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APPROXIMATION TO OPTIMAL STOPPING RULES FOR GUMBEL RANDOM VARIABLES WITH UNKNOWN LOCATION AND SCALE PARAMETERS

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Abstract. An optimal stopping rule is a rule that stops the sampling process at a sample size n that maximizes the expected reward. In this paper we will study the approximation to optimal stopping rule for Gumbel random variables, because the Gumbel-type distribution is the most commonly referred to in discussions of extreme values. Let $X_1, X_2, \cdots X_n, \cdots$ be independent, identically distributed Gumbel random variables with unknown location and scale parameters,α and β. If we define the reward sequence $Y_n = \max\{X_1, X_2, \cdots, X_n\} - cn$ for $c > 0$, the optimal stopping rule for Y_n depends on the unknown location and scale parameters α and β. We propose an adaptive stopping rule that does not depend on the unknown location and scale parameters and show that the difference between the optimal expected reward and the expected reward using the proposed adaptive stopping rule vanishes as c goes to zero. Also, we use simulation in statistics to verify the results.

1. INTRODUCTION

Extreme value statistics is the study of rare events that lie beyond common experience. The main contributions for extreme value theory are a set of limiting results which enable one to analyze unusual events. It can be applied to extremes in many fields, including nature, engineering, sport and economics. Accurate assessments of the probabilities of extreme events are sought in a diversity of applications from environmental impact assessment ([6, 9, 10, 19]) to financial risk management ([8, 14, 22]) and Internet traffic modeling ([16, 18]). The well-established branch of statistics has been employed in insurance problems for many years, but has only recently been applied in risk management settings. Its proponents argue that the tools

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provide many supplement or even substitute for the industry-standard approach to risk measurement.

The Gumbel-type distribution is the most commonly referred to in discussions of extreme values. The purpose of this paper is to find the approximation to optimal stopping rule for Gumbel random variables with unknown location and scale parameters, α and β in the hope to maximize the expected reward in the sampling process.

An optimal stopping rule is a rule that stops the sampling process at a sample size n that maximizes the expected reward. Let $X_1, X_2, \cdots, X_n, \cdots$ be independent, identically distributed Gumbel random variables with unknown location and scale parameters, α and β. The X_i is observed sequentially and we are allowed to stop observing at any stage. If we stop at the nth observation then we will receive a reward Y_n, where Y_n is a measurable function of $X_1, X_2, \cdots, X_n$. Optimal stopping rule depends on the distribution of the X_i which has the consequence that determination of an optimal stopping rule requires complete knowledge of the underlying distribution for the data. If only partial information is available, e.g. some parameter values are unknown, then it becomes necessary to use an adaptive stopping rule to approximate the optimal rule.

In this paper, we assume that the X_i is an independent Gumbel random variable with common probability density function

$$g(x;\alpha,\beta) = \frac{1}{\beta}\exp(-\frac{x-\alpha}{\beta})\exp(-\exp(-\frac{x-\alpha}{\beta})),\ -\infty < x < \infty, \alpha \in R, \beta > 0$$

where β and α, respectively, scale and location parameters. Let $\max\{X_1, X_2, \cdots, X_n\}$ be the reward for the first n trials and let $c > 0$ be the cost for each trial. Then we will consider reward or net gain functions of the form $Y_n = \max\{X_1, X_2, \cdots, X_n\} - cn$. Such reward function arises in the context of sampling with recall. Discussion of their motivations and utility can be found in [5] or [7].

The problem of finding an adaptive stopping rule to approximate stopping rule has been studied by [1] that proved that in certain cases involving unknown location parameters, the ratio of the expected reward under an adaptive stopping rule to the optimal expected reward will approach one as c goes to zero. [15] assumed that X_i is exponential distributed random variable with unknown mean. [21] considered the case where the X_i has common density function $(\alpha-1)x^{-\alpha}I_{[1,\infty]}$ with unknown α , where $I_A(\bullet)$ denotes the indicator function for the set A. [12] considered exponential distributed random variables with unknown location and scale parameters. Under the distribution discussed by [12, 15, 21] the optimal stopping rules have closed forms. [20] considered the case where the X_i is normal with unknown mean and [11] generalized [20]'s results to include the case where both the mean and variance are unknown. [13] treated the situation when the X_i is Gamma distribution with unknown scale parameter, while [2] generalized [13]'s results to include the case where both the location and scale parameters are unknown. In the situations of [2, 11, 13, 20] the optimal stopping rules no longer have a closed form and adaptive stopping rules were used to approximate the optimal stopping rules.

In this paper, we define the optimal stopping rule as

$$\tau_c^* = \inf\{n \geq 1 : X_n \geq \gamma_c\} \tag{1}$$

where γ_c satisfies $\boldsymbol{E}(X_1 - \gamma_c)^+ = c$, and $(X_1 - \gamma_c)^+ = \max\{X_1 - \gamma_c, 0\}$. The stopping rule τ_c^* , maximizes $\boldsymbol{E}(Y_\tau)$ over all stopping rules τ with $\boldsymbol{E}(Y_\tau^-) < \infty$ where $Y_\tau^- = \min\{Y_\tau, 0\}$ and the expected reward is $\boldsymbol{E}(Y_{\tau_c^*}) = \boldsymbol{E}(X_{\tau_c^*}) - c\boldsymbol{E}(\tau_c^*) = \gamma_c$. For more details see [4, p. 56-58].

However, in order to use the optimal stopping rule τ_c^* it is necessary to know γ_c, which in turn requires knowledge of distribution of X_i. If only partial information about the distribution is available, it would be desirable to find an adaptive stopping rule to approximate the optimal rule τ_c^* and the optimal reward $\boldsymbol{E}(Y_{\tau_c^*})$ as well. Throughout the rest of this paper we assume that the X_i is independent Gumbel random variable with common probability density function

$$g(x;\alpha,\beta) = \frac{1}{\beta}\exp(-\frac{x-\alpha}{\beta})\exp(-\exp(-\frac{x-\alpha}{\beta})), \quad -\infty < x < \infty, \alpha \in R, \beta > 0$$

We define the function $\boldsymbol{E}(X_1 - \alpha - x)^+ = f(x,\beta)$, and we can obtain

$$\begin{aligned} f(x,\beta) &= \int_{\alpha+x}^{\infty}(y-\alpha-x)\frac{1}{\beta}\exp(-\frac{y-\alpha}{\beta})\exp(-\exp(-\frac{y-\alpha}{\beta}))dy \\ &= \int_0^{e^{-\frac{x}{\beta}}}(\alpha-\beta\ln z)\exp(-z)dz - (\alpha+x)(1-\exp(-\exp(-\frac{x}{\beta})))). \end{aligned} \tag{2}$$

Let γ_c satisfy

$$f(\gamma_c - \alpha, \beta) = c. \tag{3}$$

In this case the optimal stopping rule τ_c^* will depend on the unknown parameters,α and β. Therefore, while α and β are replaced by its estimator $\widehat{\alpha}_n$ and $\widehat{\beta}_n$, we obtain an adaptive stopping rule $\widehat{\tau}_c$ which is

$$\widehat{\tau}_c = \inf\{n \geq n_c : X_n \geq \widehat{\gamma}_{c,n}\}, \tag{4}$$

where $\widehat{\gamma}_{c,n}$ satisfies

$$f(\widehat{\gamma}_{c,n} - \widehat{\alpha}_n, \widehat{\beta}_n) = c. \tag{5}$$

Where

$$\widehat{\beta}_n = \sqrt{\frac{6}{\pi^2}\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}, \widehat{\alpha}_n = \overline{X}_n - \widehat{\beta}_n d$$

and n_c is a function of c. d is a constant, $d = -\int_0^\infty \ln z \exp(-z)dz \doteq 0.577216$.

