

## WEAK AND STRONG CONVERGENCE FOR COMMON FIXED POINTS OF A PAIR OF NONEXPANSIVE AND ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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**Abstract.** In this paper, we establish some weak and strong convergence theorems of modified Ishikawa iteration methods with errors with respect to a pair of nonexpansive and asymptotically nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify a few results due to Chang [2], Osilike and Aniagbosor [10], Rhoades [11] and Schu [13, 14] and others.

### 1. INTRODUCTION

In 1972, Goebel and Kirk [4] introduced the concept of asymptotically nonexpansive mappings and proved that if  $K$  is a nonempty closed bounded subset of a uniformly convex Banach space  $E$ , then every asymptotically nonexpansive self-mapping on  $K$  has a fixed point. After that, a number of authors studied a few iterative approximation methods of fixed points for asymptotically nonexpansive mappings. In 1991, Schu [13], [14] introduced the modified Ishikawa iteration methods and modified Mann iteration methods and proved that the modified Mann iteration sequence converges strongly to some fixed points of asymptotically nonexpansive mappings in Hilbert spaces. Rhoades [11] extended the results in [13] to uniformly convex Banach spaces and to modified Ishikawa iteration methods. Chang [2], Liu, Kim and Kim [6], Liu, Kim and Chun [7], Liu and Kang [8] and Osilike and Aniagbosor [10] and others also established some strong and weak convergence

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theorems of modified Ishikawa iteration methods with errors and three-step iteration methods with errors for asymptotically nonexpansive mappings. Very recently, Agarwal, Cho, Li and Hunag [1] have studied convergence of the Ishikawa iteration process with errors for a couple of quasi-contractive mappings in  $q$ -uniformly smooth Banach spaces.

In this paper, we introduce a new iterative method, called modified Ishikawa iteration method with errors with respect to a pair of mappings, and establish some strong and weak convergence theorems of the modified Ishikawa iteration method with errors with respect to a pair of nonexpansive and asymptotically nonexpansive mappings in nonempty closed convex subsets of uniformly convex Banach spaces. The results presented in this paper generalize, improve and unify a few results due to Chang [2], Osilike and Aniagbosor [10], Rhoades [11] and Schu [13, 14] and others.

## 2. PRELIMINARIES

Let  $E$  be a uniformly convex Banach space,  $K$  be a nonempty subset of  $E$  and  $S, T : K \rightarrow K$  be two mappings.  $F(S, T)$  denotes the set of common fixed points of  $S$  and  $T$ ,  $I$  stands for the identity mapping on  $E$ . Let  $J : E \rightarrow 2^E$  be the normalized duality mapping defined by

$$J(x) = \{f \in E^*, \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \forall x \in E.$$

Let us recall the following concepts and results.

**Definition 2.1** [3]. A mapping  $T : K \rightarrow K$  is said to be

- (i) asymptotically nonexpansive if there exists a sequence  $\{k_n\}_{n \geq 1} \subset [1, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = 1$  such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in K, n \geq 1;$$

- (ii) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K;$$

- (iii) uniformly  $L$ -Lipschitzian if there exists a constant  $L \geq 1$  satisfying

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall x, y \in K, n \geq 1;$$

- (iv) semi-compact if  $K$  is closed and for any bounded sequence  $\{x_n\}_{n \geq 1}$  in  $K$  with  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ , there exists a subsequence  $\{x_{n_i}\}_{i \geq 1} \subset \{x_n\}_{n \geq 1}$  and  $x \in K$  such that  $\lim_{i \rightarrow \infty} x_{n_i} = x$ .

It is well known that each nonexpansive mapping is asymptotically nonexpansive with  $k_n \equiv 1$  for all  $n \geq 1$ , and each asymptotically nonexpansive mapping with a sequence  $\{k_n\}_{n \geq 1} \subset [1, \infty)$  and  $\lim_{n \rightarrow \infty} k_n = 1$  is uniformly  $L$ -Lipschitzian with  $L = \sup\{k_n : n \geq 1\}$ , and the converses are not true.

**Definition 2.2.** A mapping  $T$  with domain  $D(T)$  and range  $R(T)$  in  $E$  is called demiclosed at a point  $p \in D(T)$  if whenever  $\{x_n\}_{n \geq 1}$  is a sequence in  $E$  which converges weakly to a point  $x \in E$  and  $\{Tx_n\}_{n \geq 1}$  converges strongly to  $p$ , then  $Tx = p$ .

