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THE DUAL BRUNN-MINKOWSKI INEQUALITIES FOR INTERSECTION BODIES AND TWO ADDITIONS

Shufeng Yuan, Jun Yuan and Gangsong Leng

Abstract. In this paper, some dual Brunn-Minkowski inequalities are established for intersection bodies for the harmonic Blaschke additions and p-radial additions.

1. INTRODUCTION

Intersection body is a basic concept in geometric tomography. The history of intersection bodies began with Busemann's theorem which has important implications for Busemann's theory of area in Finsler spaces [1]. Intersection bodies were first explicitly defined and named by Lutwak in the important paper [10], and played a key role in the ultimate solution of Busemann-Petty problem [2, 4-7, 12]. The duality between intersection bodies and projection bodies was first made clear in [4, 10], and the notion of mixed intersection bodies has been raised in [8, 11]. The idea of an intersection body is a relatively new one, and rather less is known about them than about projection bodies. The object of this paper is to establish the dual Brunn-Minkowski inequalities for intersection bodies for the harmonic Blaschke additions and p-radial additions.

Let φ_o^n be the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. For $K, L \in \varphi_o^n$, let $K\hat{+}L$ denote the harmonic Blaschke addition of K and L, $K\widetilde{+}_pL$ the p-radial addition of K and L. IK denotes the intersection body of K, $\widetilde{V}_1(K)$ the dual volume of K and V(K) the volume of K .

Our main results are the following two theorems.

Theorem 1.1. *Let $K, L \in \varphi_o^n, n \geq 2$, then*

$$\frac{\widetilde{V}_1(I(K\hat{+}L))^{\frac{n+1}{n-1}}}{V(K\hat{+}L)} \geq \frac{\widetilde{V}_1(IK)^{\frac{n+1}{n-1}}}{V(K)} + \frac{\widetilde{V}_1(IL)^{\frac{n+1}{n-1}}}{V(L)},$$

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with equality if and only if L is a dilatate of K.

Theorem 1.2. *Let $K, L \in \varphi_o^n$, $n \geq 2$.*

(i) *If $1 \leq p \leq n-1$, then*

$$V(I(K\tilde{+}_p L))^{\frac{p}{(n-1)n}} \leq V(IK)^{\frac{p}{(n-1)n}} + V(IL)^{\frac{p}{(n-1)n}}.$$

(ii) *If $p \geq (n-1)n$, then*

$$V(I(K\tilde{+}_p L))^{\frac{p}{(n-1)n}} \geq V(IK)^{\frac{p}{(n-1)n}} + V(IL)^{\frac{p}{(n-1)n}}.$$

The equalities of the above inequalities hold if and only if L is a dilatate of K.

This paper, except for the introduction, is divided into three sections. In Section 2, we provide some basic definitions and notations. We will give the proofs of the sharping of Theorem 1.1 and Theorem 1.2 in Sections 3 and 4 respectively.

2. Basic Definitions and Notations

As usual , S^{n-1} denotes the unit sphere, B the unit ball, and o the origin in Euclidean n-space $\mathbb{R}^n$. If u is a unit vector , that is , an element of S^{n-1} , we denote by $u^{\perp}$ the $(n-1)$-dimensional subspace orthogonal to u and $\mathbb{R}^+$ the positive real number set.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(K,\cdot) : S^{n-1} \longrightarrow \mathbb{R}$, defined for $u \in S^{n-1}$, by

$$\rho(K,u) = \rho_K(u) = Max\{\lambda \geq 0 : \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is continuous and positive , K will be called a star body.

Let φ_o^n denote the set of star bodies in $\mathbb{R}^n$ containing the origin in their interiors. Two star bodies $K, L \in \varphi_o^n$ are said to be dilatate (of one another) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$.

