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THE LIFE CONTRIBUTION OF KOSTIA BEIDAR IN RING AND NEARRING THEORY

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Abstract. Kostia Beidar published more than 120 research papers and solved many well-known problems. Our goal is to mention just some of his brilliant results in ring and nearring theory, and also a brief history of his life.

1. Research Contributions

Kostia Beidar obtained many nice results in different areas of mathematics. When he was a student of Moscow State University he proved that a generalized polynomial identity on a semiprime ring can be lifted to a maximal right ring of quotients, a very important result in the theory of generalized identities! Most of Kostia's results are connected with ring and nearring theory, but one can also mention his contribution to Hopf algebras, Jordan and Lie algebras, linear algebra and mathematical physics.

In this article we shall mention just 5 areas of Kostia's research. It is very difficult to imagine these areas without Kostia's brilliant results.

1.1. Radical Theory

It follows from the Hilbert Nullstellensatz that the Jacobson radical of a finitely generated commutative algebra over a field is nilpotent. It was proved by Kemer that a similar result holds for PI algebras when the field is of characteristic zero while Braun removed this restriction on the characteristic of the underlying field. Moreover, a theorem of Amitsur asserts that Jacobson radical of a finitely generated algebra over an uncountable field is nil. In 1956 Amitsur posed the problem whether generally the Jacobson radical of a finitely generated algebra is nil [2].

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One of the most famous open problems in Ring Theory, known as Koethe's conjecture, states that the sum of two nil left ideals (of a ring) is nil. A positive answer to Amitsur's question would lead to a positive solution of Koethe's conjecture, so Amitsur's problem was very attractive for many algebraists.

In 1981 Beidar [3] constructed an example which gave a negative answer to Amitsur's question. It was not just a solution of an old open problem, it was also a very good approximation of Koethe's conjecture "from below."

Koethe's conjecture has a positive solution if and only if for each nil ring R the ring of polynomials $R[x]$ is Jacobson radical. Recently Smoktunowicz constructed an example of a nil ring R such that $R[x]$ is not nil, which is another approximation of Koethe's conjecture "from below." In 1998 Puczyłowski and Smoktunowicz showed that for every nil ring R the ring $R[x]$ is Brown-McCoy radical, i.e. it cannot be homomorphically mapped onto a ring with identity. This is an approximation of Koethe's conjecture "from above." If $R[x]$ could be mapped onto a ring with a nonzero idempotent for some nil ring R, then this ring R could serve as a negative solution of Koethe's conjecture. Recently Beidar, Fong and Puczyłowski [15] showed that this is impossible, namely, for every nil ring R the ring $R[x]$ is Behrens radical. This result answers a question posed by Puczyłowski and Smoktunowicz and up to our knowledge it is the best known approximation of Koethe's conjecture "from above."

The reader who is interested in Koethe's conjecture is referred to the survey paper by Puczyłowski [27].

1.2. The Method of Orthogonal Completions

A ring R is called *prime* if for any nonzero $a, b \in R$ there exists an $r \in R$ such that $arb \neq 0$. A ring S is called *semiprime* if for any $a \in S$ there exists $s \in S$ such that $asa \neq 0$.

As one can see from these definitions the classes of prime and semiprime are very wide. We just mention here for example that standard operator algebras (and in particular the algebra $B(X)$ of all bounded linear operators on a Banach space X) are prime and C^*-algebras are semiprime.

In the study of semiprime rings it turns out to be useful whether one can reduce the problem to the case of prime rings (where the solution is already known). However if one will try to factor a semiprime ring by a prime ideal (which is the simplest way of how to get a prime ring from a semiprime ring) this idea may fail in many situations. For example, it is well-known that every polynomial identity on a prime ring R is an identity on its maximal right ring of quotients $Q(R)$. It is a natural question to prove the same result for a semiprime ring S. However the idea of direct reduction does not work here since there is no homomorphism $Q(S) \to Q(S/I)$ for a prime ideal I of S.

The method of orthogonal completions developed by Beidar and Mikhalev [19, 20] shows that many difficulties of reducing the "semiprime case" to the "prime case" can be successfully overcome in the study of semiprime rings with (generalized) polynomial identities, involutions, and derivations.

The method of orthogonal completions can be found in Chapter 3 of the book by Beidar, Martindale and Mikhalev [18].

1.3. Functional Identities

A functional identity on a ring R is, roughly speaking, an equation on R with functions involved. For instance, in a ring extension Q of R with center C, the following is an example of a functional identity:

$$f_1(x)y + f_2(y)x + xg_1(y) + yg_2(x) = 0, \tag{1}$$

where f_1, f_2, g_1 and g_2 are maps from R into Q. A goal is to find possible solutions f_1, f_2, g_1, and g_2 such that (1) holds for all $x, y \in R$. Natural solutions are found when f_1, f_2, g_1 and g_2 are of the forms

$$\begin{aligned} f_1(x) &= xp + \mu(x), \quad f_2(y) = yq + \nu(y), \\ g_1(y) &= -py - \nu(y), \quad g_2(x) = -qx - \mu(x), \end{aligned} \tag{2}$$

where $p, q \in Q$, and $\mu : R \to C$ and $\nu : R \to C$. Note that in this case the functions f_i and g_i, $i = 1, 2$, do not depend on the structure of R, and so (2) are often referred to as the *standard solutions* of the identity (1).

