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ON b-WEAKLY COMPACT OPERATORS ON BANACH LATTICES

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Abstract. In this paper every b-weakly compact operator is shown to factor through a KB-space. Also we give some necessary and sufficient conditions for a continuous operator $T : E \to X$ from a Banach lattice into a Banach space to be a b-weakly compact. Moreover, we investigated the order structure of b-weakly compact operator.

1. Introduction and Preliminaries

The notions b-weakly compactness of operators from a Banach lattice into a Banach space, b-order boundedness of sets in a Riesz space and b-order boundedness of operators between Riesz spaces were introduced in [3].

Let L be a Riesz space. Let $L^\sim$ and $L^{\sim\sim}$ denote order dual of L and second order dual of L, respectively. The canonical embedding $Q_L : L \to L^{\sim\sim}$ is defined by

$$Q_L(x) = \hat{x}\ ;\ \ \hat{x}(f) = f(x),\ \ f \in L^\sim$$

for each $x \in L,\ \hat{x}$ is an order bounded and order continuous linear functional on $L^\sim$. The canonical embedding is a lattice preserving operator. If $L^\sim$ separates the points of L then Q_L is also one-to-one, and hence L can be considered as a Riesz subspace of $L^{\sim\sim}$. Since all Banach lattices have separating order duals, we will not distinguish between a Banach lattice E and its image in E''.

Definition 1. Let A be a subset of L. If $Q_L(A)$ is order bounded in $L^{\sim\sim}$, then A is said to b-order bounded in L.

It is clear that every order bounded subset of L is b-order bounded. However, the converse is not true in general. For example, $A = \{e_n :\ n \in \mathbb{N}\}$ is b-order

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bounded in c_0 but A is not order bounded in c_0, where e_n is sequence of reals with all terms zero except for the n'th which is 1.

Definition 2. A Riesz space L is said to have property (b) if every $b-$order bounded subset of L is order bounded in L [3].

Every order dual of Riesz space has property b [3].

Definition 3. An operator T between Riesz spaces L, M is called b-order bounded, if it maps b-order bounded subsets of L into b-order bounded subsets of M.

It is clear that every order bounded operator between Riesz spaces is b-order bounded operator.

Definition 4. Let E be a Banach lattice and X be a Banach space. An operator $T : E \to X$ is said to be b-weakly compact whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X. The collection of b-weakly compact operators will be denoted by $W_b(E, X)$.

Let $W(E, X)$ and $W_o(E, X)$ denote the spaces of all weakly compact operators and of all order weakly compact operators from E into X respectively. Clearly we have $W(E, X) \subset W_b(E, X) \subset W_0(E, X)$. On the other hand, it is clear that the equality $W(E, X) = W_b(E, X)$ holds whenever E is an AM-space. Let F be a Banach lattice. $\mathcal{L}(E, F)$ and $\mathcal{L}_b(E, F)$ denote the spaces of all bounded and of all order bounded operators from E into F respectively. For brevity, $\mathcal{L}(E, E)$ will be denoted by $\mathcal{L}(E)$. $W_b^r(E, F)$ denotes the linear span of the positive b-weakly compact operators from E into F.

$I_{x''}$ denotes the principal ideal generated by $x'' \in E''$ and $Y_{x''}$ denotes the Riesz space $I_{x''} \cap E$ for each x'' in E''_+. It is clear that for each x'' in E''_+ $Y_{x''}$ is an AM-space with the norm,

$$\| u \|_\infty = \inf\{\lambda > 0 : | u | \leq \lambda x''\} \text{ for each } u \in Y_{x''}.$$

Let T be a continuous operator from Banach lattice E into Banach space X and A be a norm bounded subset of E'. We define two Riesz seminorms on E

$q_T(x) = \sup\{\| T(y) \| : | y | \leq | x |\},\ x \in E$ and

$\rho_A(x) = \sup\{| f | (| x |) : f \in A\},\ x \in E.$

For all other undefined terms and notations we will adhere to the conventions in [2] and [6].

2. Characterization of b-Weakly Compact Operators

In this section we give some characterizations for a b-weakly compact operator.

