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APOSTOL-EULER POLYNOMIALS OF HIGHER ORDER AND GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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Abstract. The purpose of this paper is to give analogous definitions of Apostol type (see T. M. Apostol [*Pacific J. Math.* 1 (1951), 161-167]) for the so-called Apostol-Euler numbers and polynomials of higher order. We establish their elementary properties, obtain several explicit formulas involving the Gaussian hypergeometric function and the Stirling numbers of the second kind, and deduce their special cases and applications that lead to the corresponding formulas of the classical Euler numbers and polynomials of higher order.

1. Introduction

Analogous definitions of the classical Bernoulli numbers and polynomials were given by Apostol (see [2, p. 165 (3.1)]). These analogues, called here the Apostol-Bernoulli numbers and polynomials, were used recently by Srivastava (see [4, p. 83-84]). Further, some generalizations of Apostol-Bernoulli polynomials were defined by Luo and Srivastava (see [3, p. 290-302]). In this section, we similarly give the analogous definitions for the classical Euler numbers and polynomials of higher order by using Apostol's idea as follows.

Definition 1. Apostol-Euler polynomials of higher order $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ are defined by means of the generating function:

$$\left(\frac{2}{\lambda e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \qquad (|z+\log\lambda|<\pi). \tag{1}$$

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Setting $\alpha = 1$ in (1), $\mathcal{E}_n(x;\lambda) = \mathcal{E}_n^{(1)}(x;\lambda)$ are called the Apostol-Euler polynomials; setting $\lambda = 1$ in (1), $E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1)$ are called the Euler polynomials of higher order (see [5, p. 66 (64)]); setting $\alpha = 1$ and $\lambda = 1$ in (1), $E_n(x) = \mathcal{E}_n^{(1)}(x;1)$ are called the classical Euler polynomials (see [5, p. 63 (39)]).

Definition 2. Apostol-Euler numbers of higher order $\mathcal{E}_n^{(\alpha)}(\lambda)$ are defined by means of the generating function:

$$\left(\frac{2e^z}{\lambda e^{2z}+1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(\lambda)\frac{z^n}{n!} \qquad (|2z+\log\lambda| < \pi). \tag{2}$$

Setting $\alpha = 1$ in (2), $\mathcal{E}_n(\lambda) = \mathcal{E}_n^{(1)}(\lambda)$ are called the Apostol-Euler numbers; setting $\lambda = 1$ in (2), $E_n^{(\alpha)} = \mathcal{E}_n^{(\alpha)}(1)$ are called the Euler numbers of higher order (see [5, p. 66 (65)]); setting $\alpha = 1$ and $\lambda = 1$ in (2), $E_n = \mathcal{E}_n^{(1)}(1)$ are called the classical Euler numbers (see [5, p. 63 (40)]).

Here α, λ are arbitrary parameter (real or complex).

2. Lementary Properties of the Apostol-euler Polynomials of Higher Order

In the present section we can readily prove each of the following results in a straightforward way by using generating functions (1) and (2).

Proposition 1.

$$\mathcal{E}_n^{(\alpha)}(\lambda) = 2^n \mathcal{E}_n^{(\alpha)}\left(\frac{\alpha}{2};\lambda\right) \quad \textit{and} \quad \mathcal{E}_n^{(0)}(x;\lambda) = x^n. \tag{3}$$

Proposition 2.

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{E}_k^{(\alpha)}(\lambda)}{2^k}\left(x-\frac{\alpha}{2}\right)^{n-k}. \tag{4}$$

Proposition 3. *Difference equation*

$$\lambda\mathcal{E}_n^{(\alpha)}(x+1;\lambda) + \mathcal{E}_n^{(\alpha)}(x;\lambda) = 2\mathcal{E}_n^{(\alpha-1)}(x;\lambda) \tag{5}$$

Proposition 4. *Differential relation*

$$\frac{\partial}{\partial x}\mathcal{E}_n^{(\alpha)}(x;\lambda) = n\mathcal{E}_{n-1}^{(\alpha)}(x;\lambda) \tag{6}$$

$$\frac{\partial^p}{\partial x^p}\mathcal{E}_n^{(\alpha)}(x;\lambda) = \frac{n!}{(n-p)!}\mathcal{E}_{n-p}^{(\alpha)}(x;\lambda) \tag{7}$$