The purpose of this paper is to find an adaptive stopping rule in the case of sequential observed Gumbel random variables with unknown location and scale parameters. Using a proposed adaptive stopping rule we prove that the difference between the optimal expected reward and the expected reward using the proposed adaptive stopping rule vanishes as c goes to zero. In the next section, we give some preliminary Lemmas which are useful in studying the behaviors of γ_c and τ_c^*. In Section 3, we study the performance of $\widehat{\tau}_c$ and show that if $n_c = \delta c^{-\theta}$ for some $\delta > 0$ and $0 < \theta < 1$,

$$\boldsymbol{E}(Y_{\tau_c^*}) - \boldsymbol{E}(Y_{\widehat{\tau}_c}) \longrightarrow 0 \quad \text{as } c \longrightarrow 0.$$

In this paper, different from the previous studies [2] and [13], we especially, in Section 4, use simulation in statistics to verify if the results from our simulation are in accord with the theorem. In addition, Shu, W.Y. [21] conducted a simulation study on approximation to optimal stopping rules with heavy tail when α is unknown. Our study is more complicated because two parameters α and β are considered, and the optimal stopping rules under Gumbel distribution do not have a closed form; that is, γ_c cannot be expressed in explicit form of c, α and β.

2. Primary Results

First we state some properties of $f(x, \beta)$ which will be needed later.

Lemma 2.1. *For fixed β, $f(x, \beta)$ is a strictly decreasing function in x.*

Proof.

$$\begin{aligned}\frac{\partial f}{\partial x} &= -\frac{\alpha + x}{\beta}\exp(-\frac{x}{\beta})\exp(-\exp(-\frac{x}{\beta})) \\ &\quad -(1 - \exp(-\exp(-\frac{x}{\beta}))) + \frac{\alpha + x}{\beta}\exp(-\frac{x}{\beta})\exp(-\exp(-\frac{x}{\beta})) \\ &= \exp(-\exp(-\frac{x}{\beta})) - 1 < 0\end{aligned}$$

We have proved that $f(x, \beta)$ is a strictly decreasing function in x.

Lemma 2.2. *For fixed x, $f(x,\beta)$ is a strictly increasing function in β .*

Proof.

$$\begin{aligned}\frac{\partial f}{\partial \beta} &= \frac{x(x+\alpha)}{\beta^2}\exp(-\frac{x}{\beta})\exp(-\exp(-\frac{x}{\beta})) \\ &\quad -\int_0^{e^{-\frac{x}{\beta}}} \ln z \exp(-z) dz. \\ &\quad -\frac{x(x+\alpha)}{\beta^2}\exp(-\frac{x}{\beta})\exp(-\exp(-\frac{x}{\beta})) \\ &= -\int_0^{e^{-\frac{x}{\beta}}} \ln z \exp(-z) dz > 0.\end{aligned}$$

We have proved that $f(x,\beta)$ is a strictly increasing function in β .

Using Lemma 2.1 and Lemma 2.2, it is easy to obtain Lemma 2.3

Lemma 2.3. *If $0 < \beta_1 < \beta_2$, and $f(x,\beta_1) = f(y,\beta_2)$ then $y > x$.*

Let γ_c satisfy $f(\gamma_c - \alpha, \beta) = c$, this implies $\boldsymbol{E}(X_1 - \gamma_c)^+ = c$ in this case. For fixed β, by lemma 2.2, we have γ_c which is a decreasing function of c.

Lemma 2.4. *For any $b > 0$, we can get $\gamma_c = o(c^{-b})$, as $c \longrightarrow 0$.*

Proof.

$$\begin{aligned}c = f(\gamma_c - \alpha, \beta) &= \int_{\gamma_c}^{\infty} (y-\gamma_c)\frac{1}{\beta}\exp(-\frac{y-\alpha}{\beta})\exp(-\exp(-\frac{y-\alpha}{\beta}))dy \\ &\le \int_{\gamma_c}^{\infty} (y-\gamma_c)\frac{1}{\beta}\exp(-\frac{y-\alpha}{\beta})dy \\ &= \beta\exp(-\frac{\gamma_c-\alpha}{\beta}) \\ &\Rightarrow c \le \beta\exp(-\frac{\gamma_c-\alpha}{\beta}) \\ &\Rightarrow \ln c \le \ln\beta - \frac{\gamma_c-\alpha}{\beta} \\ &\Rightarrow \gamma_c \le \alpha - \beta(\ln c - \ln\beta).\end{aligned}$$

For any $b > 0$, we can get $\gamma_c = o(c^{-b})$, as $c \longrightarrow 0$.

Therefore, we can choose c_0, which is small enough such that for all $c \in (0, c_0)$, $\gamma_c - \alpha > 0$,and obtain Lemma 2.5.

Lemma 2.5. *For $0 < c < c_0$, we have $\boldsymbol{P}(X_1 \geq \gamma_c)e\beta \geq c$*

Proof. From the equality as the following

$$\boldsymbol{P}(X_1 \geq \gamma_c) = \int_{\gamma_c}^{\infty} \frac{1}{\beta}\exp(-\frac{x-\alpha}{\beta})\exp(-\exp(-\frac{x-\alpha}{\beta}))dx$$

From this integral, we know $x \geq \gamma_c$ and $-\frac{x-\alpha}{\beta} \leq -\frac{\gamma_c-\alpha}{\beta} < 0$. Therefore, we obtain $\exp(-\frac{x-\alpha}{\beta}) < 1$ and $\exp(-\exp(-\frac{x-\alpha}{\beta})) > 1/e$. Hence

$$\begin{aligned}\boldsymbol{P}(X_1 \geq \gamma_c) &\geq \frac{1}{e}\exp(-\frac{\gamma_c-\alpha}{\beta}) \\ &\geq \frac{c}{e\beta}.\end{aligned}$$

Therefore, we obtain $\boldsymbol{P}(X_1 \geq \gamma_c)e\beta \geq c$.

Lemma 2.6. *Let τ_c^* be as defined in (1) . Then $\{(c\tau_c^*)^p : 0 \leq c \leq c_0\}$ is uniformly integrable for all $p > 0$.*

Proof. Since τ_c^* is a geometric random variable, we have $c\boldsymbol{E}(\tau_c^*) = c[\boldsymbol{P}(X_1 \geq \gamma_c)]^{-1}$. Using Lemma 2.5 for all $c \in (0, c_0)$, we obtain $\sup_{0\leq c\leq c_0} c\boldsymbol{E}(\tau_c^*) \leq e\beta$. This implies
$\sup_{0\leq c\leq c_0} \boldsymbol{E}(c\tau_c^*)^p \leq M_p(e\beta)^p$, where M_p only depends on p.

3. Performance of $\widehat{\tau}_c$

For the rest of this section, we define $d = -\int_0^\infty \ln z \exp(-z)dz \doteq 0.577216$. Unlike τ_c^*, the adaptive stopping rule $\widehat{\tau}_c$ defined by (4) and (5) is not a geometric random variable. The key to study the behavior of $\widehat{\tau}_c$ is to approximate $\widehat{\tau}_c$ by $\tau_{c,b}^+$ and $\tau_{c,b}^-$ which are defined as follows:

$$\tau_{c,b}^+ = \inf\{n \geq 1 : X_n \geq \gamma_{c,b}^+\} \tag{6}$$

and

$$\tau_{c,b}^- = \inf\{n \geq 1 : X_n \geq \gamma_{c,b}^-\} \tag{7}$$

where $\gamma_{c,b}^+$ and $\gamma_{c,b}^-$ satisfy

$$f(\gamma_{c,b}^+ - \alpha - 24c^b\beta d^3/\pi^2, (1 + 12c^b d^2/\pi^2)\beta) = c, \tag{8}$$

and

$$f(\gamma_{c,b}^{-} - \alpha + 24c^b\beta d^3/\pi^2, (1 - 12c^b d^2/\pi^2)\beta) = c, \tag{9}$$

respectively. By lemma 2.3, we have $\gamma_{c,b}^{-} < \gamma_c < \gamma_{c,b}^{+}$. For fixed positive β, the function $f(x, \beta)$ is a function of x only and denotes $f(x, \beta) = h(x)$. From (2),(3),(8) and (9) it is easy to obtain lemma 3.1.

Lemma 3.1. *For fixed positive β, and $\beta > 0$, we have*

(*a*) $\gamma_c - \alpha = h^{-1}(c)$;

(*b*) $\gamma_{c,b}^{+} - \alpha = (1 + 12c^b d^2/\pi^2)h^{-1}(\frac{c}{1+12c^b d^2/\pi^2}) + 24c^b\beta d^3/\pi^2$;

(*c*) $\gamma_{c,b}^{-} - \alpha = (1 - 12c^b d^2/\pi^2)h^{-1}(\frac{c}{1-12c^b d^2/\pi^2}) - 24c^b\beta d^3/\pi^2$.