Proposition 1. *Let E be a Banach lattice, X be Banach space and $T : E \to X$ be a continuous operator, the following statements are equivalent:*

(i) *T is b-weakly compact operator.*

(ii) *For each b-order bounded disjoint sequence $\{x_n\}$ of E_+ $\lim q_T(x_n) = 0$.*

(iii) *$\{T(x_n)\}$ is norm convergent for every b-order bounded increasing sequence $\{x_n\} \subseteq E_+$. ($i.e.$ T is of type B [7].)*

Proof. $(i) \Longrightarrow (iii)$ Let $T : E \to X$ be b-weakly compact. Let $\{x_n\}$ be a b-order bounded increasing sequence of E_+. We choose $x'' \in E''_+$ with $0 \leq x_n \uparrow x''$ in E''. Let $T_{x''}$ be the restriction of the operator T to $Y_{x''}$. It is clear that $T_{x''}$ is weakly compact. Accordingly, $T'_{x''} : X' \to Y'_{x''}$ is also weakly compact. Thus if W is the closed unit ball of X', $B = T'_{x''}(W)$ is relatively weakly compact. Theorem 2.5.5 in [6] implies that the sequence $\{x_n\}$ in $Y_{x''}$ is ρ_B-Cauchy. Hence, $\{T(x_n)\}$ is norm convergent in X.

$(iii) \Longrightarrow (i)$ is obvious.

$(i) \Longleftrightarrow (ii)$ It follows from Theorem 2.5.5. in []6. ∎

Since the dual of Banach lattice has property (b), 3.5 Proposition in [7] is given as a result of preceding proposition.

The preceding proposition coupled with Theorem 3.4.11 and Theorem 3.5.8 in [6] yields the following characterization.

Proposition 2. *Let $T : E \to X$ be a continuous operator from a Banach lattice with order continuous norm into a Banach space, then T is b-weakly compact if and only if T admits a factorization through a $KB-$space F*

$$\begin{array}{ccc} E & \xrightarrow{T} & X \\ & {\scriptstyle Q}\searrow \quad \nearrow{\scriptstyle S} & \\ & F & \end{array}$$

where Q is an interval preserving lattice homomorphism.

Corollary 1. *Let E be a Banach lattice with order continuous norm, X be a Banach space and $T : E \to X$ be a continuous operator, then the following statements are equivalent:*

(i) *T is a b-weakly compact.*

(ii) $\lim \parallel T(x_n) \parallel = 0$ *for every b-order bounded sequence $\{x_n\} \subset E_+$ satisfying $x_n \overset{w}{\to} 0$.*

Proposition 3. *Let E be a Banach lattice with order continuous norm and weakly sequentialy continuous lattice operations. Let X be a Banach space and $T : E \to X$ be a continuous operator, then the following assertions are equivalent:*

(i) *T is b-weakly compact*

(ii) *If $\{x_n\}$ is a b-order bounded $\sigma(E, E')$-Cauchy sequence, then $\{T(x_n)\}$ is $\| \cdot \|$-convergent.*

Proof. $(i) \Longrightarrow (ii)$ Let $\{x_n\}$ be a b-order bounded $\sigma(E, E')$-Cauchy sequence of E. If $\{T(x_n)\}$ is not norm Cauchy sequence of X, then there exist some $\epsilon > 0$ and a subsequence $\{y_n\}$ of $\{x_n\}$ satisfying $\| T(y_{n+1} - y_n) \| > \epsilon$ for all $n \in \mathbb{N}$. Since $\{y_{n+1} - y_n\}$ convergences weakly to zero and lattice operations in E are weakly sequentially continous, we see that $(y_{n+1} - y_n)^+ \to 0$ and $(y_{n+1} - y_n)^- \to 0$ weakly. Therefore, $\lim \| T(y_{n+1} - y_n) \| = 0$, which is impossible. Thus, $\{T(x_n)\}$ is a norm Cauchy sequence, and hence is norm convergent in X.

$(ii) \Longrightarrow (i)$ This assertion follows from proposition 2.8 in [3]. ∎

Proposition 4. *Let E be a Banach lattice with order continuous norm, X be a Banach space and $T : E \to X$ be a continuous operator, then the following statements are equivalent:*

(i) *T is b-weakly compact operator.*

(ii) *For each $x'' \in E''_+$ and $\epsilon > 0$ there exist $0 \leq y''' \in (Y_{x''})'''$ and $\delta > 0$ such that $| y | \leq x''$ and $y'''(| y |) < \delta$ imply $\| T(y) \| < \epsilon$.*

Proof. $(i) \Longrightarrow (ii)$ Let $T : E \to X$ be b-weakly compact operator. Fix $\epsilon > 0$ and $0 < x'' \in E''$. We can assume that $Y_{x''} \neq \{0\}$. Denote by $T_{x''}$, the restriction of the operator T to $Y_{x''}$. $T_{x''} : Y_{x''} \to X$ is weakly compact operator. Accordingly, $T'''_{x''} : X''' \to (Y_{x''})'''$ is also weakly compact. Thus, if W is the closed unit ball of X''' then $T'''_{x''}(W)$ is relatively weakly compact in $(Y_{x''})'''$. Then by Theorem 13.10 in [2], there exists some $0 \leq y''' \in (Y_{x''})'''$ such that $\| (| T'''_{x''}(f) | - y''')^+ \| < \epsilon/2 \| x'' \|$ holds for all $f \in W$.