Proposition 5.

$$\int_a^b \mathcal{E}_n^{(\alpha)}(x;\lambda)\,\mathrm{d}x = \frac{\mathcal{E}_{n+1}^{(\alpha)}(b;\lambda) - \mathcal{E}_{n+1}^{(\alpha)}(a;\lambda)}{n+1} \tag{8}$$

Proposition 6. *Addition formula*

$$\mathcal{E}_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k^{(\alpha)}(x;\lambda)\mathcal{E}_{n-k}^{(\beta)}(y;\lambda) \tag{9}$$

Proposition 7.

$$\mathcal{E}_n^{(\alpha)}(\alpha-x;\lambda) = \frac{(-1)^n}{\lambda^\alpha}\mathcal{E}_n^{(\alpha)}(x;\lambda^{-1}) \tag{10}$$

$$\mathcal{E}_n^{(\alpha)}(\alpha+x;\lambda) = \frac{(-1)^n}{\lambda^\alpha}\mathcal{E}_n^{(\alpha)}(-x;\lambda^{-1}) \tag{11}$$

Proposition 8. *Two recursion formulas*

$$\mathcal{E}_{n+1}^{(\alpha)}(x;\lambda) = x\mathcal{E}_n^{(\alpha)}(x;\lambda) - \frac{\alpha\lambda}{2}\mathcal{E}_n^{(\alpha+1)}(x+1;\lambda) \tag{12}$$

$$\mathcal{E}_n^{(\alpha+1)}(x;\lambda) = \frac{2}{\alpha}\mathcal{E}_{n+1}^{(\alpha)}(x;\lambda) + \frac{2(\alpha-x)}{\alpha}\mathcal{E}_n^{(\alpha)}(x;\lambda) \tag{13}$$

3. Several Explicit Formulas for the Apostol-euler Polynomials and Numbers of Higher Order

Recently, Srivastava and Todorov gave the following two explicit formulas for the Bernoulli polynomials and numbers of higher order in terms of the Gaussian hypergeometric function and the Stirling numbers of the second kind, respectively (see [5, p. 62, Eq. (28) and p. 63, Eq. (37)], see also [6]).

$$\begin{aligned} B_n^{(\alpha)}(x) &= \sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{2k}(x+j)^{n-k} \\ &\quad \times F[k-n,k-\alpha;2k+1;j/(x+j)] \end{aligned} \tag{14}$$

and

$$B_n^{(\alpha)} = \sum_{k=0}^{n} (-1)^k \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \binom{n+k}{k}^{-1} S(n+k,k). \tag{15}$$

In this section, we apply a similar technique in order to derive some analogous representations for the Apostol-Euler polynomials and numbers of higher order. Now we state and prove our main results given below.

Theorem 1. *If n is positive integer and α and λ are arbitrary real or complex parameters, then we have*

$$\begin{aligned} \mathcal{E}_n^{(\alpha)}(x;\lambda) = \;& 2^\alpha \sum_{l=0}^{n} \binom{n}{l} \binom{\alpha+l-1}{l} \lambda^l (\lambda+1)^{-\alpha-l} \\ & \times \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^l (x+k)^{n-l} F[l-n, l; l+1; k/(x+k)] \end{aligned} \tag{16}$$

where $F[a,b;c;z]$ denotes the Gaussian hypergeometric function $[5,\ p.\,44\,(4)]$.

Proof. We differentiate both side of the generating relation (1) with respect to the variable z. By using Leibniz's rule, we thus get

$$\begin{aligned} \mathcal{E}_n^{(\alpha)}(x;\lambda) = \;& D_z^n \left\{ \left(\frac{2}{\lambda e^z + 1} \right)^\alpha e^{xz} \right\} \Big|_{z=0}, \quad D_z = \frac{d}{dz}. \\ = \;& 2^\alpha \sum_{s=0}^{n} \binom{n}{s} x^{n-s} D_z^s \left\{ \Big[(\lambda+1) + \lambda(e^z - 1) \Big]^{-\alpha} \right\} \Big|_{z=0}. \end{aligned} \tag{17}$$