Let $L_j \in \varphi_o^n (1 \leq j \leq n)$. The dual mixed volume $\widetilde{V}(L_1, \ldots, L_n)$is defined by

$$\widetilde{V}(L_1, \ldots, L_n) = \frac{1}{n}\int_{S^{n-1}} \rho_{L_1}(u) \ldots \rho_{L_n}(u) du. \tag{2.1}$$

As with mixed volumes, we use the notation $\widetilde{V}(L_1, i_1; \ldots; L_n, i_n)$ to denote the dual mixed volume in which the set L_j appears i_j times.

For a special case, it will be convenient to relax the restriction on the numbers i_j ([3], p. 363). Define, for $i \in \mathbb{R}$,

$$\widetilde{V}_i(L_1, L_2) = \widetilde{V}(L_1, n-i, L_2, i) = \frac{1}{n}\int_{S^{n-1}} \rho_{L_1}(u)^{n-i} \rho_{L_2}(u)^i du. \tag{2.2}$$

Let $L \in \varphi_o^n$, $i \in \mathbb{R}$. The i-dual volume $\widetilde{V}_i(L)$ and i-dual quermassintegral $\widetilde{W}_{n-i}(L)$ of L are defined by

$$\widetilde{V}_i(L) = \widetilde{W}_{n-i}(L) = \widetilde{V}(L, i; B, n-i) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^i du. \tag{2.3}$$

In particular, we call $\widetilde{V}_1(L)$ *the dual volume of* L.

When $i = n$, we have

$$\widetilde{V}_n(L) = \frac{1}{n} \int_{S^{n-1}} \rho_L(u)^n du = \lambda_n(L), \tag{2.4}$$

where $\lambda_n(L)$ denote the n-dimensional Lebesgue measure of L according to the formula for volume in polar coordinates.

Let $K \in \varphi_o^n, n \geq 2$. The intersection body IK of K is a star body such that

$$\rho_{IK}(u) = v(K \cap u^{\perp}) = \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \rho_K(v)^{n-1} d\lambda_{n-2}(v), \tag{2.5}$$

where v is (n-1)-dimensional volume .

If $K_1, \ldots, K_{n-1} \in \varphi_o^n, n \geq 2$, then the mixed intersection body $I(K_1, \ldots, K_{n-1})$ of star bodies $K_1, \ldots, K_{n-1}$ is defined by

$$\begin{aligned} \rho_{I(K_1,\ldots,K_{n-1})}(u) &= \widetilde{v}(K_1 \cap u^{\perp}, \ldots, K_{n-1} \cap u^{\perp}) \\ &= \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \rho_{K_1}(v) \ldots \rho_{K_{n-1}}(v) d\lambda_{n-2}(v), \end{aligned} \tag{2.6}$$

where $\widetilde{v}$ is (n-1)-dimensional dual mixed volume .

If $K_1 = \ldots = K_{n-i-1} = K$, $K_{n-i} = \ldots = K_{n-1} = L$, then $I(K_1, \ldots, K_{n-1})$ will be denoted as $I_i(K, L)$. When $K = B$, $I_i(B, L)$ is called as the intersection body of order i of L which will often be written as $I_i K$. Specially, $I_{n-1}L$=IL.

Corresponding to the relaxation of restriction on number i of the dual mixed volume $\widetilde{V}_i(L_1, L_2)$, we can expand number i of $I_i(K, L)$ to the real set $\mathbb{R}$. Let $K, L \in \varphi_o^n, n \geq 2, i \in \mathbb{R}$. The mixed intersection body $I_i(K, L)$ of K, L is defined by

$$\begin{aligned} \rho_{I_i(K,L)}(u) &= \widetilde{v}(K \cap u^{\perp}, n-i-1; L \cap u^{\perp}, i) \\ &= \frac{1}{n-1} \int_{S^{n-1} \cap u^{\perp}} \rho_K(v)^{n-i-1} \rho_L(v)^i d\lambda_{n-2}(v). \end{aligned} \tag{2.7}$$