**Definition 2.3** [9]. A Banach space  $E$  is called to satisfy Opial's condition if for each sequence  $\{x_n\}_{n \geq 1}$  in  $E$  which converges weakly to a point  $x \in E$

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in E - \{x\}.$$

**Definition 2.4.** Let  $K$  be a nonempty convex subset of a normed linear space  $E$  and  $S, T : K \rightarrow K$  be two mappings. For an arbitrary  $x_1 \in K$ , the modified Ishikawa iteration sequence with errors  $\{x_n\}_{n \geq 1}$  with respect to  $S$  and  $T$  is defined by

$$(2.1) \quad \begin{aligned} y_n &= a'_n Sx_n + b'_n T^n x_n + c'_n v_n, \\ x_{n+1} &= a_n Sx_n + b_n T^n y_n + c_n u_n, \quad \forall n \geq 1, \end{aligned}$$

where  $\{u_n\}_{n \geq 1}$  and  $\{v_n\}_{n \geq 1}$  are given bounded sequences in  $K$ ,  $\{a_n\}_{n \geq 1}$ ,  $\{b_n\}_{n \geq 1}$ ,  $\{c_n\}_{n \geq 1}$ ,  $\{a'_n\}_{n \geq 1}$ ,  $\{b'_n\}_{n \geq 1}$  and  $\{c'_n\}_{n \geq 1}$  are sequences in  $[0, 1]$  satisfying

$$(2.2) \quad a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \quad \forall n \geq 1.$$

**Remark 2.1.** In case  $S = I$  in (2.2), then the sequence  $\{x_n\}_{n \geq 1}$  generated in (2.1) reduces to the usual modified Ishikawa iteration sequence with errors.

**Remark 2.2.** For the usual Ishikawa iteration process with errors, Rhoades [12] pointed out that "Not knowing the precise values  $u_n$  and  $v_n$  makes it impossible to continue the iteration process". In order to solve the problem, that is, in order to continue the iteration process, we have to know the precise values  $u_n$  and  $v_n$  before calculating the usual Ishikawa iteration process with errors. Hence we state that the sequences  $\{u_n\}_{n \geq 1}$  and  $\{v_n\}_{n \geq 1}$  in the Definition 2.4 are given bounded sequences in  $K$ .

**Lemma 2.1** ([5]). *Let  $E$  be a Banach space satisfying Opial's condition and  $K$  be a nonempty closed convex subset of  $E$ . If  $T : K \rightarrow K$  is an asymptotically nonexpansive mapping, then  $I - T$  is demiclosed at zero.*

**Lemma 2.2** ([14]). *Let  $E$  be a uniformly convex Banach space,  $\{t_n\}_{n \geq 1} \subseteq [b, c] \subset (0, 1)$ ,  $\{x_n\}_{n \geq 1}$  and  $\{y_n\}_{n \geq 1}$  be sequences in  $E$ . If  $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$ ,  $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$  and  $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = a$  for some constant  $a \geq 0$ , then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .*

**Lemma 2.3** ([3]). *Let  $E$  be a normed linear space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall x, y \in E, \quad j(x + y) \in J(x + y).$$

**Lemma 2.4** ([15]). *Let  $p > 1$  and  $r > 0$  be two constants. Then a Banach space  $E$  is uniformly convex if and only if there exists a continuous and strictly increasing convex function  $g : [0, \infty) \rightarrow [0, \infty)$  with  $g(0) = 0$  such that*

$$\|ax + (1 - a)y\|^p \leq a\|x\|^p + (1 - a)\|y\|^p - w_p(a)g(\|x - y\|)$$

for each  $a \in [0, 1]$  and  $x, y \in B(\theta, r) = \{u : \|u\| \leq r \text{ and } u \in E\}$ , where

$$w_p(a) = a^p(1 - a) + a(1 - a)^p.$$

**Lemma 2.5** ([10]). *Let  $\{a_n\}_{n \geq 1}$ ,  $\{b_n\}_{n \geq 1}$  and  $\{c_n\}_{n \geq 1}$  be sequences of non-negative numbers satisfying the inequality*

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \quad \forall n \geq 1.$$

If  $\sum_{n=1}^{\infty} c_n < \infty$  and  $\sum_{n=1}^{\infty} b_n < \infty$ , then  $\lim_{n \rightarrow \infty} a_n$  exists. In particular, if  $\{a_n\}_{n \geq 1}$  has a subsequence which converges to zero, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

### 3. MAIN RESULTS

**Lemma 3.1.** *Let  $K$  be a nonempty convex subset of a normed linear space  $E$ . Let  $S : K \rightarrow K$  be a mapping and  $T : K \rightarrow K$  be uniformly  $L$ -Lipschitzian. Then*

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + L(L^2 + 2L + 2)\|x_n - T^n x_n\| \\ &\quad + L(L + 1)[(L + 1)\|Sx_n - x_n\| + c_n\|u_n - x_n\| + Lb_n c'_n\|v_n - x_n\|] \end{aligned}$$

for any  $n \geq 1$ , where  $\{x_n\}_{n \geq 1}$  is defined by (2.1).