Applying the series expansion:

$$(a+w)^{-\alpha} = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} a^{-\alpha-k} (-w)^k \qquad (|w| < |a|) \tag{18}$$

and the well-known formula (see [5, p. 58 (15)])

$$(e^z - 1)^k = k! \sum_{r=k}^{\infty} S(r,k) \frac{z^r}{r!} \tag{19}$$

we have

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = 2^\alpha \sum_{s=0}^{n} \binom{n}{s} x^{n-s} \sum_{l=0}^{s} \binom{\alpha+l-1}{l} (\lambda+1)^{-\alpha-l} (-\lambda)^l l! S(s,l). \tag{20}$$

We now change the order of summation in the above double series, and make use of the following formula: (see [5, p. 58 (20)])

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n, \tag{21}$$

so that

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}(x;\lambda)=&\ 2^\alpha\sum_{l=0}^{n}\binom{n}{l}\binom{\alpha+l-1}{l}(\lambda+1)^{-\alpha-l}\lambda^l x^{n-l}\\&\times\sum_{k=0}^{l}(-1)^k\binom{l}{k}k^lF[l-n,1;l+1;-k/x]\end{aligned} \tag{22}$$

in terms of the Gaussian hypergeometric function.

Finally, if we apply the known transformation [1, 15.3.4]:

$$F[a,b;c;z]=(1-z)^{-a}F[a,c-b;c;z/(z-1)]$$

in (22), we are led immediately to the explicit formula (16) asserted by Theorem 1. ■

Corollary 1. *For $\alpha=1$ in Theorem 1, we obtain the following explicit formula for the Apostol-Euler polynomials:*

$$\begin{aligned}\mathcal{E}_n(x;\lambda)=&\ 2\sum_{l=0}^{n}\binom{n}{l}\lambda^l(\lambda+1)^{-l-1}\\&\times\sum_{k=0}^{l}(-1)^k\binom{l}{k}k^l(x+k)^{n-l}F[l-n,l;l+1;k/(x+k)]\end{aligned} \tag{23}$$

Theorem 2. *If n is positive integer, and if α and λ are arbitrary real or complex parameters, then*

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}(\lambda)=&\ 2^\alpha\sum_{j=0}^{n}(-1)^j\binom{n+\alpha}{j+\alpha}\binom{\alpha+j-1}{j}(\lambda+1)^{-j-\alpha}\\&\sum_{i=0}^{j}\binom{j}{i}\lambda^{j-i}(j-2i)^n\end{aligned} \tag{24}$$

Proof. From the generating relation (2), we apply a similar proof process of Theorem 1, note the elementary combinatorial identities:

$$\binom{m}{l}\binom{l}{n}=\binom{m}{n}\binom{m-n}{m-l}\quad\text{and}\quad\sum_{\nu=0}^{s}\binom{\eta+\nu}{\nu}=\binom{\eta+s+1}{s}, \tag{25}$$

and we obtain the formula (24) immediately. ■

Corollary 2. *For $\alpha = 1$ in Theorem 2, we obtain the following formula of the Apostol-Euler numbers:*

$$\mathcal{E}_n(\lambda) = 2\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}(\lambda+1)^{-j-1}\sum_{i=0}^{j}\binom{j}{i}\lambda^{j-i}(j-2i)^n. \tag{26}$$

Lemma 1.

$$\mathcal{E}_n^{(\alpha)}\left(\frac{x}{2};\lambda\right) = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{-x}{2}\right)^{n-k}\mathcal{E}_k^{(\alpha)}(x;\lambda). \tag{27}$$

Proof. The generating relation (1) yields

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}\left(\frac{x}{2};\lambda\right)\frac{z^n}{n!} &= \left(\frac{2}{\lambda e^z+1}\right)^{\alpha}e^{\frac{xz}{2}} = \left(\frac{2}{\lambda e^z+1}\right)^{\alpha}e^{xz}\cdot e^{\frac{-xz}{2}}\\ &= \sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}\cdot\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\left(\frac{-x}{2}\right)^{n-k}\mathcal{E}_k^{(\alpha)}(x;\lambda)\right]\frac{z^n}{n!}.\end{aligned} \tag{28}$$

We now compare the coefficients of $\frac{z^n}{n!}$ on both side of (28), and we obtain the formula (27) immediately. ■