3. Inequalities for the Harmonic Blaschke Linear Combinations

Suppose $K, L \in \varphi_o^n$, and in this paper, we use $\lambda, \mu \geq 0$ denote that λ and μ are nonnegative real numbers and not both zero. To define the harmonic Blaschke linear combination, $\lambda K \hat{+} \mu L$, first define $\xi > 0$ by ([9])

$$\xi^{1/(n+1)} = \frac{1}{n}\int_{S^{n-1}}[\lambda V(K)^{-1}\rho(K,u)^{n+1} + \mu V(L)^{-1}\rho(L,u)^{n+1}]^{n/(n+1)}dS(u). \tag{3.1}$$

The body $\lambda K \hat{+} \mu L \in \varphi_o^n$ is defined as the body whose radial function is given by

$$\xi^{-1}\rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1}\rho(K,\cdot)^{n+1} + \mu V(L)^{-1}\rho(L,\cdot)^{n+1}. \tag{3.2}$$

In fact, we will establish two inequalities more general than Theorem 1.1 as follows.

Theorem 3.1. *Let $K, L, K_1, \ldots, K_{n-1} \in \varphi_o^n, n \geq 2, C = (K_1, \ldots, K_{n-1})$, $\lambda, \mu \geq 0$, then*

$$\frac{\widetilde{V}_1(C, I(\lambda K \hat{+} \mu L))^{\frac{n+1}{n-1}}}{V(\lambda K \hat{+} \mu L)} \geq \frac{\lambda}{V(K)} \cdot \widetilde{V}_1(C, IK)^{\frac{n+1}{n-1}} + \frac{\mu}{V(L)} \cdot \widetilde{V}_1(C, IL)^{\frac{n+1}{n-1}}, \tag{3.3}$$

with equality if and only if L is a dilatate of K.

Theorem 3.2. *Let $L, K_1, K_2 \in \varphi_o^n, n \geq 2, \lambda, \mu \geq 0$, $0 < i < n+1, i \in \mathbb{R}$, then*

$$\frac{\widetilde{V}_1(I_i(L, \lambda K_1 \hat{+} \mu K_2))^{\frac{n+1}{i}}}{V(\lambda K_1 \hat{+} \mu K_2)} \geq \frac{\lambda}{V(K_1)} \cdot \widetilde{V}_1(I_i(L, K_1))^{\frac{n+1}{i}} + \frac{\mu}{V(K_2)} \cdot \widetilde{V}_1(I_i(L, K_2))^{\frac{n+1}{i}}, \tag{3.4}$$

equality holds if and only if K_1 is a dilatate of K_2.

Remark. According to the definition of $I_i(K, L)$, we can expand number i to the real set $\mathbb{R}$. If $i \geq n+1$, then the reverse inequality of (3.4) holds.

To prove Theorem 3.1 and Theorem 3.2, we establish the following two lemmas.

Lemma 3.3. *If $L, K_1, K_2 \in \varphi_o^n, n \geq 2, \lambda, \mu \geq 0$, and $0 < i < n+1$, $i \in \mathbb{R}$, then*

$$\frac{\rho(I_i(L, \lambda K_1 \hat{+} \mu K_2), \cdot)^{\frac{n+1}{i}}}{V(\lambda K_1 \hat{+} \mu K_2)} \geq \frac{\lambda}{V(K_1)} \cdot \rho(I_i(L, K_1), \cdot)^{\frac{n+1}{i}} + \frac{\mu}{V(K_2)} \cdot \rho(I_i(L, K_2), \cdot)^{\frac{n+1}{i}}, \tag{3.5}$$

equality holds if and only if K_1 is a dilatate of K_2.