*Proof.* Set  $A_n = \|x_n - T^n x_n\|$ ,  $B_n = \|Sx_n - x_n\|$  for each  $n \geq 1$ . It follows that

$$(3.1) \quad \|y_n - x_n\| \leq a'_n B_n + b'_n A_n + c'_n \|v_n - x_n\|,$$

$$\begin{aligned}
(3.2) \quad \|x_{n+1} - x_n\| &\leq a_n B_n + b_n (\|T^n y_n - T^n x_n\| \\
&\quad + \|T^n x_n - x_n\|) + c_n \|x_n - u_n\| \\
&\leq a_n B_n + b_n L \|x_n - y_n\| + b_n A_n + c_n \|x_n - u_n\|
\end{aligned}$$

and

$$\begin{aligned}
(3.3) \quad \|x_{n+1} - T x_{n+1}\| &\leq A_{n+1} + L \|T^n x_{n+1} - x_{n+1}\| \\
&\leq A_{n+1} + L (\|T^n x_{n+1} - T^n x_n\| \\
&\quad + \|T^n x_n - x_n\| + \|x_n - x_{n+1}\|) \\
&\leq A_{n+1} + L^2 \|x_{n+1} - x_n\| + L \|T^n x_n - x_{n+1}\| \\
&\leq A_{n+1} + L(L+1) \|x_{n+1} - x_n\| + L A_n
\end{aligned}$$

for all  $n \geq 1$ . Substituting (3.1) and (3.2) into (3.3), we obtain that

$$\begin{aligned}
\|x_{n+1} - T x_{n+1}\| &\leq A_{n+1} + L^2(L+2L+2)A_n + L(L+1)^2 B_n \\
&\quad + L(L+1)[c_n \|u_n - x_n\| + L b_n c'_n \|v_n - x_n\|]
\end{aligned}$$

for any  $n \geq 1$ . This completes the proof of Lemma 2.1.

**Remark 3.1.** Lemma 1.2 in [10], Lemma 1.4 in [11] and Lemma 1.4 in [14] are special cases of Lemma 3.1.

**Lemma 3.2.** *Let  $K$  be a nonempty convex subset of a normed linear space  $E$ . Let  $S : K \rightarrow K$  be a nonexpansive mapping and  $T : K \rightarrow K$  be an asymptotically nonexpansive mapping with a sequence  $\{k_n\}_{n \geq 1} \subseteq [1, \infty)$  satisfying  $\lim_{n \rightarrow \infty} k_n = 1$  and  $F(S, T) \neq \emptyset$ . If the following conditions*

$$(3.4) \quad \sum_{n=1}^{\infty} c_n < \infty,$$

$$(3.5) \quad \sum_{n=1}^{\infty} b_n c'_n < \infty$$

and

$$(3.6) \quad \sum_{n=1}^{\infty} (k_n - 1) < \infty$$

hold, then  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists for any  $q \in F(S, T)$ , where  $\{x_n\}_{n \geq 1}$  is defined by (2.1).

*Proof.* Let  $q \in F(S, T)$ . Since  $\{u_n - q\}_{n \geq 1}$  and  $\{v_n - q\}_{n \geq 1}$  are bounded, it follows that  $M = \sup\{\|u_n - q\|, \|v_n - q\| : n \geq 1\} < \infty$ . In view of (2.1) we infer that

$$\begin{aligned}
(3.7) \quad \|x_{n+1} - q\| &\leq a_n \|x_n - q\| + b_n k_n \|y_n - q\| + c_n \|u_n - q\| \\
&\leq a_n \|x_n - q\| + b_n k_n (a'_n \|x_n - q\| + b'_n k_n \|x_n - q\| \\
&\quad + c'_n \|v_n - q\|) + c_n M \\
&\leq [a_n + b_n k_n (a'_n + b'_n k_n)] \|x_n - q\| + (b_n c'_n k_n + c_n) M \\
&\leq [1 - b_n + b_n k_n (1 - b'_n + b'_n k_n)] \|x_n - q\| + (b_n c'_n k_n + c_n) M \\
&\leq [1 + (k_n - 1)(1 + k_n)] \|x_n - q\| + (b_n c'_n k_n + c_n) M
\end{aligned}$$

for each  $n \geq 1$ . It follows from (3.4)-(3.7) and Lemma 2.5 that  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists. This completes the proof.

**Remark 3.2.** Lemma 3.2 generalizes Lemma 3 in [10] and Lemma 1.2 in [14].

**Lemma 3.3.** Let  $K$  be a nonempty convex subset of a uniformly convex Banach space  $E$ . Let  $S : K \rightarrow K$  be a nonexpansive mapping and  $T : K \rightarrow K$  be an asymptotically nonexpansive mapping with a sequence  $\{k_n\}_{n \geq 1} \subseteq [1, \infty)$  satisfying  $\lim_{n \rightarrow \infty} k_n = 1$ ,  $F(S, T) \neq \emptyset$ , (3.4) – (3.6) and

$$(3.8) \quad \|x - Ty\| \leq \|Sx - Ty\|, \quad \forall x, y \in K.$$

If the sequence  $\{x_n\}_{n \geq 1}$  satisfies (2.1),

$$(3.9) \quad \limsup_{n \rightarrow \infty} b'_n k_n < 1,$$

and

$$(3.10) \quad 0 < a \leq b_n \leq b < 1, \quad \forall n \geq 1,$$

where  $a$  and  $b$  are constants, then  $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ .

*Proof.* Let  $q \in F(S, T)$ . Lemma 3.2 ensures that  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exists. Set  $\lim_{n \rightarrow \infty} \|x_n - q\| = d$ . Since  $\{u_n\}_{n \geq 1}$  and  $\{v_n\}_{n \geq 1}$  are bounded sequences, it follows that

$$M = \sup\{\|u_n - q\|, \|v_n - q\|, \|x_n - v_n\|, \|x_n - u_n\| : n \geq 1\} < \infty.$$

Observe that

$$\begin{aligned}
(3.11) \quad \lim_{n \rightarrow \infty} \|x_{n+1} - q\| &= \lim_{n \rightarrow \infty} \|(1 - b_n - c_n)Sx_n + b_n T^n y_n + c_n u_n - q\| \\
&= \lim_{n \rightarrow \infty} \|(1 - b_n)[Sx_n - q - c_n(Sx_n - u_n)] \\
&\quad + b_n[T^n y_n - q - c_n(Sx_n - u_n)]\|.
\end{aligned}$$

From the nonexpansivity of  $S$  and (3.4), we deduce that

$$(3.12) \quad \begin{aligned} & \limsup_{n \rightarrow \infty} \|Sx_n - q - c_n(Sx_n - u_n)\| \\ & \leq \limsup_{n \rightarrow \infty} [(1 - c_n)\|x_n - q\| + c_n\|u_n - q\|] \\ & \leq d. \end{aligned}$$

Because  $T$  is asymptotically nonexpansive, by (2.1) we derive that

$$(3.13) \quad \begin{aligned} & \|T^n y_n - q - c_n(Sx_n - u_n)\| \\ & \leq k_n\|y_n - q\| + c_n\|Sx_n - q\| + c_n\|u_n - q\| \\ & \leq k_n(a'_n\|x_n - q\| + b'_n k_n\|x_n - q\|) + (c'_n k_n + c_n)M + c_n\|x_n - q\| \\ & \leq (k_n^2 + c_n)\|x_n - q\| + (c'_n k_n + c_n)M \end{aligned}$$

for all  $n \geq 1$ . In view of (3.4)-(3.6) and (3.13), we conclude that

$$(3.14) \quad \limsup_{n \rightarrow \infty} \|T^n y_n - q - c_n(Sx_n - u_n)\| \leq d.$$

On account of (3.10)-(3.12), (3.14) and Lemma 2.2, we see that

$$(3.15) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \|Sx_n - T^n y_n\| \\ & = \lim_{n \rightarrow \infty} \|[Sx_n - q - c_n(Sx_n - u_n)] - [T^n y_n - q - c_n(Sx_n - u_n)]\| \\ & = 0, \end{aligned}$$

which implies that

$$(3.16) \quad \lim_{n \rightarrow \infty} \|x_n - T^n y_n\| = 0$$

by (3.8). Clearly

$$\|Sx_n - x_n\| \leq \|Sx_n - T^n y_n\| + \|x_n - T^n y_n\|, \quad \forall n \geq 1.$$

Thus (3.15) and (3.16) mean that

$$(3.17) \quad \lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0.$$

It is easy to verify that

$$\begin{aligned} \|x_n - T^n x_n\| & \leq \|x_n - T^n y_n\| + k_n\|y_n - x_n\| \\ & \leq \|x_n - T^n y_n\| + k_n(a'_n\|Sx_n - x_n\| + b'_n\|T^n x_n - x_n\| + c'_n M) \end{aligned}$$

for each  $n \geq 1$ . It follows from (3.9) that there exists a positive integer  $N$  such that  $k_n b'_n < 1$  for all  $n > N$ . This yields that

$$(3.18) \quad \|x_n - T^n x_n\| \leq \frac{1}{1 - k_n b'_n} (\|x_n - T^n y_n\| + k_n \|Sx_n - x_n\| + M k_n c'_n)$$

for all  $n \geq N$ . According to (3.5), (3.10), (3.16)-(3.18) and (3.20), we conclude that

$$(3.19) \quad \lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0.$$

It follows from (3.4)-(3.5), (3.17), (3.19) and Lemma 3.1 that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0.$$

This completes the proof.