Proof. From (2) , we have $c = f(\gamma_c - \alpha, \beta) = \beta \int_{\frac{\gamma_c - \alpha}{\beta}}^{\infty} z \exp(-z) \exp(-\exp(-z))dz$ $- (\gamma_c - \alpha) \int_{\frac{\gamma_c - \alpha}{\beta}}^{\infty} \exp(-z) \exp(-\exp(-z))dz \equiv h(\gamma_c - \alpha)$.
For fixed β,

$$f(\gamma_c - \alpha, \beta) = h(\gamma_c - \alpha) = c,$$

hence $\gamma_c = h^{-1}(c) + \alpha$.

For (b), from (2) and (8), we have

$$\begin{aligned}
& f(\gamma_{c,b}^{+} - \alpha - 24c^b\beta d^3/\pi^2, (1 + 12c^b d^2/\pi^2)\beta) \\
= \ & (1 + 12c^b d^2/\pi^2)\beta \int_{\frac{\gamma_{c,b}^{+} - \alpha - 24c^b\beta d^3/\pi^2}{(1+12c^b d^2/\pi^2)\beta}}^{\infty} z \exp(-z) \exp(-\exp(-z))dz \\
& -(\gamma_{c,b}^{+} - \alpha - 24c^b\beta d^3/\pi^2) \int_{\frac{\gamma_{c,b}^{+} - \alpha - 24c^b\beta d^3/\pi^2}{(1+12c^b d^2/\pi^2)\beta}}^{\infty} \exp(-z) \exp(-\exp(-z))dz \\
= \ & (1 + 12c^b d^2/\pi^2)h(\frac{\gamma_{c,b}^{+} - \alpha - 24c^b\beta d^3/\pi^2}{1 + 12c^b d^2/\pi^2}) = c.
\end{aligned}$$

We obtain $\gamma_{c,b}^{+} - \alpha = (1 + 12c^b d^2/\pi^2)h^{-1}(\frac{c}{1+12c^b d^2/\pi^2}) + 24c^b\beta d^3/\pi^2$. Similarly we can obtain
$\gamma_{c,b}^{-} - \alpha = (1 - 12c^b d^2/\pi^2)h^{-1}(\frac{c}{1-12c^b d^2/\pi^2}) - 24c^b\beta d^3/\pi^2$.

Lemma 3.2. *For any $b > 0$, $0 \le \gamma_{c,b}^{+} - \gamma_{c,b}^{-} = o(c^{b/4})$ as $c \longrightarrow 0$.*

Proof. Since

$$h^{-1}(\frac{c}{1 + 12c^b d^2/\pi^2}) \ge h^{-1}(\frac{c}{1 - 12c^b d^2/\pi^2}),$$

and by Lemma 3.1, we have

$$
\begin{aligned}
& 0 \le \gamma_{c,b}^{+} - \gamma_{c,b}^{-} - 48c^b\beta d^3/\pi^2 \\
= \; & h^{-1}(\frac{c}{1+12c^bd^2/\pi^2}) - h^{-1}(\frac{c}{1-12c^bd^2/\pi^2}) \\
& + \frac{12c^bd^2}{\pi^2}\left[h^{-1}(\frac{c}{1+12c^bd^2/\pi^2}) + h^{-1}(\frac{c}{1-12c^bd^2/\pi^2})\right] \\
\le \; & h^{-1}(\frac{c}{1+12c^bd^2/\pi^2}) - h^{-1}(\frac{c}{1-12c^bd^2/\pi^2}) \\
& + \frac{24c^bd^2}{\pi^2}h^{-1}(\frac{c}{1+12c^bd^2/\pi^2}).
\end{aligned}
$$

Using the Mean-Value theorem, we get

$$
\begin{aligned}
& h^{-1}(\frac{c}{1+12c^bd^2/\pi^2}) - h^{-1}(\frac{c}{1-12c^bd^2/\pi^2}) \\
= \; & (h^{-1})^{'}(cx^*)\left(\frac{-24c^{b+1}d^2/\pi^2}{1-144c^{2b}d^4/\pi^4}\right) \\
= \; & \left(\frac{24c^{b+1}d^2/\pi^2}{1-144c^{2b}d^4/\pi^4}\right)\left\{1-\exp(-\exp\left(-\frac{h^{-1}(cx^*)}{\beta}\right))\right\}^{-1},
\end{aligned}
$$

where $x^* \in (\frac{1}{1+12c^bd^2/\pi^2}, \frac{1}{1-12c^bd^2/\pi^2})$ and $(h^{-1})^{'}$ is the first derivative of h^{-1}. Using

$$
h^{-1}(cx^*) \le h^{-1}(\frac{c}{1+12c^bd^2/\pi^2}) = \frac{\gamma_{c,b}^{+} - \alpha - 24c^b\beta d^3/\pi^2}{1+12c^bd^2/\pi^2},
$$

and letting $c^{'} = \frac{c}{1+12c^bd^2/\pi^2}$ in Lemma 3.1, we have $\gamma_{c^{'}} - \alpha = h^{-1}(c^{'}) = \frac{\gamma_{c,b}^{+}-\alpha-24c^b\beta d^3/\pi^2}{1+12c^bd^2/\pi^2}$. Replacing c by $c^{'}$ in Lemma 2.5 and using Lemma 2.4, we get

$$
\begin{aligned}
\gamma_{c,b}^{+} - \gamma_{c,b}^{-} \; & \le \; \frac{24c^{1+b}d^2/\pi^2}{\boldsymbol{P}(X_1 \ge \gamma_{c^{'}})(1-144c^{2b}d^4/\pi^4)} + 24c^bd^2(\gamma_{c^{'}}-\alpha)/\pi^2 + 48c^b\beta d^3/\pi^2 \\
& = \; \frac{24c^bd^2/\pi^2c^{'}}{\boldsymbol{P}(X_1 \ge \gamma_{c^{'}})(1-12c^bd^2/\pi^2)} \\
& \quad + 24c^bd^2(\gamma_{c^{'}}-\alpha)/\pi^2 + 48c^b\beta d^3/\pi^2 \\
& \le \; \frac{24c^bd^2e\beta}{1-12c^bd^2/\pi^2} + 24c^bd^2\pi^{-2}o(c^{-b/4}) + 48c^b\beta d^3/\pi^2 \le o(c^{b/4}).
\end{aligned}
$$

Therefore $0 \le \gamma_{c,b}^{+} - \gamma_{c,b}^{-} = o(c^{b/4})$as $c \longrightarrow 0$.

Since $\tau^+_{c,b}$ and $\tau^-_{c,b}$ are geometric distributed, it is easy to obtain Lemma 3.3.

Lemma 3.3.

(i) $\{(c\tau^+_{c,b})^p : 0 < c \le c_0\}$ *is uniformly integrable for all* $p > 0$.

(ii) $\{(c\tau^-_{c,b})^p : 0 < c \le c_0\}$ *is uniformly integrable for all* $p > 0$.

Now, for all $b > 0$ define

$$L^{(1)}_{c,b} = \sup\{n \ge 1 : \left|\widehat{\beta}_n - \beta\right| \ge c^b \left(\frac{12\beta d^2}{\pi^2}\right)\}; \tag{10}$$

$$L^{(2)}_{c,b} = \sup\{n \ge 1 : |\widehat{\alpha}_n - \alpha| \ge 2c^b \left(\frac{12\beta d^3}{\pi^2}\right)\}, \tag{11}$$

$$L^{(2)}_{c,b,1} = \sup\{n \ge 1 : \left|\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\alpha + \beta d)^2 - \frac{\beta^2\pi^2}{6}\right| \ge c^b\beta^2 d^2\}; \tag{12}$$

and

$$L^{(1)}_{c,b,1} = \sup\{n \ge 1 : \left|\overline{X}_n - \alpha - \beta d\right| \ge c^b \frac{\beta^2 d^2}{4(\alpha + \beta d)}\}. \tag{13}$$

Lemma 3.4.