Now put $\delta = \epsilon/2$ and let $| y | \leq x''$ satisfy $y'''(| y |) < \delta$. Then for each $f \in W$, we have

$$\begin{aligned} | f(T''_{x''}(y) | &= | T'''_{x''} f(y) | \\ &\leq | T'''_{x''} f | (| y |) \\ &= (| T'''_{x''} f | - y''')^+(| y |) + (y''' \wedge | T'''_{x''} f | (| y |) \\ &\leq \| (| T'''_{x''} f | - y''')^+ \| \cdot \| y \| + y'''(| y |) \\ &< \epsilon, \end{aligned}$$

which implies that $\| T(y) \| < \epsilon$ holds, as desired.

$(ii) \Longrightarrow (i)$ Let B denote the band generated by E in E'' and $x'' \in B_+$. We choose a net $\{x_\alpha\}$ in E with $0 \leq x_\alpha \uparrow x''$. Taking into account that $x_\alpha \xrightarrow{w^*} x''$ in E''

and $T'' : E'' \to X''$ is w^*-continuous, we see that $Tx_\alpha \xrightarrow{w^*} T''x''$ also holds. Since $\| x_\alpha \|_\infty \leq 1$ and $\{x_\alpha\}$ is increasing net in $Y_{x''}$, there exists a positive element y'''' in $Y''''_{x''}$ with $0 \leq x_\alpha \uparrow y''''$ in $Y''''_{x''}$. Therefore, $f(x_\alpha) \to y''''(f)$ for all $f \in (Y'''_{x''})_+$. Now let $\epsilon > 0$. Choose $\delta > 0$ and $0 \leq y''' \in Y'''_{x''}$ such that $| \; y \; | \leq x''$ and $y'''(| \; y \; |) < \delta$ imply $\| \; T(y) \; \| < \epsilon$. Next pick some α_0 so that $y'''(| \; x_\alpha - x_\beta \; |) < 2\delta$ holds for all $\alpha, \beta \geq \alpha_0$. Fix some $\beta \geq \alpha_0$ and note that if $g \in X'$ with $\| \; g \; \| \leq 1$, then

$$
\begin{aligned}
\| \; (T''x'' - Tx_\beta)(g) \; \| &= \lim_{\alpha \geq \alpha_0} | \; g(T(x_\alpha - x_\beta) \; | \\
&\leq \limsup_{\alpha} \| \; T(x_\alpha - x_\beta) \; \| \\
&< \epsilon
\end{aligned}
$$

holds, the latter implies that $\| \; T''(x'') - T(x_\beta) \; \| < \epsilon$. This shows that $T''x''$ lies in the norm closure of X in X''. Since X is a Banach space, we see that $T''(x'') \in X$ hence, $T''(B) \subseteq X$ holds. By Proposition 2.11 in [3], T is b-weakly compact. ∎

Neither the adjoint of b-weakly compact operator nor a continuous operator with a b-weakly compact adjoint have to be b-weakly compact in general. One can just put the identity operators on l_1 and c_0 respectively.

Recall that a continuous operator $T : X \to E$ from a Banach space into a Banach lattice is semicompact whenever for $\epsilon > 0$ there exists some $u \in E_+$ satisfying

$$\| \; (| \; T(x) \; | - u)^+ \; \| < \epsilon$$

for all $x \in X$ with $\| \; x \; \| \leq 1$.

A continuous operator, whose adjoint is semicompact, from a Banach lattice with order continuous norm into a Banach lattice is b-weakly compact.

Corollary 2. *Let $T : E \to F$ be a continuous operator from a Banach lattice with order continuous norm into a Banach lattice. If the adjoint of T is semicompact, then T is b-weakly compact.*

However, as the next example shows, the converse of this result is not true in general.

Example 1. The identity operator $I : l_2 \to l_2$ is a b-weakly compact but its adjoint $I : l_2 \to l_2$ is not a semicompact.