Proof. By (3.1), (3.2) and the polar coordinate formula for volume (2.4), we can get $\xi = V(\lambda K \hat{+} \mu L)$. Hence from (3.2), we obtain

$$V(\lambda K \hat{+} \mu L)^{-1}\rho(\lambda K \hat{+} \mu L, \cdot)^{n+1} = \lambda V(K)^{-1}\rho(K, \cdot)^{n+1} + \mu V(L)^{-1}\rho(L, \cdot)^{n+1}. \tag{3.6}$$

For $u \in S^{n-1}$. By(2.7), (3.6), we have

$$\begin{aligned}\rho(I_i(L, \lambda K_1 \hat{+} \mu K_2), u) &= \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho(\lambda K_1 \hat{+} \mu K_2, v)^i \rho(L, v)^{n-i-1} dv \\ &= \frac{1}{n-1} \cdot V(\lambda K_1 \hat{+} \mu K_2)^{\frac{i}{n+1}} \int_{S^{n-1}\cap u^\perp} \\ &\quad \left(\frac{\lambda}{V(K_1)} \cdot \rho(K_1, v)^{n+1} + \frac{\mu}{V(K_2)} \cdot \rho(K_2, v)^{n+1}\right)^{\frac{i}{n+1}} \\ &\quad \rho(L, v)^{n-i-1} dv\end{aligned}$$

Since $0 < i < n+1$, applying Minkowski integral inequality($0 < p = \frac{i}{n+1} < 1$) and (2.7), we have

$$\begin{aligned}&\rho(I_i(L, \lambda K_1 \hat{+} \mu K_2), u) \\ &\geq V(\lambda K_1 \hat{+} \mu K_2)^{\frac{i}{n+1}} \\ &\quad \cdot \left[\frac{\lambda}{V(K_1)}\left(\frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho(K_1, v)^i \rho(L, v)^{n-i-1} dv\right)^{\frac{n+1}{i}}\right. \\ &\quad \left. + \frac{\mu}{V(K_2)}\left(\frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho(K_2, v)^i \rho(L, v)^{n-i-1} dv\right)^{\frac{n+1}{i}}\right]^{\frac{i}{n+1}} \\ &= V(\lambda K_1 \hat{+} \mu K_2)^{\frac{i}{n+1}} \cdot \left(\frac{\lambda}{V(K_1)} \cdot \rho(I_i(L, K_1), u)^{\frac{n+1}{i}}\right. \\ &\quad \left. + \frac{\mu}{V(K_2)} \cdot \rho(I_i(L, K_2), u)^{\frac{n+1}{i}}\right)^{\frac{i}{n+1}}.\end{aligned}$$

Then,

$$\frac{\rho(I_i(L, \lambda K_1 \hat{+} \mu K_2), \cdot)^{\frac{n+1}{i}}}{V(\lambda K_1 \hat{+} \mu K_2)} \geq \frac{\lambda}{V(K_1)} \cdot \rho(I_i(L, K_1), \cdot)^{\frac{n+1}{i}} + \frac{\mu}{V(K_2)} \cdot \rho(I_i(L, K_2), \cdot)^{\frac{n+1}{i}}$$

By the equality conditions of Minkowski integral inequality, the equality of (3.5) holds if and only if $\frac{\lambda}{V(K_1)} \cdot \rho(K_1, \cdot)^{n+1}$ and $\frac{\mu}{V(K_2)} \cdot \rho(K_2, \cdot)^{n+1}$ are proportional, that is, K_1 is a dilatate of K_2. ■

Taking for $i = n-1$ in Lemma 3.3, we have

Lemma 3.4 *Let* $K, L \in \varphi_o^n, n \geq 2, \lambda, \mu \geq 0$*, then*

$$\rho(I(\lambda K \hat{+} \mu L), \cdot)^{\frac{n+1}{n-1}} \over V(\lambda K \hat{+} \mu L)} \geq \frac{\lambda \rho(IK, \cdot)^{\frac{n+1}{n-1}}}{V(K)} + \frac{\mu \rho(IL, \cdot)^{\frac{n+1}{n-1}}}{V(L)}, \tag{3.7}$$

with equality if and only if L *is a dilatate of* K*.*

Proof of Theorem 3.1. Let $\rho(C, \cdot) = \rho(K_1, \cdot) \ldots \rho(K_{n-1}, \cdot), a = \frac{\lambda \cdot V(\lambda K \hat{+} \mu L)}{V(K)}$, $b = \frac{\mu \cdot V(\lambda K \hat{+} \mu L)}{V(L)}$.