(i) $\{(c^{2b}L^{(1)}_{c,b})^p : 0 < c < c_0\}$ *is uniformly integrable for all* $p > 0$.

(ii) $\{(c^{2b}L^{(2)}_{c,b})^p : 0 < c < c_0\}$ *is uniformly integrable for all* $p > 0$.

Proof. For the case of $\alpha + \beta d > 0$, we can choose c which is small enough such that $(\alpha + \beta d)^2 - c^b\beta^2 d^2 > 0$, then

$$\begin{aligned}
& \{|\overline{X}_n^2 - (\alpha + \beta d)^2| < c^b\beta^2 d^2\} \\
= \; & \{(\alpha + \beta d)^2 - c^b\beta^2 d^2 < \overline{X}_n^2 < (\alpha + \beta d)^2 + c^b\beta^2 d^2\} \\
= \; & \{(\alpha+\beta d)^2(1-c^b\beta^2 d^2/(\alpha+\beta d)^2) < \overline{X}_n^2 < (\alpha+\beta d)^2(1+c^b\beta^2 d^2/(\alpha+\beta d)^2)\} \\
\supseteq \; & \{(\alpha+\beta d)[1-c^b\beta^2 d^2/(\alpha+\beta d)^2]^{1/2} < \overline{X}_n < (\alpha+\beta d)[1+c^b\beta^2 d^2/(\alpha+\beta d)^2]^{1/2}\}.
\end{aligned}$$

Using $0 < x < 1$, the inequalities $1 + x/4 \le (1 + x)^{1/2}$ and $(1 - x)^{1/2} \le 1 - x/4$, we have

$$\begin{aligned}
& \{|\overline{X}_n^2 - (\alpha + \beta d)^2| < c^b\beta^2 d^2\} \\
\supseteq \; & \{(\alpha + \beta d)(1 - \frac{c^b\beta^2 d^2}{4(\alpha + \beta d)^2}) < \overline{X}_n < (\alpha + \beta d)(1 + \frac{c^b\beta^2 d^2}{4(\alpha + \beta d)^2})\} \\
= \; & \{|\overline{X}_n - \alpha - \beta d| < \frac{c^b\beta^2 d^2}{4(\alpha + \beta d)}\}.
\end{aligned}$$

This implies

$$\{|\overline{X}_n^2-(\alpha+\beta d)^2|\geq c^b\beta^2 d^2\}\subseteq\{|\overline{X}_n-\alpha-\beta d|\geq\frac{c^b\beta^2d^2}{4(\alpha+\beta d)}\}. \tag{14}$$

From (14), we have

$$\begin{aligned}
&\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_i^2-\overline{X}_n^2-\frac{\beta^2\pi^2}{6}\right|\geq 2c^b\beta^2d^2\right\}\\
\subseteq\;&\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_i^2-(\alpha+\beta d)^2-\frac{\beta^2\pi^2}{6}\right|\geq c^b\beta^2d^2\right\}\bigcup\left\{|\overline{X}_n^2-(\alpha+\beta d)^2|\geq c^b\beta^2d^2\right\}\\
\subseteq\;&\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_i^2-(\alpha+\beta d)^2-\frac{\beta^2\pi^2}{6}\right|\geq c^b\beta^2d^2\right\}\bigcup\left\{|\overline{X}_n-\alpha-\beta d|\geq\frac{c^b\beta^2d^2}{4(\alpha+\beta d)}\right\}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left\{|\widehat{\beta}_n^2-\beta^2|\geq c^b\left(\frac{12\beta^2d^2}{\pi^2}\right)\right\}\\
=\;&\left\{\frac{\pi^2}{6}|\widehat{\beta}_n^2-\beta^2|\geq 2c^b\beta^2d^2\right\}\\
\subseteq\;&\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_i^2-(\alpha+\beta d)^2-\frac{\beta^2\pi^2}{6}\right|\geq c^b\beta^2d^2\right\}\bigcup\left\{|\overline{X}_n-\alpha-\beta d|\geq\frac{c^b\beta^2d^2}{4(\alpha+\beta d)}\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left\{\left|\widehat{\beta}_n^2-\beta^2\right|\geq c^b\left(\frac{12\beta^2d^2}{\pi^2}\right)\right\}\\
=\;&\left\{\widehat{\beta}_n^2\geq\beta^2\left(1+c^b\frac{12d^2}{\pi^2}\right)\right\}\bigcup\left\{\widehat{\beta}_n^2\leq\beta^2\left(1-c^b\frac{12d^2}{\pi^2}\right)\right\},
\end{aligned}$$

and using $0<x<1$, the inequalities $(1+x)^{1/2}\leq 1+x$ and $(1-x)^{1/2}\geq 1-x$, we have

$$\begin{aligned}
&\left\{|\widehat{\beta}_n^2-\beta^2|\geq c^b\left(\frac{12\beta^2d^2}{\pi^2}\right)\right\}\\
=\;&\left\{\widehat{\beta}_n\geq\beta\left(1+c^b\frac{12d^2}{\pi^2}\right)^{1/2}\right\}\bigcup\left\{\widehat{\beta}_n\leq\beta\left(1-c^b\frac{12d^2}{\pi^2}\right)^{1/2}\right\}\\
\supseteq\;&\left\{\widehat{\beta}_n\geq\beta\left(1+c^b\frac{12d^2}{\pi^2}\right)\right\}\bigcup\left\{\widehat{\beta}_n\leq\beta\left(1-c^b\frac{12d^2}{\pi^2}\right)\right\}\\
=\;&\left\{\left|\widehat{\beta}_n-\beta\right|\geq c^b\beta\left(\frac{12d^2}{\pi^2}\right)\right\}.
\end{aligned}$$

This implies

$$
\begin{aligned}
&\left\{\left|\widehat{\beta}_n-\beta\right| \geq c^b\beta\left(\frac{12d^2}{\pi^2}\right)\right\} \\
\subseteq\ &\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_i^2-(\alpha+\beta d)^2-\frac{\beta^2\pi^2}{6}\right| \geq c^b\beta^2 d^2\right\} \\
&\bigcup\left\{\left|\overline{X}_n-\alpha-\beta d\right| \geq \frac{c^b\beta^2 d^2}{4(\alpha+\beta d)}\right\}.
\end{aligned}
$$

Hence, we have

$$
\left\{L_{c,b}^{(1)}>j\right\} \subseteq \left\{L_{c,b,1}^{(1)}>j\right\}\bigcup\left\{L_{c,b,1}^{(2)}>j\right\},
$$

and this implies

$$
\boldsymbol{P}\left(L_{c,b}^{(1)}>j\right) \leq \boldsymbol{P}\left(L_{c,b,1}^{(1)}>j\right)+\boldsymbol{P}\left(L_{c,b,1}^{(2)}>j\right). \tag{15}
$$

By Theorem 7 in Chow and Lai(1975),

$$
\left\{\left(c^{2b}L_{c,b,1}^{(1)}\right)^p : 0<c<c_0\right\} \tag{16}
$$

and

$$
\left\{\left(c^{2b}L_{c,b,1}^{(2)}\right)^p : 0<c<c_0\right\} \tag{17}
$$

are uniformly integrable for all $p>0$. From (15), (16), and (17), we have

$$
\left\{\left(c^{2b}L_{c,b}^{(1)}\right)^p : 0<c<c_0\right\}
$$

is uniformly integrable for all $p>0$.
For part (ii), since

$$
\begin{aligned}
&\left\{\left|\widehat{\alpha}_n-\alpha\right| \geq 2c^b\left(\frac{12\beta d^3}{\pi^2}\right)\right\} \\
=\ &\left\{\left|(\overline{X}_n-\alpha-\beta d)-d(\widehat{\beta}_n-\beta)\right| \geq 2c^b\left(\frac{12\beta d^3}{\pi^2}\right)\right\} \\
\subseteq\ &\left\{\left|\overline{X}_n-\alpha-\beta d\right| \geq c^b\left(\frac{12\beta d^3}{\pi^2}\right)\right\}\bigcup\left\{d\left|\widehat{\beta}_n-\beta\right| \geq c^b\left(\frac{12\beta d^3}{\pi^2}\right)\right\} \\
=\ &\left\{\left|\overline{X}_n-\alpha-\beta d\right| \geq c^b\left(\frac{12\beta d^3}{\pi^2}\right)\right\}\bigcup\left\{\left|\widehat{\beta}_n-\beta\right| \geq c^b\left(\frac{12\beta d^2}{\pi^2}\right)\right\}
\end{aligned}
$$