Recall that an operator $T : X \to Y$ between two Banach spaces is a Dunford-Pettis operator whenever $x_n \xrightarrow{w} 0$ in X implies $\lim \| \; T(x_n) \; \| = 0$.

Every Dunford-Pettis operator maps b-order bounded sets onto relatively weakly compact sets.

Proposition 5. *Every Dunford-Pettis operator $T : E \to X$ from a Banach lattice E into a Banach space X is b-weakly compact.*

A b-weakly compact operator need not be a Dunford-Pettis operator. For instance, the identity operator $I : L_1[0,1] \to L_1[0,1]$ is b-weakly compact but it is not Dunford Pettis operator.

3. Order Structure of b-Weakly Compact Operators

In [1], the idea of a generalized sublattice was introduced. There it is said that $(\mathcal{J}, \leq)$ is a partially ordered vector spaces and $\mathcal{Z}$ a subspace of $\mathcal{J}$, then $\mathcal{Z}$ is a generalized sublattice of $\mathcal{J}$ if $(\mathcal{Z}, \leq)$ is a lattice and for each $x, y \in \mathcal{Z}$ the supremum of x and y calculated in $\mathcal{Z}$ is also their supremum in $\mathcal{J}$. For example the compact operators from $C([0,1])$ into c_0 (which form a lattice) as a subset of $\mathcal{L}(C([0,1]), c_0)$ (which is not a lattice).

The following example shows that on Dedekind complete Banach lattices, b-weakly compact operators do not form a lattice.

Example 2. The well known operator $T : L_1[0,1] \to c_0$ defined by

$$T(f) = (\int_0^1 f(x)\sin x dx, \int_0^1 f(x)\sin 2x dx, \cdots)$$

is a b-weakly compact operator but it is not order bounded. Therefore, $W_b(L_1[0,1], c_0)$ is not a lattice.

The next example due to Z.L. Chen and A.W. Wickstead in [4] shows that the order bounded, b-weakly compact operators from a Banach lattice into a Dedekind complete Banach lattice do not form a lattice.

Example 3. Let $E = C([0,1])$, $F = l_\infty(F_n)$ where $F_n = (l_\infty, \| \cdot \|)$ and $\| (\lambda_k) \| = max\{\| (\lambda_k) \|_\infty, n \limsup(| \lambda_k |)\}$ for all $(\lambda_k) \in l_\infty$. Then for each $n \in \mathbb{N}$, F_n is a Dedekind complete AM-space, hence so is F. Define $T_n : E \to F_n$ by $T_n(f) = (2^n \cdot \int_{I_n} f \cdot r_k dt)_{k=1}^\infty \in F_n$ for all $f \in E$, where r_n is the n'th Radamacher function on $[0,1]$ and $I_n = (2^{-n}, 2^{-n+1})$.

Now define $T : E \to F$ by $T(f) = (\frac{1}{n} T_n(f))_{n=1}^\infty$. Then T is a weakly compact operator, so T is a b-weakly compact operator and its modulus $| T |$ exists and $| T |$

is not order weakly compact hence not b-weakly compact so $W_b(E,F)$ is not a lattice.

By Corollary 2.9 in [3] we see that the linear span of the positive b-weakly compact operators from a Banach lattice into a Dedekind complete Banach lattice is a generalized sublattice of the space of all b-weakly compact operators.

Proposition 6. *Let E and F be two Banach lattices with F Dedekind complete. Then $W_b^r(E,F)$ is a generalized sublattice of $W_b(E,F)$.*

We note that $W_b^r(E,F)$ is an ideal in $\mathcal{L}_b(E,F)$, but $W_b^r(E,F)$ is not a band in $\mathcal{L}_b(E,F)$ in general.

The following proposition gives us some sufficient conditions for the composition of two operators to be a b-weakly compact. The proof of the following proposition is routine.

Proposition 7. *Let E, F, G be Banach lattices and $E \xrightarrow{T} F \xrightarrow{S} G$ be operators, then we have that*

(1) *If T is a b-order bounded operator and S is a b-weakly compact operator then ST is a b-weakly compact operator.*

(2) *If S is continuous and T is a b-weakly compact operator then ST is a b-weakly compact operator.*

(3) *If F has a continuous norm and T is a continuous operator with $T''(B_E) \subset B_F$ and S is a b-weakly compact operator then ST is a b-weakly compact operator, where $B_E(B_F)$ is the band generated by $E(F)$ in $E''(F'')$.*

The above proposition also informs us that the b-weakly compact operators on a Banach lattice E form a left-sided ring ideal of $\mathcal{L}(E)$.

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