**Remark 3.3.** Lemma 3.3 extends Lemma 3.3 in [10], Lemma 4 in [11] and Theorem 1 in [13].

**Theorem 3.1.** *Let  $E$  be a uniformly convex Banach space satisfying Opial's condition and  $K$  be a nonempty closed convex subset of  $E$ . Let  $S : K \rightarrow K$  be a nonexpansive mapping and  $T : K \rightarrow K$  be an asymptotically nonexpansive mapping with a sequence  $\{k_n\}_{n \geq 1} \subseteq [1, \infty)$  satisfying  $\lim_{n \rightarrow \infty} k_n = 1$ ,  $F(S, T) \neq \emptyset$ , (3.4) – (3.6) and (3.8) – (3.10). Then the modified Ishikawa iteration sequence with errors  $\{x_n\}_{n \geq 1}$  with respect to  $S$  and  $T$  given by (2.1) converges weakly to a common fixed point of  $S$  and  $T$ .*

*Proof.* It follows from Lemma 3.2 that  $\{x_n\}_{n \geq 1}$  is bounded. Hence  $\{x_n\}_{n \geq 1}$  has a subsequence  $\{x_{n_j}\}_{j \geq 1}$ , which converges weakly to  $p$ . Since  $\{x_{n_j}\}_{j \geq 1} \subseteq K$  and  $K$  is weakly closed, it follows that  $p \in K$ . In light of Lemma 2.1 we deduce that  $I - T$  and  $I - S$  are demiclosed at zero. Thus Lemma 3.3 ensures that  $(I - T)p = (I - S)p = 0$ . That is,  $p \in F(S, T)$ . Suppose that  $\{x_n\}_{n \geq 1}$  does not converge weakly to  $p$ . Then there exists another subsequence  $\{x_{m_k}\}_{k \geq 1}$  of  $\{x_n\}_{n \geq 1}$  which converges weakly to some  $q \neq p$ . It is clear that  $q \in F(S, T)$ ,  $\lim_{n \rightarrow \infty} \|x_n - p\|$  and  $\lim_{n \rightarrow \infty} \|x_n - q\|$  exist. Let  $a = \lim_{n \rightarrow \infty} \|x_n - p\|$ ,  $b = \lim_{n \rightarrow \infty} \|x_n - q\|$ . Because  $E$  satisfies Opial's condition, we obtain that

$$\begin{aligned} a &= \liminf_{j \rightarrow \infty} \|x_{n_j} - p\| < \liminf_{j \rightarrow \infty} \|x_{n_j} - q\| = \\ b &= \liminf_{k \rightarrow \infty} \|x_{m_k} - q\| < \liminf_{k \rightarrow \infty} \|x_{m_k} - p\| = a, \end{aligned}$$

which is a contradiction. Hence  $p = q$  and  $\{x_n\}_{n \geq 1}$  converges weakly to  $p \in F(S, T)$ . This completes the proof.



**Lemma 3.4.** *Let  $K$  be a nonempty bounded convex subset of a normed linear space  $E$ . Let  $S : K \rightarrow K$  be a mapping and  $T : K \rightarrow K$  be a uniformly  $L$ -Lipschitzian mapping. Then*

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| \leq & \|x_{n+1} - T^{n+1}x_{n+1}\| + L(3 + 2L + L^2)\|x_n - T^n x_n\| \\ & + L(L+1)(\|Sx_n - x_n\| + c_n\|u_n - Sx_n\| + Lb_n c'_n\|v_n - Sx_n\|) \end{aligned}$$

for every  $n \geq 1$ , where  $\{x_n\}_{n \geq 1}$  is defined by (2.1).

*Proof.* Put

$$(3.20) \quad A_n = c_n(u_n - Sx_n), \quad B_n = c'_n(v_n - Sx_n), \quad \forall n \geq 1.$$

Then the sequence  $\{x_n\}_{n \geq 1}$  defined by (2.1) can be rewritten as

$$(3.21) \quad \begin{aligned} y_n &= (1 - b'_n)Sx_n + b'_n T^n x_n + B_n, \\ x_{n+1} &= (1 - b_n)Sx_n + b_n T^n y_n + A_n, \quad \forall n \geq 1. \end{aligned}$$

The rest of the proof is exactly the same as that of Lemma 3.1, and is omitted. This completes the proof.

**Remark 3.4.** Lemma 3.4 is an improvement of Lemma 3 in [2] and Lemma 1.2 in [13].

**Lemma 3.5.** *Let  $K$  be a nonempty bounded closed convex subset of a real Banach space  $E$ . Let  $S : K \rightarrow K$  be a nonexpansive mapping and  $T : K \rightarrow K$  be a uniformly  $L$ -Lipschitzian and asymptotically nonexpansive mapping with a sequence  $\{k_n\}_{n \geq 1} \subseteq [1, \infty)$  satisfying  $\lim_{n \rightarrow \infty} k_n = 1$ ,  $F(S, T) \neq \emptyset$ , (3.4)-(3.6), (3.8), (3.10) and*

$$(3.22) \quad \limsup_{n \rightarrow \infty} b'_n < L^{-1}.$$

Then

$$\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = \lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0,$$

where  $\{x_n\}_{n \geq 1}$  is defined by (2.1).