Therefore, we have

$$\left\{L_{c,b}^{(2)} > j\right\} \subseteq \left\{L_{c,b}^{(1)} > j\right\} \bigcup \left\{L_{c,b,2}^{(1)} > j\right\},$$

where

$$L_{c,b,2}^{(1)} = \sup\left\{n \geq 1 : \left|\overline{X}_n - \alpha - \beta d\right| \geq c^b \left(\frac{12\beta d^3}{\pi^2}\right)\right\}.$$

This implies

$$\boldsymbol{P}\left(L_{c,b}^{(2)} > j\right) \leq \boldsymbol{P}\left(L_{c,b}^{(1)} > j\right) + \boldsymbol{P}\left(L_{c,b,2}^{(1)} > j\right).$$

Therefore, we obtain

$$\left\{\left(c^{2b} L_{c,b}^{(2)}\right)^p : 0 < c < c_0\right\}$$

is uniformly integrable for all $p > 0$.

For the case of $\alpha + \beta d = 0$ and $\alpha + \beta d < 0$,(13) can be simplified as follows:

$$L_{c,b,1}^{(1)} = \sup\{n \geq 1 : \left|\overline{X}_n\right| \geq c^b \beta^2 d^2\}$$

and

$$L_{c,b,1}^{(1)} = \sup\{n \geq 1 : \left|\overline{X}_n - \alpha - \beta d\right| \geq c^b \frac{\beta^2 d^2}{-4(\alpha + \beta d)}\}$$

respectively. Similarly,we can prove

$$\left\{\left(c^{2b} L_{c,b}^{(2)}\right)^p : 0 < c < c_0\right\}$$

is uniformly integrable for all $p > 0$.

Lemma 3.5. $\{(c\widehat{\tau}_c)^p : 0 < c < c_0\}$ *is uniformly integrable for all* $p > 0$ *as* $c \longrightarrow 0$

Proof. For K sufficiently large,$c < c_0$, we have $Kc^{-1} > 2n_c$. Treating $Kc^{-1}/2$ as an integer, we get $\boldsymbol{P}(c\widehat{\tau}_c > K) \leq \boldsymbol{P}(L_{c,b}^{(1)} \geq Kc^{-1}/2) + \boldsymbol{P}(L_{c,b}^{(2)} \geq Kc^{-1}/2) + \boldsymbol{P}(c\widehat{\tau}_c > K, L_{c,b}^{(1)} \leq Kc^{-1}/2, L_{c,b}^{(2)} \leq Kc^{-1}/2)$. From the definitions of $L_{c,b}^{(1)}, L_{c,b}^{(2)}$ and $\widehat{\tau}_c$, we have

$$\begin{aligned}
& \{c\widehat{\tau}_c > K, L_{c,b}^{(1)} \leq Kc^{-1}/2, L_{c,b}^{(2)} \leq Kc^{-1}/2\} \\
\subseteq\ & \{Kc^{-1}/2 < n < Kc^{-1}, X_n < \widehat{\gamma}_{c,n}, |\widehat{\beta}_n - \beta| \\
\leq\ & 12c^b \beta d^2/\pi^2, |\widehat{\alpha}_n - \alpha| \leq 24c^b \beta d^3/\pi^2\} \\
=\ & \{Kc^{-1}/2 < n < Kc^{-1}, X_n < \widehat{\gamma}_{c,n}, (1 - 12c^b d^2/\pi^2)\beta \leq \widehat{\beta}_n \\
\leq\ & (1 + 12c^b d^2/\pi^2)\beta, \alpha - 24c^b \beta d^3/\pi^2 \\
\leq\ & \widehat{\alpha}_n \leq \alpha + 24c^b \beta d^3/\pi^2\}.
\end{aligned}$$

Since $\widehat{\gamma}_{c,n}$ satisfies $f(\widehat{\gamma}_{c,n}-\widehat{\alpha}_n,\widehat{\beta}_n)=c$, and we can get $f(\widehat{\gamma}_{c,n}-\alpha-24c^b\beta d^3/\pi^2, \widehat{\beta}_n)>c$ by Lemma 2.1. From Lemma 2.2, we have $f(\widehat{\gamma}_{c,n}-\alpha-24c^b\beta d^3/\pi^2,(1+12c^bd^2/\pi^2)\beta)>c$. Because of $f(\gamma^+_{c,b}-\alpha-24c^b\beta d^3/\pi^2,(1+12c^bd^2/\pi^2)\beta)=c$, we obtain $\gamma^+_{c,b}-\alpha-24c^b\beta d^3/\pi^2>\widehat{\gamma}_{c,n}-\alpha-24c^b\beta d^3/\pi^2$. This implies $\gamma^+_{c,b}>\widehat{\gamma}_{c,n}$, and

$$\begin{aligned}&\{c\widehat{\tau}_c>K, L^{(1)}_{c,b}\le Kc^{-1}/2, L^{(2)}_{c,b}\le Kc^{-1}/2\}\\ \subseteq\ &\{Kc^{-1}/2<n<Kc^{-1}, X_n<\gamma^+_{c,b}\}\\ \subseteq\ &\{\tilde{\tau}^+_{c,b}>Kc^{-1}/2\}\end{aligned}$$

where $\tilde{\tau}^+_{c,b}\equiv\inf\{m\ge 1: X_{m+Kc^{-1}/2}\ge\gamma^+_{c,b}\}$. Therefore

$$\begin{aligned}\boldsymbol{P}(c\widehat{\tau}_c>K)\ &\le\ \boldsymbol{P}(cL^{(1)}_{c,b}\ge K/2)+\boldsymbol{P}(cL^{(2)}_{c,b}\ge K/2)+\boldsymbol{P}(c\tilde{\tau}^+_{c,b}>K/2)\\ &=\ \boldsymbol{P}(cL^{(1)}_{c,b}\ge K/2)+\boldsymbol{P}(cL^{(2)}_{c,b}\ge K/2)+\boldsymbol{P}(c\tau^+_{c,b}>K/2).\end{aligned}$$

From Lemma 3.3 and Lemma 3.4, $\{(c\widehat{\tau}_c)^p: 0<c<c_0\}$ is uniformly integrable for all $p>0$ as $c\longrightarrow 0$.

Lemma 3.6. *Let $\widehat{\tau}_c$ be as defined in (4) and (5) with $n_c=\delta c^{-\theta}$, $\delta>0$ and $0<\theta<1$. For $0<b<\frac{\theta}{2}$ and as $c\longrightarrow 0$, we have $\boldsymbol{E}(\widehat{\tau}_c)\le o(1)+(n_c-1)+\boldsymbol{E}(\tau^+_{c,b})$.*

Proof. Using Lemma 3.4 for $p>(\theta/2-b)^{-1}$ and Lemma 3.5 for $p=2$, we get

$$\begin{aligned}\boldsymbol{E}(\widehat{\tau}_c)\ &\le\ \boldsymbol{E}(\widehat{\tau}_c\boldsymbol{I}_{[L^{(1)}_{c,b}\ge n_c]})+\boldsymbol{E}(\widehat{\tau}_c\boldsymbol{I}_{[L^{(2)}_{c,b}\ge n_c]})+\boldsymbol{E}(\widehat{\tau}_c\boldsymbol{I}_{[L^{(1)}_{c,b}<n_c,L^{(2)}_{c,b}<n_c]})\\ &\le\ \left[\boldsymbol{E}(\widehat{\tau}^2_c)\right]^{1/2}\{\Big[\boldsymbol{P}(L^{(1)}_{c,b}\ge n_c)\Big]^{1/2}\\ &\quad+\Big[\boldsymbol{P}(L^{(2)}_{c,b}\ge n_c)\Big]^{1/2}\}+\boldsymbol{E}(\inf\{n\ge n_c: X_n\ge\ \gamma^+_{c,b}\})\\ &\le\ \left[\boldsymbol{E}(\widehat{\tau}^2_c)\right]^{1/2}n_c^{-p/2}\{\Big[\boldsymbol{E}(L^{(1)}_{c,b})^p\Big]^{1/2}+\Big[\boldsymbol{E}(L^{(2)}_{c,b})^p\Big]^{1/2}\}+(n_c-1)+\boldsymbol{E}(\tau^+_{c,b})\\ &\le\ \left[c^2\boldsymbol{E}(\widehat{\tau}^2_c)\right]^{1/2}(c^{-bp}+c^{-bp})c^{(\theta/2)p-1}o(1)+(n_c-1)+\boldsymbol{E}(\tau^+_{c,b})\\ &=\ o(1)+(n_c-1)+\boldsymbol{E}(\tau^+_{c,b}).\end{aligned}$$

The proof is completed.