In 1995 Brešar proved that for prime rings standard solutions are the only possible solutions of (1).

In 1998 Beidar [4] considered functional identities of the form

$$\sum_{i=1}^{r} E_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r)x_i + \sum_{j=1}^{r} x_j F_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_r) = 0$$

on R, where E_i and F_j ($i, j \in \{1, \ldots, r\}$) are functions from R^{r-1} to Q, and showed that they have only standard solutions on a prime ring R, provided that R contains a transcendental element or an algebraic element of degree $\geq r$.

This result can be considered as a break-through in the study of functional identities. It is not surprising that Beidar's paper [4] was recognized by Thomson ISI as a fast-breaking paper in the field of Mathematics. In [16] Beidar and Martindale considered functional identities in prime rings with involution. In [8, 9] Beidar and Chebotar developed the theory of functional identities on sets of arbitrary rings. Papers [3, 9, 9, 16] are "building blocks" of the new theory known as the theory of functional identities.

The reader is referred to the survey papers [21, 22] for acquaintance with the theory of functional identities and its applications.

1.4. Herstein's Lie Map Conjectures

In his 1961 AMS hour talk Herstein posed several problems he deemed worthy of attention. Problem 1 was about Jordan and n-Jordan maps. Here we shall quote [24, p. 528] starting from Problem 2.

> *2. The very important and interesting problem of finding all the Lie mappings of simple (or, perhaps, even prime) rings. That is, characterize all additive mappings of a simple ring R into R' such that $\phi(ab - ba) = \phi(a)\phi(b) - \phi(b)\phi(a)$. In the case of matrices these are, roughly speaking, of the form $\psi + \tau$ where ψ is either an automorphism or the negative of an anti-automorphism and where τ is a trace-like mapping into the scalars.*
>
> *We would conjecture that a similar result holds for general simple rings. In this full generality little progress has been made. However, as a test case, in characteristic 2, in joint paper with Kleinfeld we proved that a Lie mapping of a simple ring which preserves cubes must be automorphism or an anti-isomorphism.*
>
> *3. The analogous question to that in Problem 2 for Lie derivations, that is, mappings d such that $d[a, b] = [d(a), b] + [a, d(b)]$ would be of interest to investigate. In an unpublished work Kaplansky has shown that if the ring has $n \times n$ matrix units, with $n \geq 3$, then any Lie derivation is an ordinary one plus an additive map into the center.*
>
> *4. Problems 2 and 3 in the setting of the simple or almost-simple Lie rings $[R, R]$ and $[R, R]/Z \cap [R, R]$ are worthy of serious attention.*
>
> *5. To settle the very difficult questions above for the skew-symmetric elements, K, of a single ring and their associated simple or almost-simple Lie rings $[K, K]$ and $[K, K]/Z \cap [K, K]$ offers a real challenge. The nature of the 8×8 matrices over fields under transpose promises interesting side difficulties.*

The description of Lie automorphisms of matrix rings over fields has been known for a long time. For example, it was mentioned in Jacobson's book "Lie algebras" (Interscience, New York, 1962). In 1951 Hua Lo-Keng described Lie automorphisms of a matrix rings over skew fields, so Herstein already had some confirmation for his conjectures when he posed them.

Since 1963 Martindale and some of his students obtained solutions of Herstein's problems provided that the rings contain some nontrivial idempotents. Lie map problems have been also considered in certain operator algebras in papers by Miers and the techniques there also rest heavily on the presence of idempotents.

Herstein's problems were completely solved by means of the theory of functional identities. In 1993 Bresar described Lie isomorphisms and Lie derivations of prime rings. In 1994 Beidar, Martindale and Mikhalev [17] described Lie isomorphisms of K. Swain, a student of Martindale, used the description of commuting triadditive maps due to Beidar, Martindale and Mikhalev, to characterize Lie derivations of K. Lie isomorphisms and Lie derivations of $[R,R]$ and $[R,R]/Z \cap [R,R]$ were described by Beidar and Chebotar in [10] and [11]. Finally Lie maps of $[K,K]$ and $[K,K]/Z \cap [K,K]$ were characterized by Beidar, Brešar, Chebotar and Martindale in [5-7].

The paper [7] is the last paper in the series on Herstein's Lie map conjectures. It was also acknowledged by Thomson ISI as a fast-breaking paper in the field of Mathematics in October, 2003.