*Proof.* Let  $\{A_n\}_{n \geq 1}$  and  $\{B_n\}_{n \geq 1}$  be defined by (3.20) and  $q \in F(S, T)$ . Note that  $K$  is a nonempty bounded closed convex subset and  $\{x_n\}_{n \geq 1}$ ,  $\{y_n\}_{n \geq 1}$ ,  $\{T^n x_n\}_{n \geq 1}$ ,  $\{T^n y_n\}_{n \geq 1}$ ,  $\{Sx_n\}_{n \geq 1}$  are in  $K$ . Then there exists  $r > 0$  such that

$$\begin{aligned} K \cup \{x_n - q, y_n - q, Sx_n - q, Sx_n - q + A_n, Sx_n - q + B_n, \\ Sx_n - u_n, Sx_n - v_n, T^n x_n - q + A_n, T^n y_n - q + B_n, T^n y_n - q + A_n\} \subset B(\theta, r) \end{aligned}$$

for any  $n \geq 1$ . Using (3.21), Lemmas 2.3 and 2.4, we get that

$$(3.23) \quad \begin{aligned} \|y_n - q\|^2 - \|x_n - q\|^2 &\leq (1 - b'_n)\|Sx_n - q\|^2 + b'_n\|T^n x_n - q\|^2 - \|x_n - q\|^2 + 2r\|B_n\| \\ &\leq b'_n(k_n^2 - 1)\|x_n - q\|^2 + 2r\|B_n\| \end{aligned}$$

for each  $n \geq 1$ . It follows from (3.21), (3.23) and Lemma 2.4 that

$$(3.24) \quad \begin{aligned} \|x_{n+1} - q\|^2 &= \|(1 - b_n)(Sx_n - q + A_n) + b_n(T^n y_n - q + A_n)\|^2 \\ &\leq (1 - b_n)\|Sx_n - q + A_n\|^2 + b_n\|T^n y_n - q + A_n\|^2 \\ &\quad - w_2(b_n)g(\|Sx_n - T^n y_n\|) \\ &\leq (1 - b_n)[\|Sx_n - q\|^2 + 2\langle A_n, j(Sx_n - q + A_n) \rangle] \\ &\quad + b_n\|T^n y_n - q\|^2 + 2\langle A_n, j(T^n y_n - q + A_n) \rangle \\ &\quad - w_2(b_n)g(\|Sx_n - T^n y_n\|) \\ &\leq (1 - b_n)(\|x_n - q\|^2 + 2r\|A_n\|) \\ &\quad + b_n(\|T^n y_n - q\|^2 + 2r\|A_n\|) \\ &\quad - b_n(1 - b_n)g(\|Sx_n - T^n y_n\|) \\ &= \|x_n - q\|^2 + b_n(\|T^n y_n - q\|^2 - \|y_n - q\|^2) \\ &\quad + b_n(\|y_n - q\|^2 - \|x_n - q\|^2) \\ &\quad + 2r\|A_n\| - b_n(1 - b_n)g(\|Sx_n - T^n y_n\|) \\ &\leq \|x_n - q\|^2 + b_n(k_n^2 - 1)\|y_n - q\|^2 + b_n b'_n(k_n^2 - 1)\|x_n - q\|^2 \\ &\quad + 2r b_n + 2r\|A_n\| - b_n(1 - b_n)g(\|Sx_n - T^n y_n\|)\|B_n\| \end{aligned}$$

for any  $j(Sx_n - q + A_n) \in J(Sx_n - q + A_n)$  and  $n \geq 1$ . Because  $\lim_{n \rightarrow \infty} k_n = 1$ , there exist a positive integer  $M$  such that  $k_n \leq 2$  for all  $n \geq M$ . Note that  $\{x_n - q\}_{n \geq 1}$  and  $\{y_n - q\}_{n \geq 1}$  belong to  $B(\theta, r)$ . Using (3.10) and (3.24), we get that

$$(3.25) \quad \begin{aligned} \|x_{n+1} - q\|^2 &\leq \|x_n - q\|^2 + 6r^2(k_n - 1) + 2r^2(c'_n + c_n) \\ &\quad - a(1 - b)g(\|Sx_n - T^n y_n\|) \end{aligned}$$

for any  $n \geq M$ . This means that

$$a(1 - b) \sum_{n=M}^{\infty} g(\|Sx_n - T^n y_n\|) \leq \|x_1 - q\|^2 + 6r^2 \sum_{n=M}^{\infty} (k_n - 1) + 2r^2 \sum_{n=M}^{\infty} (c'_n + c_n) < \infty$$

by (3.4)-(3.6) and (3.10). It follows that

$$(3.26) \quad \lim_{n \rightarrow \infty} g(\|Sx_n - T^n y_n\|) = 0.$$

Note that  $g : [0, \infty) \rightarrow [0, \infty)$  is continuous and strictly increasing with  $g(0) = 0$ . (3.26) yields that