Lemma 3.7. *Let $\widehat{\tau}_c$ be as defined in (4) and (5) with $n_c=\delta c^{-\theta}$, $\delta>0$ and $0<\theta<1$. Then for $0<b<\theta/2$, as $c\longrightarrow 0$,*

$$\boldsymbol{E}(\widehat{\tau}_c)\ge\boldsymbol{E}(\tau^-_{c,b})+(n_c-1)-o(1).$$

Proof. Let $L^{(1)}_{c,b}$, $L^{(2)}_{c,b}$ be as defined in (10) and (11) respectively.

$$\begin{aligned}\boldsymbol{E}(\widehat{\tau}_c) &\geq \boldsymbol{E}(\widehat{\tau}_c \boldsymbol{I}_{[L^{(1)}_{c,b}<n_c, L^{(2)}_{c,b}<n_c]}) \\ &\geq \boldsymbol{E}([\inf\{n \geq n_c : X_n \geq \gamma^-_{c,b}\}]\boldsymbol{I}_{[L^{(1)}_{c,b}<n_c, L^{(2)}_{c,b}<n_c]}) \\ &\geq \boldsymbol{E}(\inf\{n \geq n_c : X_n \geq \gamma^-_{c,b}\}) \\ &\quad -\boldsymbol{E}([\inf\{n \geq n_c : X_n \geq \gamma^-_{c,b}\}][\boldsymbol{I}_{[L^{(1)}_{c,b}\geq n_c]} + \boldsymbol{I}_{[L^{(2)}_{c,b}\geq n_c]}]).\end{aligned}$$

Taking $p = 2$ in Lemma 3.3 and $p > (\theta/2 - b)^{-1}$ in Lemma 3.4, we have

$$\begin{aligned}\boldsymbol{E}(\widehat{\tau}_c) &\geq (n_c - 1) + \boldsymbol{E}(\tau^-_{c,b}) - \{\boldsymbol{E}[(n_c - 1) + \tau^-_{c,b}]^2\}^{1/2}\{[n_c^{-p}\boldsymbol{E}(L^{(1)}_{c,b})^p]^{1/2} \\ &\quad +[n_c^{-p}\boldsymbol{E}(L^{(2)}_{c,b})^p]^{1/2}\} \\ &\geq (n_c - 1) + \boldsymbol{E}(\tau^-_{c,b}) - \{O(c^{-2\theta}) + O(c^{-\theta-1}) + O(c^{-2})\}^{1/2} O(c^{(\theta/2-b)p}) \\ &\geq (n_c - 1) + \boldsymbol{E}(\tau^-_{c,b}) - O(c^{(\theta/2-b)p-1}) \\ &\geq (n_c - 1) + \boldsymbol{E}(\tau^-_{c,b}) - o(c^q), \quad \text{for some } q > 0.\end{aligned}$$

The proof is completed.

From Lemma 3.4, it is easy to obtain Lemma 3.8.

Lemma 3.8. *Let $L^{(1)}_{c,b}$ and $L^{(2)}_{c,b}$ be as defined in (10) and (11) with $n_c = \delta c^{-\theta}$ for some $\delta > 0$ and $0 < \theta < 1$. Then for $b \in (0, \theta/2)$,*

$$(i) \sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j|\boldsymbol{I}_{[L^{(2)}_{c,b}\geq j]}) \longrightarrow 0, \quad \text{as } c \longrightarrow 0;$$

$$(ii) \sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j|\boldsymbol{I}_{[L^{(1)}_{c,b}\geq j]}) \longrightarrow 0, \quad \text{as } c \longrightarrow 0.$$

Proof. For (i), since

$$\begin{aligned}&\sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j|\boldsymbol{I}_{[L^{(2)}_{c,b}\geq j]}) \\ \leq\ &\sum_{j=n_c}^{\infty} \{\boldsymbol{E}(X_j^2)\boldsymbol{P}(L^{(2)}_{c,b} \geq j)\}^{1/2} \\ \leq\ &\sum_{j=n_c}^{\infty} \{\boldsymbol{E}(X_j^2)\boldsymbol{E}(L^{(2)}_{c,b})^{2p_1}/j^{2p_1}\}^{1/2}\end{aligned}$$

$$
\begin{aligned}
&\leq \{\boldsymbol{E}(X_1^2)\}^{1/2}\{\boldsymbol{E}[(c^{2b}L_{c,b}^{(2)})^{2p_1}]\}^{1/2}\sum_{j=n_c}^{\infty} c^{-2bp_1}j^{-p_1} \\
&\leq \{\boldsymbol{E}(X_1^2)\}^{1/2}\{\boldsymbol{E}[(c^{2b}L_{c,b}^{(2)})^{2p_1}]\}^{1/2}c^{-2bp_1}O(n_c^{-p_1+1}) \\
&\leq O(c^{-2bp_1-\theta+p_1\theta}) \\
&= O(c^{p_1(\theta-2b)-\theta}).
\end{aligned}
$$

Therefore taking p_1 such that $p_1(\theta-2b)-\theta>0$, we have

$$
\sum_{j=n_c}^{\infty}\boldsymbol{E}(|X_j|\boldsymbol{I}_{[L_{c,b}^{(2)}\geq j]}) \longrightarrow 0, \text{ as } c\to 0.
$$

Similarly, we can prove part (ii).

Lemma 3.9. $\boldsymbol{E}(X_{\widehat{\tau}_c}) \geq \boldsymbol{E}(X_1\boldsymbol{I}_{[X_1\geq\gamma_{c,b}^{+}]})\boldsymbol{E}(\tau_{c,b}^{-})+o(1)$ *as* $c\longrightarrow 0$.

Proof. Let $L_{c,b}^{(1)}$, $L_{c,b}^{(2)}$ be as defined in (10) and (11) respectively.

$$
\begin{aligned}
\boldsymbol{E}(X_{\widehat{\tau}_c}) &= \sum_{j=n_c}^{\infty}\boldsymbol{E}(X_j\boldsymbol{I}_{[\widehat{\tau}_c=j]}) \\
&\geq \sum_{j=n_c}^{\infty}\boldsymbol{E}(X_j\boldsymbol{I}_{[\widehat{\tau}_c=j,L_{c,b}^{(1)}<j,L_{c,b}^{(2)}<j]})+o(1) \\
&\geq \sum_{j=n_c}^{\infty}\boldsymbol{E}(X_j\boldsymbol{I}_{[\widehat{\tau}_c=j,(1-12c^bd^2/\pi^2)\beta\leq\widehat{\beta}_n\leq(1+12c^bd^2/\pi^2)\beta,|\widehat{\alpha}_n-\alpha|<2c^b\left(\frac{12\beta d^3}{\pi^2}\right)} \\
&\qquad \text{for all } n\geq j]) \\
&\geq \sum_{j=n_c}^{\infty}\boldsymbol{E}(X_j\boldsymbol{I}_{[\widehat{\tau}_c=j,(1-12c^bd^2/\pi^2)\beta\leq\widehat{\beta}_n\leq(1+12c^bd^2/\pi^2)\beta,|\widehat{\alpha}_n-\alpha|<2c^b\left(\frac{12\beta d^3}{\pi^2}\right)} \\
&\qquad \text{for all } n\geq j, X_j\geq\gamma_{c,b}^{+}]) \\
&\geq \sum_{j=n_c}^{\infty}\boldsymbol{E}(X_j\boldsymbol{I}_{[\widehat{\tau}_c\geq j,(1-12c^bd^2/\pi^2)\beta\leq\widehat{\beta}_n\leq(1+12c^bd^2/\pi^2)\beta,|\widehat{\alpha}_n-\alpha|<2c^b\left(\frac{12\beta d^3}{\pi^2}\right)} \\
&\qquad \text{for all } n\geq j, X_j\geq\gamma_{c,b}^{+}]) \\
&= \sum_{j=n_c}^{\infty}\boldsymbol{E}(X_j\boldsymbol{I}_{[X_j\geq\gamma_{c,b}^{+}]}\boldsymbol{I}_{[\widehat{\tau}_c\geq j,L_{c,b}^{(1)}<j,L_{c,b}^{(2)}<j]}) \\
&\geq \sum_{j=n_c}^{\infty}\boldsymbol{E}(X_j\boldsymbol{I}_{[X_j\geq\gamma_{c,b}^{+}]}\{\boldsymbol{I}_{[\widehat{\tau}_c\geq j]}-\boldsymbol{I}_{[\widehat{\tau}_c\geq j,L_{c,b}^{(1)}\geq j]}-\boldsymbol{I}_{[\widehat{\tau}_c\geq j,L_{c,b}^{(2)}\geq j]}\})
\end{aligned}
$$