1.5. Nearring Theory

A nearring as a set N together with two operations "$+$" and "$\cdot$" such that (i) $(N,+)$ is a not necessarily abelian group, (ii) $(N,\cdot)$ is a semigroup, and (iii) $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in N$. Roughly speaking, one can realize that a nearring satisfies all the axioms of a ring except possibly two: commutativity of addition and one of the two distributive laws. The prototype example of an honest nearring can be obtained by taking a group $(G,+)$, not necessarily abelian, and forming $M(G) = \{f \mid f : G \to G\}$, the set of all mappings from G to G itself. Under pointwise addition and composition of mappings, $M(G)$ is a nearring whenever the order of the group G is greater than one.

Nearrings have nice applications in coding theory, cryptography and combinatorics. Let us mentioned just a few results of Kostia Beidar in the nearring theory.

In the paper [12], Beidar, Fong and Ke determined that when a finite Ferrero pair (N,Φ) is circular, then Φ is a metacyclic group. Along the line, they solved the problems posed by Clay [23] on the existence of finite circular planar nearrings determined by Ferrero pairs (N,Φ) such that either N is nonabelian, or Φ-is nonabelian. Since there exists a correspondence between Ferrero pairs and Frobenius groups, these results can be stated in group theoretic terms.

In [13] Beidar, Fong and Ke solved an old problem by Meldrum-Zeller on regularity of simple centralizer nearrings. They proved that if the automorphism group is nilpotent, then the question has a positive answer. On the other hand they also showed that there exists a counterexample in the class of solvable groups.

The maximal right ring of quotients is an important tool in studying semiprime rings. In the nearring case, the classical right nearring of quotients was studied for some time which is an analogue to the classical right ring of quotients. However, even for 3-semiprime nearrings, the classical right nearrings of quotients may not exist. In [14], Beidar, Fong and Ke established the maximal right nearrings of quotients for 3-semiprime nearrings, which always exist and have nice properties.

Finally, with joint efforts, it is also shown that there is no nontrivial derivations on the nearring $M_0(G) = \{f \in M_G \mid f(0) = 0\}$ where G is an additively written group.

The reader is referred to the books by Clay [23], Meldrum [25] or Pilz [26] for terminology, definitions and basic facts of nearrings.

2. A Brief History

Personal Data

Date of Birth: October 21, 1951, Vladikavkaz, USSR.
Citizenship: Russian.
Family: wife Lidia, daughters Ekaterina (date of birth: July 15, 1986) and Tatiana (date of birth: May 17 1989).

Degrees:

- D. Sci., Leningrad State University, Leningrad, USSR, 1991.
- Ph. D., Moscow State University, Moscow, USSR, 1978.
- M. A., Moscow State University, Moscow, USSR, 1974.

Positions Held:

- National Cheng-Kung University, Tainan, Taiwan; Distinguished Professor, 2002–2004.
- Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan; Full Professor, 1995–2004.
- Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan; Associative Professor, 1994–1995.
- Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan; Visiting Scholar, 1993–1994.
- Department of Mechanics and Mathematics, Moscow State University, Moscow; Associate Professor, 1990–1994.
- Research Institute of Automatization of A Control in Nonindustrial Branches, Moscow; Senior Researcher, 1987–1990.
- Research Institute of Economy of A Construction, Moscow; Senior Researcher, 1981–1987.
- North-Caucasian Mountain’s-Metallurgical Institute, Ordjonikidze, Assistant Professor, 1977–1981.

Ph. D. students:

M. A. Chebotar (jointly with A. V. Mikhalev), W.-R. Wu (jointly with W.-F. Ke), T.-S. Chen (jointly with W.-F. Ke).

In 1991 Kostia met Yuen Fong at Szekszard's Conference on Radicals in Hungary. Due to Richard Wiegandt's extremely strong recommendation, in the fall of 1993 Kostia was invited to Tainan for a one-year visiting position. Kostia came with his family. That was a wonderful and fruitful year for both the host and the visitors. He subsequently obtained a regular position in the National Cheng Kung University and maintained his residence in Tainan for the rest of his life.

During his stay in Tainan, Kostia wrote approximately 80 papers and completed the book "Rings with Generalized Identities" jointly written with Martindale and Mikhalev. Throughout his life, Kostia made efforts to seek and foster collaborations with other mathematicians. His presence in NCKU made Tainan one of the research centers in ring and nearring theories, and attracted many mathematicians all over the world.

During his life Kostia published more than 120 papers, gave more than 70 conference and colloquium lectures and had more than 15 visiting positions in the universities around the world. As an indication of Kostia's leadership role in research we wish to include here a passage from the letter of acceptance by American Mathematical Society on May 19, 1996 written by the 1994 Fields Medalist Professor Efim Zelmanov:

> *I am very happy to accept your paper "On Frobenius algebras and quantum Yang-Baxter equation" for publication in Transactions of American Mathematical Society.*
>
> *I find your construction of solutions of the quantum Yang-Baxter equation by associative Frobenius algebra* **important**, **interesting** *and* **exciting**. *I am sure it will have* **far reaching consequences** *both for the study of QYBE and for the theory of Frobenius algebras. Please accept my congratulations and thank you for submitting to our Journal.*

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