$$(3.27) \quad \lim_{n \rightarrow \infty} \|Sx_n - T^n y_n\| = 0.$$

On account of (3.8) and (3.27), we know that

$$(3.28) \quad \lim_{n \rightarrow \infty} \|x_n - T^n y_n\| = 0.$$

It follows from (3.27) and (3.28) that

$$(3.29) \quad \lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0.$$

By virtue of (3.20) and (3.21), we deduce that for each  $n \geq 1$

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - T^n y_n\| + \|T^n y_n - y_n\| \\ &\leq \|x_n - T^n y_n\| + (1 - b'_n)\|Sx_n - T^n y_n\| + b'_n L \|x_n - y_n\| + \|B_n\|, \end{aligned}$$

which implies that

$$(3.30) \quad \|x_n - y_n\| \leq \frac{1}{1 - b'_n L} [\|x_n - T^n y_n\| + (1 - b'_n)\|Sx_n - T^n y_n\| + c'_n r], \quad \forall n \geq 1.$$

Thus (3.22) ensures that there exists a positive integer  $N$  such that  $b'_n < L^{-1}$  for each  $n \geq N$ . By (3.5), (3.10), (3.27), (3.28) and (3.30), we obtain that

$$\begin{aligned} \|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|x_n - T^n y_n\| \\ &\leq L \|x_n - y_n\| + \|x_n - T^n y_n\| \\ &\leq \frac{L}{1 - b'_n L} [\|x_n - T^n y_n\| + (1 - b'_n)\|Sx_n - T^n y_n\| \\ &\quad + c'_n r] + \|x_n - T^n y_n\| \\ &\rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . That is,

$$(3.31) \quad \lim_{n \rightarrow \infty} \|T^n x_n - x_n\| = 0.$$

In view of (3.4), (3.5), (3.10), (3.29), (3.31) and Lemma 3.4, we know that  $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ . This completes the proof.

**Remark 3.5.** Lemma 3.5 improves Lemma 4 in [2], Theorem 1 in [13] and Lemma 1.4 in [14].

**Theorem 3.2.** *Let  $E$  be a real uniformly convex Banach space,  $K$  be a nonempty bounded closed convex subset of  $E$ . Let  $S : K \rightarrow K$  be a nonexpansive mapping and  $T : K \rightarrow K$  be a uniformly  $L$ -Lipschitzian and asymptotically non-expansive mapping with a sequence  $\{k_n\}_{n \geq 1} \subseteq [1, \infty)$  satisfying  $\lim_{n \rightarrow \infty} k_n = 1$ ,  $F(S, T) \neq \emptyset$  and (3.8). Suppose that (3.4)-(3.6), (3.10) and (3.22) hold. If  $T^m$  is semi-compact for some positive  $m$ , then the modified Ishikawa iteration sequence with errors  $\{x_n\}_{n \geq 1}$  with respect to  $S$  and  $T$  given by (2.1) converges strongly to a common fixed point of  $S$  and  $T$ .*

*Proof.* Lemma 3.5 means that  $\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = \lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$ . Notice that

$$\|T^m x_n - x_n\| \leq \sum_{i=0}^{m-1} \|T^{i+1} x_n - T^i x_n\| \leq (1 + \sum_{i=1}^{m-1} k_i) \|Tx_n - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

that is,  $\lim_{n \rightarrow \infty} \|T^m x_n - x_n\| = 0$ . Since  $T^m$  is semi-compact, there exists a subsequence  $\{x_{n_i}\}_{i \geq 1} \subset \{x_n\}_{n \geq 1}$  such that  $x_{n_i} \rightarrow q \in K$  as  $i \rightarrow \infty$ . By the continuity of  $S$  and  $T$ , and Lemma 3.5, we conclude that

$$\begin{aligned} \|Sq - q\| &= \lim_{i \rightarrow \infty} \|Sx_{n_i} - x_{n_i}\| = \lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0 \\ &= \lim_{n \rightarrow \infty} \|Tx_n - x_n\| = \lim_{i \rightarrow \infty} \|Tx_{n_i} - x_{n_i}\| = \|Tq - q\|. \end{aligned}$$

That is,  $q$  is a common fixed point of  $S$  and  $T$  in  $K$ . From (3.25) we know that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + 6r^2(k_n - 1) + 2r^2(c'_n + c_n)$$

for all  $n \geq 1$ . Then (3.4)-(3.6), (3.10) and Lemma 2.5 guarantee that  $\lim_{n \rightarrow \infty} \|x_n - q\|^2 = 0$ . That is,  $\lim_{n \rightarrow \infty} x_n = q$ . This completes the proof.