$$\geq \sum_{j=n_c}^{\infty} \boldsymbol{E}(X_j \boldsymbol{I}_{[X_j \geq \gamma_{c,b}^+]} \boldsymbol{I}_{[\widehat{\tau}_c \geq j]}) - \sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j| \boldsymbol{I}_{[L_{c,b}^{(1)} \geq j]})$$
$$- \sum_{j=n_c}^{\infty} \boldsymbol{E}(|X_j| \boldsymbol{I}_{[L_{c,b}^{(2)} \geq j]}).$$

From Lemma 3.8, it is easy to obtain

$$\begin{aligned}
\boldsymbol{E}(X_{\widehat{\tau}_c}) &\geq \sum_{j=n_c}^{\infty} \boldsymbol{P}\{\widehat{\tau}_c \geq j\} \boldsymbol{E}(X_j \boldsymbol{I}_{[X_j \geq \gamma_{c,b}^+]}) + o(1) \\
&= \boldsymbol{E}(X_1 \boldsymbol{I}_{[X_1 \geq \gamma_{c,b}^+]}) \left[\boldsymbol{E}(\widehat{\tau}_c) - (n_c - 1)\right] + o(1).
\end{aligned}$$

By Lemma 3.7, we have

$$\boldsymbol{E}(X_{\widehat{\tau}_c}) \geq \boldsymbol{E}(X_1 \boldsymbol{I}_{[X_1 \geq \gamma_{c,b}^+]}) \boldsymbol{E}(\tau_{c,b}^-) + o(1).$$

Lemma 3.10. *For all* $b > 0$, $\gamma_c(1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^+)/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^-)) \longrightarrow 0$ *as* $c \longrightarrow 0$.

Proof. Note that

$$\begin{aligned}
\boldsymbol{P}(X_1 \geq \gamma_{c,b}^+) &= \boldsymbol{P}(X_1 - \alpha \geq \gamma_{c,b}^+ - \alpha) \\
&= \int_{\gamma_{c,b}^+ - \alpha}^{\infty} \frac{1}{\beta} \exp(-\frac{y}{\beta}) \exp(-\exp(-\frac{y}{\beta})) dy. \\
&\geq \frac{1}{e} \exp(-\frac{\gamma_{c,b}^+ - \alpha}{\beta}).
\end{aligned}$$

and $\gamma_c\{1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^+)/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^-)\} \leq \gamma_c\{\boldsymbol{P}(\gamma_{c,b}^- \leq X_1 \leq \gamma_{c,b}^+)\}/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^+)$. Using the Mean Value theorem to compute $\boldsymbol{P}(\gamma_{c,b}^- \leq X_1 \leq \gamma_{c,b}^+)$, we have

$$\begin{aligned}
\boldsymbol{P}(\gamma_{c,b}^- \leq X_1 \leq \gamma_{c,b}^+) &\leq (\gamma_{c,b}^+ - \gamma_{c,b}^-) \frac{1}{\beta} \exp(-\frac{\gamma_{c,b}^- - \alpha}{\beta}) \exp(-\exp(-\frac{\gamma_{c,b}^+ - \alpha}{\beta})) \\
&\leq (\gamma_{c,b}^+ - \gamma_{c,b}^-) \frac{1}{\beta} \exp(-\frac{\gamma_{c,b}^- - \alpha}{\beta}).
\end{aligned}$$

Therefore

$$\gamma_c(1 - \boldsymbol{P}(X_1 \geq \gamma_{c,b}^+)/\boldsymbol{P}(X_1 \geq \gamma_{c,b}^-)) \leq \gamma_c(\gamma_{c,b}^+ - \gamma_{c,b}^-) \frac{e}{\beta} \exp(\frac{\gamma_{c,b}^+ - \gamma_{c,b}^-}{\beta}) \tag{18}$$

By Lemma 2.4 and Lemma 3.2, take $b < p/4$ and we have

$$\begin{aligned} 0 &\le \gamma_c(1 - \boldsymbol{P}(X_1 \ge \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \ge \gamma_{c,b}^{-})) \\ &\le o(c^{-b})o(c^{p/4})o(1) \longrightarrow 0 \text{ as } \quad c \longrightarrow 0. \end{aligned}$$

Lemma 3.11. $\boldsymbol{E}(X_1 \boldsymbol{I}_{[\gamma_c \le X_1 \le \gamma_{c,b}^{+}]})\boldsymbol{E}(\tau_{c,b}^{-}) \longrightarrow 0$ *as* $c \longrightarrow 0$.

Proof.

$$\begin{aligned} \boldsymbol{E}(X_1 \boldsymbol{I}_{[\gamma_c \le X_1 \le \gamma_{c,b}^{+}]})\boldsymbol{E}(\tau_{c,b}^{-}) &\le \boldsymbol{E}(X_1 \boldsymbol{I}_{[\gamma_{c,b}^{-} \le X_1 \le \gamma_{c,b}^{+}]})\boldsymbol{E}(\tau_{c,b}^{-}) \\ &\le \gamma_{c,b}^{+}\boldsymbol{P}(\gamma_{c,b}^{-} \le X_1 \le \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \ge \gamma_{c,b}^{-}). \end{aligned}$$

Using Lemma 3.2, we have $\gamma_{c,b}^{+} = \gamma_c + o(c^{b/4})$, therefore

$$\begin{aligned} 0 &\le \boldsymbol{E}(X_1 \boldsymbol{I}_{[\gamma_c \le X_1 \le \gamma_{c,b}^{+}]})\boldsymbol{E}(\tau_{c,b}^{-}) \\ &\le (\gamma_c + o(c^{b/4}))(1 - \boldsymbol{P}(X_1 \ge \gamma_{c,b}^{+})/\boldsymbol{P}(X_1 \ge \gamma_{c,b}^{-})). \end{aligned}$$

Using Lemma 3.10 it is easy to obtain the result.

Lemma 3.12. $c\{\boldsymbol{E}(\widehat{\tau}_c) - \boldsymbol{E}(\tau_{c,b}^{-})\} \longrightarrow 0$ *as* $c \longrightarrow 0$.

Proof. From Lemma 3.6., we have

$$\begin{aligned} c\{\boldsymbol{E}(\widehat{\tau}_c) - \boldsymbol{E}(\tau_{c,b}^{-})\} &\le c\{\boldsymbol{E}(\tau_{c,b}^{+}) - \boldsymbol{E}(\tau_{c,b}^{-})\} + c(n_c - 1) + o(1) \\ &= c\{\boldsymbol{E}(\tau_{c,b}^{+}) - \boldsymbol{E}(\tau_{c,b}^{-})\} + o(1) \\ &= \frac{c\boldsymbol{P}(\gamma_{c,b}^{-} \le X_1 \le \gamma_{c,b}^{+})}{\boldsymbol{P}(X_1 \ge \gamma_{c,b}^{+})\boldsymbol{P}(X_1 \ge \gamma_{c,b}^{-})} + o(1) \\ &\le \frac{c\boldsymbol{P}(\gamma_{c,b}^{-} \le X_1 \le \gamma_{c,b}^{+})}{\boldsymbol{P}(X_1 \ge \gamma_{c,b}^{+})\boldsymbol{P}(X_1 \ge \gamma_c)} + o(1). \end{aligned}$$

Using (18), Lemma 2.5 and Lemma 3.10, we obtain
$c\{\boldsymbol{E}(\widehat{\tau}_c) - \boldsymbol{E}(\tau_{c,b}^{-})\} \longrightarrow 0$ as $c \longrightarrow 0$.