Theorem 3. *If n is a positive integer, and if α and λ are arbitrary real or complex parameters, then*

$$\begin{aligned}\mathcal{E}_n^{(\alpha)}(\lambda) = {} & (-1)^n\sum_{k=0}^{n}\binom{n}{k}2^{k+\alpha}\alpha^{n-k}\\ & \sum_{l=0}^{k}\binom{\alpha+l-1}{l}(-1)^l(\lambda+1)^{-\alpha-l}l!S(k,l)\end{aligned} \tag{29}$$

Proof. By [1, 15.1.20]

$$F[a,b;c;1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (c \neq 0,-1,-2,\ldots,\Re(c-a-b)>0),$$

which (for $a = l-n, b = l$, and $c = l+1$) readily yields

(30) $$F[l-n,l;l+1;1]=\binom{n}{l}^{-1}, \quad (0\leq l\leq n)$$

In view of (30), the special case of our formula (16) when $x=0$ gives the representation as follows:

(31) $$\mathcal{E}_n^{(\alpha)}(0;\lambda)=2^{\alpha}\sum_{l=0}^{n}\binom{\alpha+l-1}{l}(\lambda+1)^{-\alpha-l}(-\lambda)^{l}l!S(n,l)$$

Finally, if we apply the formulas (31) (replace λ by λ^{-1}), (10) (set $x=0$), (27) (set $x=\alpha$), and (3), we are led immediately to the formula (29) asserted by Theorem 3. ■

Corollary 3. *For $\alpha=1$ in Theorem 3, the following formula for the Apostol-Euler numbers involving the Stirling numbers of the second kind holds true:*

(32) $$\mathcal{E}_n(\lambda)=(-1)^{n}\sum_{k=0}^{n}\binom{n}{k}2^{k+1}\sum_{l=0}^{k}(-1)^{l}(\lambda+1)^{-l-1}l!S(k,l)$$

4. New Formulas for Euler Polynomials and Numbers of Higher Order

In this section, we will give some special cases and applications of $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ and $\mathcal{E}_n^{(\alpha)}(\lambda)$, thereby obtaining the corresponding formulas for the Euler polynomials and numbers of higher order, including the classical Euler polynomials and numbers. These results will further develop and supplement the contents of the recent monograph by H.M. Srivastava and Junesang Choi (see [5, p. 59-66]), concerning the Euler polynomials and numbers of higher order.

By setting $\lambda=1$ in (16) and (29), we (respectively) obtain the following formulas of the classical Euler polynomials and numbers of higher order:

(33) $$\begin{aligned}E_n^{(\alpha)}(x)=&\sum_{l=0}^{n}\binom{n}{l}\binom{\alpha+l-1}{l}2^{-l}\\&\times\sum_{k=0}^{l}(-1)^{k}\binom{l}{k}k^{l}(x+k)^{n-l}F[l-n,l;l+1;k/(x+k)]\end{aligned}$$

and

(34) $$E_n^{(\alpha)}=(-1)^{n}\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\sum_{l=0}^{k}\binom{\alpha+l-1}{l}(-1)^{l}2^{k-l}l!S(k,l)$$

Further, by setting $\alpha = 1$ in (33) and (34), we (respectively) obtain the following formulas of the classical Euler polynomials and numbers:

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} 2^{-l} \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^l (x+k)^{n-l} F[l-n, l; l+1; k/(x+k)] \tag{35}$$

and

$$E_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \sum_{l=0}^{k} (-1)^l 2^{k-l} l! S(k,l) \tag{36}$$

In addition, by setting $\lambda = 1$ in (24), we obtain the following formula of the Euler numbers of higher order:

$$E_n^{(\alpha)} = \sum_{j=0}^{n} (-1)^j \binom{n+\alpha}{j+\alpha} \binom{\alpha+j-1}{j} 2^{-j} \sum_{i=0}^{j} \binom{j}{i} (j-2i)^n \tag{37}$$

Further, by setting $\alpha = 1$ in (37), we obtain the following formula of the classical Euler numbers:

$$E_n = \sum_{j=0}^{n} (-1)^j \binom{n+1}{j+1} 2^{-j} \sum_{i=0}^{j} \binom{j}{i} (j-2i)^n \tag{38}$$

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