**Remark 3.6.** Theorems 3.1 and 3.2 extend, improve and unify Theorems 1.1 and 1.2 in [2], Theorems 1 and 2 in [10], Theorems 2 and 3 in [11], Theorem 1.5 in [13] and Theorems 2.1 and 2.2 in [14] and in the following ways:

- (i) the identity mapping in [2], [10], [11], [13] and [14] is replaced by the more general nonexpansive mapping;
- (ii) the usual modified Mann iteration methods in [14], the usual modified Ishikawa iteration methods in [11] and [13] and the usual modified Ishikawa iterations methods with errors in [2] and [10] are extended to the modified Ishikawa iteration methods with errors with respect to a pair of mappings;

(iii) the conditions (3.8) and (3.10) are weaker than the condition

$$0 < \varepsilon \leq a_n \leq 1 - \varepsilon, \quad \forall n \geq 1$$

imposed on Theorems 1.1 and 1.2 in [2], Theorem 1 in [10], Theorems 2 and 3 in [11] and Theorem 1.5 in [13];

(iv) the Hilbert space in [13] is extended to the more general uniformly convex Banach space.

The following example shows that Theorem 3.2 extend substantially Theorem 1.2 in [2], Theorem 2 in [10], Theorem 3 in [11], Theorem 1.5 in [13] and Theorem 2.2 in [14].

**Example 3.1.** Let  $E$  be the real line with the usual norm  $|\cdot|$  and let  $K = [-1, 1]$ . Define  $S$  and  $T : K \rightarrow K$  by

$$Tx = \begin{cases} -2 \sin \frac{x}{2}, & x \in [0, 1], \\ 2 \sin \frac{x}{2}, & x \in [-1, 0) \end{cases} \quad \text{and} \quad Sx = \begin{cases} x, & x \in [0, 1], \\ -x, & x \in [-1, 0) \end{cases}.$$

Obviously  $F(S, T) = \{0\}$  and  $T$  is semi-compact. Now we check that  $T$  is nonexpansive. In fact, if  $x$  and  $y \in [0, 1]$  or  $x$  and  $y \in [-1, 0)$ , then

$$|Tx - Ty| = 2 \left| \sin \frac{x}{2} - \sin \frac{y}{2} \right| \leq |x - y|;$$

if  $x \in [0, 1]$  and  $y \in [-1, 0)$  or  $x \in [-1, 0)$  and  $y \in [0, 1]$ , then

$$|Tx - Ty| = 2 \left| \sin \frac{x}{2} + \sin \frac{y}{2} \right| = 4 \left| \sin \frac{x+y}{4} \cos \frac{x-y}{4} \right| \leq |x+y| \leq |x-y|.$$

That is,  $T$  is nonexpansive. It follows that  $T$  is uniformly 1-Lipschitzian and asymptotically nonexpansive with  $k_n \equiv 1$  for each  $n \geq 1$ . Similarly, we can verify that  $S$  is nonexpansive. In order to show that  $S$  and  $T$  satisfy (3.8), we have to consider the following cases:

**Case 1.** Suppose that  $x$  and  $y \in [0, 1]$ . It follows that

$$|x - Ty| = \left| x + 2 \sin \frac{y}{2} \right| = |Sx - Ty|;$$

**Case 2.** Suppose that  $x$  and  $y \in [-1, 0)$ . Then we easily see that

$$|x - Ty| = \left| x - 2 \sin \frac{y}{2} \right| \leq \left| -x - 2 \sin \frac{y}{2} \right| = |Sx - Ty|;$$

**Case 3.** Suppose that  $x \in [-1, 0)$  and  $y \in [0, 1]$ . It is easy to verify that

$$|x - Ty| = \left| x + 2 \sin \frac{y}{2} \right| \leq \left| -x + 2 \sin \frac{y}{2} \right| = |Sx - Ty|;$$

**Case 4.** Suppose that  $x \in [0, 1]$  and  $y \in [-1, 0)$ . It follows that

$$|x - Ty| = \left| x - 2 \sin \frac{y}{2} \right| = |Sx - Ty|.$$

Hence (3.8) is satisfied. Suppose that  $\{u_n\}_{n \geq 1}$ ,  $\{v_n\}_{n \geq 1}$  are given sequences in  $K$ ,

$$\begin{aligned} a &= \frac{3}{5}, \quad b = \frac{6}{7}, \quad a_n = \frac{2}{5} - \frac{1}{3n+2} - \frac{1}{6n^2}, \quad a'_{2n} = 1 - \frac{1}{3n} - \frac{1}{2n^2+3}, \\ a'_{2n-1} &= \frac{1}{2} - \frac{1}{3n+2} - \frac{1}{2n^2+3}, \quad b_n = \frac{3}{5} + \frac{1}{3n+2}, \quad b'_{2n} = \frac{1}{3n}, \\ b'_{2n-1} &= \frac{1}{3n+2} + \frac{1}{2}, \quad c_n = \frac{1}{6n^2} \quad \text{and} \quad c'_{2n} = c'_{2n-1} = \frac{1}{2n^2+3} \end{aligned}$$

for any  $n \geq 1$ . Thus the conditions of Theorem 3.2 are fulfilled. Therefore, the modified Ishikawa iteration sequences with errors  $\{x_n\}_{n \geq 1}$  with respect to  $S$  and  $T$  defined by (2.1) converges strongly to 0. But the results in [2] and [10], [11], [13] and [14] are not applicable because  $S \neq I$ .

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