Theorem. *Let* $\widehat{\tau}_c$ *be as defined in (4) and (5) with* $n_c = \delta c^{-\theta}$ *for some* $\delta > 0$ *and* $0 < \theta < 1$. *Then*

$$\boldsymbol{E}(Y_{\tau_c^*}) - \boldsymbol{E}(Y_{\widehat{\tau}_c}) \longrightarrow 0 \quad \textit{as } c \longrightarrow 0.$$

That is, the expected loss due to not knowing α and β vanishes when we use the approximating rule $\widehat{\tau}_c$ as $c \longrightarrow 0$.

Proof.

$$
\begin{aligned}
0 &\leq \boldsymbol{E}(Y_{\tau_c^*}) - \boldsymbol{E}(Y_{\widehat{\tau}_c}) = \gamma_c - \boldsymbol{E}(X_{\widehat{\tau}_c}) + c\boldsymbol{E}(\widehat{\tau}_c) \\
&\leq \gamma_c - \boldsymbol{E}(X_1 \boldsymbol{I}_{[X_1 \geq \gamma_{c,b}^+]})\boldsymbol{E}(\tau_{c,b}^-) + c\boldsymbol{E}(\widehat{\tau}_c) \\
&\leq \gamma_c - \{\boldsymbol{E}(X_1 \boldsymbol{I}_{[X_1 \geq \ \gamma_c]}) - \boldsymbol{E}(X_1 \boldsymbol{I}_{[\gamma_c \leq X_1 \leq \gamma_{c,b}^+]})\}\boldsymbol{E}(\tau_{c,b}^-) + c\boldsymbol{E}(\widehat{\tau}_c) + o(1).
\end{aligned}
$$

From the result in Lemma 3.9, the second inequality holds. Using Lemma 3.11 and the equality

$$\boldsymbol{E}(X_1 \boldsymbol{I}_{[X_1 \geq \gamma_c]}) = c + \gamma_c \boldsymbol{P}(X_1 \geq \gamma_c),$$

we have

$$
\begin{aligned}
0 &\leq \boldsymbol{E}(Y_{\tau_c^*}) - \boldsymbol{E}(Y_{\widehat{\tau}_c}) \\
&\leq \gamma_c\{1 - \boldsymbol{P}(X_1 \geq \ \gamma_c)/\boldsymbol{P}(X_1 \geq \ \gamma_{c,b}^-)\} + c\{\boldsymbol{E}(\widehat{\tau}_c) - \boldsymbol{E}(\tau_{c,b}^-)\} + o(1) \\
&\leq \gamma_c\{1 - \boldsymbol{P}(X_1 \geq \ \gamma_{c,b}^+)/\boldsymbol{P}(X_1 \geq \ \gamma_{c,b}^-)\} + c\{\boldsymbol{E}(\widehat{\tau}_c) - \boldsymbol{E}(\tau_{c,b}^-)\} + o(1).
\end{aligned}
$$

By Lemma 3.10 and Lemma 3.12, we obtain

$$0 \quad \leq \quad \boldsymbol{E}(Y_{\tau_c^*}) - \boldsymbol{E}(Y_{\widehat{\tau}_c}) \longrightarrow 0 \text{ as } c \longrightarrow 0.$$

The main result is proven.

4. Simulation Study

In this section, we use simulation to compare $E(Y_{\tau_c^*})$ and $E(Y_{\widehat{\tau}_{c,n}})$. Assuming that α, β and c are known, we use the numerical method to compute the theoretical values of $E(Y_{\tau_c^*})$. Let $n_c = [c^{-\theta}] + 1$ and we try to find out the differences in different c and θ. The combinations of (α, β),c and θ under our simulation are as followed:

(1) $(\alpha, \beta) = (3, 2)$ and (1,1);

(2) c =0.01, 0.001 and 0.0001;

(3) $\theta = 0.4$ and 0.5.

For each combination of (α, β), θ and c, we generate the sequence of $X_1, X_2 \cdots X_n, \cdots$ to evaluate $Y_{\tau_c^*}$ and $Y_{\widehat{\tau}_{c,n}}$. In our simulation, 1000 data sets are generated. In these 1000 times of simulation, we obtain 1000 values of $Y_{\widehat{\tau}_{c,n}}$ and thus compute the mean of $Y_{\tau_c^*} - Y_{\widehat{\tau}_{c,n}}$.

As seen from Table 1, both values of $E(Y_{\tau_c^*})$ and the sample mean of $Y_{\tau_c^*}$ show that the optimal reward grows larger as c becomes smaller. We also find the sample mean of $Y_{\tau_c^*}$ under 1000 times of simulation is very close to $E(Y_{\tau_c^*})$, and the sample mean of $Y_{\tau_c^*} - Y_{\widehat{\tau}_{c,n}}$ approaches to 0 as the value of c becomes smaller. The sample mean of $Y_{\tau_c^*} - Y_{\widehat{\tau}_{c,n}}$ when $\theta = 0.4$, is larger than that when $\theta = 0.5$. And it is more robust and effective to evaluate the estimate of (α, β) when $\theta = 0.5$ than when $\theta = 0.4$.

From the theorem we know the $E(Y_{\tau_c^*}) - E(Y_{\widehat{\tau}_{c,n}}) \geq 0$, but in simulation results Avg. $Y_{\tau_c^*} - Y_{\widehat{\tau}_{c,n}}$ is negative when $c = 0.0001$. It is because as $\widehat{\gamma}_{c,n} > \gamma_c$, we get $\widehat{\tau}_{c,n} > \tau_c^*$ and $X_{\tau_c^*} - X_{\widehat{\tau}_{c,n}} < 0$ by the definition. Moreover, from $Y_{\tau_c^*} = X_{\tau_c^*} - c\tau_c^*$ and $Y_{\widehat{\tau}_{c,n}} = X_{\widehat{\tau}_{c,n}} - c\widehat{\tau}_{c,n}$, we know that Avg. $Y_{\tau_c^*} - Y_{\widehat{\tau}_{c,n}}$ is possibly negative in simulation. When $c = 0.0001$, it took more than one week for the computer to run 1000 times of simulation. If we want to get more accurate results, we need to increase the times of simulation, but that will definitely take a very long period of time. In general, the results show that the value of $E(Y_{\tau_c^*}) - E(Y_{\widehat{\tau}_{c,n}})$ approaches to 0 as $c \longrightarrow 0$. The result accords with the theorem we have proved in Section 3.

Table 1. As different $c, (\alpha, \beta)$ and θ are concerned, a comparison of bias under the 1000 times of simulation

(α, β)	c	θ	$E(Y_{\tau_c^*})$	Ave. $Y_{\tau_c^*}$	Ave. $Y_{\tau_c^*} - Y_{\widehat{\tau}_{c,n}}$
(3,2)	.0100	.50	13.59413	13.47871	0.97365
	.0010	.50	18.20155	18.08640	0.04780
	.0001	.50	22.80695	22.73885	-0.01626
(3,2)	.0100	.40	13.59413	13.54595	1.18501
	.0010	.40	18.20155	18.13743	0.16041
	.0001	.40	22.80695	22.82204	0.09603
(1,1)	.0100	.50	5.60267	5.60619	0.55970
	.0010	.50	7.90751	7.94844	0.23996
	.0001	.50	10.21032	10.12079	-0.11683
(1,1)	.0100	.40	5.60267	5.61478	0.74359
	.0010	.40	7.90751	7.90466	0.27135
	.0001	.40	10.21032	10.17189	-0.04366

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