $\psi_1$, that is, there exists a mapping $\psi_1 : \mathcal{B} \times \mathcal{B} \to \mathbb{R}^+$ such that*

$$\|Df(x, y) := f(xy) - f(x)y - xf(y)\| \le \psi_1(x, y) \tag{2.15}$$

*for all $x, y \in \mathcal{B}$. Assume that there exist nonzero rational numbers $a, b$ with $|\lambda := a + b| \ne 0, 1$ such that the limit*

$$\lim_{n\to\infty} \frac{\psi_1(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} \qquad \left(\lim_{n\to\infty} |\lambda|^{2n} \psi_1\left(\frac{x}{\lambda^n}, \frac{y}{\lambda^n}\right), \quad respectively\right)$$

*converges to zero for all $x, y \in \mathcal{B}$. Then the mapping $f$ is in fact a linear derivation and maps the algebra into its radical.*

*Proof.* If we consider $\psi_2, \psi_3 := 0$ in Theorem 2.2, we obtain directly the desired result. ∎

It is well-known that all linear derivations on commutative semi-simple Banach algebras are zero [22]. We remark that every linear mapping $f$ on a commutative semi-simple Banach algebra which is an $\varepsilon$-*approximate* derivation, that is,

$$\|f(xy) - f(x)y - xf(y)\| \le \varepsilon,$$

is also zero. In general we obtain the following result: Let $\mathcal{B}$ be a commutative semi-simple Banach algebra. Suppose that there exist nonzero rational numbers $a, b$ with $|\lambda := a+b| \neq 0, 1$ for which a linear mapping $f : \mathcal{B} \to \mathcal{B}$ satisfies

$$\begin{aligned} \|f(xy) - f(x)y - xf(y)\| &\leq \psi_1(x, y), \\ \lim_{n\to\infty} \frac{\psi_1(\lambda^n x, \lambda^n y)}{|\lambda|^{2n}} &= 0 \end{aligned}$$

for all $x, y \in \mathcal{B}$. Then by Theorem 2.2 there exists a unique linear mapping $g : \mathcal{B} \to \mathcal{B}$ such that

$$g(xy) - g(x)y - xg(y) = 0$$

for all $x, y \in \mathcal{B}$, and

$$\|f(x) - g(x)\| \leq \psi_2 := 0$$

for all $x \in \mathcal{B}$. Hence $f = g$ is in fact a linear derivation and it is identically zero by Thomas' result [22].

## References


1. C. Borelli, On Hyers-Ulam stability for a class of functional equations, *Aequationes Math.*, **54** (1997), 74-86.
2. D.G. Bourgin, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.*, **57** (1951), 223-237.
3. M. Brešar, Jordan derivations on semi-prime rings, *Proc. Amer. Math. Soc.*, **104** (1988), 1003-1006.
4. M. Brešar and J. Vukman, On left derivations and related mappings, *Proc. Amer. Math. Soc.*, **110** (1990), 7-16.
5. M. Brešar, Centralizing mappings and derivations in prime rings, *J. Algebra*, **156** (1993), 385-394.
6. G. L. Forti, An existence and stability theorem for a class of functional equations, *Stochastica*, **4** (1980), 23-30.
7. G. L. Forti, Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations, *J. Math. Anal. Appl.*, **295** (2004), 127-133.
8. P. Găvruta, A generalization of the Hyers-Ulam-Rassias Stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431-436.
9. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.*, **27** (1941), 222-224.

10. D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.*, **44** (1992), 125-153.
11. K. W. Jun and Y. H. Lee, On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality, *Math. Ineq. Appl.*, **4(1)** (2001), 93-118.
12. Y. S. Jung and K. H. Park, On the stability of the functional equation $f(x+y+xy) = f(x) + f(y) + xf(y) + yf(x)$, *J. Math. Anal. Appl.*, **274** (2002), 659-666.
13. H. M. Kim and I. S. Chang, Stability of the functional equations related to a multiplicative derivation, *J. Appl. Math. & Computing*, **11** (2003), 413-421.
14. G. Maksa, *18 Problem, The Thirty-fourth International Symposium on Functional Equations*, Aequationes Math., June 10-June 19 (1996), Wisla-Jawornik, Poland.
15. Z. Páles, Remark 27, In 'Report on the 34th ISFE, *Aequationes Math.*, **53** (1997), 200-201.
16. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.
17. Th. M. Rassias, On the stability of the quadratic functional equation, *Mathematica*, to appear.
18. P. Šemrl, The functional equation of multiplicative derivation is superstable on standard operator algebras, *Integ. Equ. Oper. Theory*, **18** (1994), 118-122.
19. I. M. Singer and J. Wermer, Derivations on commutative normed algebras, *Math. Ann.*, **129** (1955), 260-264.
20. A. M. Sinclair, Jordan homomorphisms and derivations on semisimple Banach algebras, *Proc. Amer. Math. Soc.*, **24** (1970), 209-214.
21. J. Tabor, Remark 20, In 'Report on the 34th ISFE, *Aequationes Math.*, **53** (1997), 194-196.
22. M. D. Thomas, The image of a derivation is contained in the radical, *Ann. of Math.*, **128** (1988), 435-460.
23. S. M. Ulam, *A collection of mathematical problems*, Interscience Publ., New York, 1960.
24. J. Vukman, A result concerning derivations in Banach algebras, *Proc. Amer. Math. Soc.*, **116** (1992), 971-975.



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