By (2.1), Lemma 3.4 ,and Minkowski integral inequality ($0 < \frac{n-1}{n+1} < 1$), we have

$$\begin{aligned}
\widetilde{V}_1(C, I(\lambda K \hat{+} \mu L)) &= \frac{1}{n} \int_{S^{n-1}} \rho(I(\lambda K \hat{+} \mu L), u) \rho(C, u) du \\
&\geq \frac{1}{n} \int_{S^{n-1}} \left(a \cdot \rho(IK, u)^{\frac{n+1}{n-1}} + b \cdot \rho(IL, u)^{\frac{n+1}{n-1}} \right)^{\frac{n-1}{n+1}} \rho(C, u) du \\
&\geq \left[\left(\frac{1}{n} \int_{S^{n-1}} a^{\frac{n-1}{n+1}} \rho(IK, u) \rho(C, u) du \right)^{\frac{n+1}{n-1}} \right. \\
&\quad \left. + \left(\frac{1}{n} \int_{S^{n-1}} b^{\frac{n-1}{n+1}} \rho(IL, u) \rho(C, u) du \right)^{\frac{n+1}{n-1}} \right]^{\frac{n-1}{n+1}} \\
&= \left(a \cdot \widetilde{V}_1(C, IK)^{\frac{n+1}{n-1}} + b \cdot \widetilde{V}_1(C, IL)^{\frac{n+1}{n-1}} \right)^{\frac{n-1}{n+1}}.
\end{aligned}$$

By the equality conditions of Minkowski integral inequality and Lemma 3.4, the equality (3.3) holds if and only if L is a dilatate of K. ■

Proof of Theorem 3.2. By (2.3), we get

$$\widetilde{V}_1(I_i(L, \lambda K_1 \hat{+} \mu K_2)) = \frac{1}{n} \int_{S^{n-1}} \rho(I_i(L, \lambda K_1 \hat{+} \mu K_2), u) du.$$

Since $0 < i < n+1$, using Lemma 3.3, Minkowski integral inequalities($0 < \frac{i}{n+1} < 1$) and (2.3), we have

$$
\begin{aligned}
&\widetilde{V}_1(I_i(L, \lambda K_1 \hat{+} \mu K_2)) \\
&\geq V(\lambda K_1 \hat{+} \mu K_2)^{\frac{i}{n+1}} \frac{1}{n} \int_{S^{n-1}} \left(\frac{\lambda}{V(K_1)} \cdot \rho(I_i(L, K_1), u)^{\frac{n+1}{i}} \right. \\
&\quad \left. + \frac{\mu}{V(K_2)} \cdot \rho(I_i(L, K_2), u)^{\frac{n+1}{i}} \right)^{\frac{i}{n+1}} du \\
&\geq V(\lambda K_1 \hat{+} \mu K_2)^{\frac{i}{n+1}} [\left(\frac{1}{n} \int_{S^{n-1}} (\frac{\lambda}{V(K_1)})^{\frac{i}{n+1}} \rho(I_i(L, K_1), u) du \right)^{\frac{n+1}{i}} \\
&\quad + \left(\frac{1}{n} \int_{S^{n-1}} (\frac{\lambda}{V(K_2)})^{\frac{i}{n+1}} \rho(I_i(L, K_2), u) du \right)^{\frac{n+1}{i}}]^{\frac{i}{n+1}} \\
&= V(\lambda K_1 \hat{+} \mu K_2)^{\frac{i}{n+1}} \left(\frac{\lambda}{V(K_1)} \cdot \widetilde{V}_1(I_i(L, K_1))^{\frac{n+1}{i}} + \frac{\mu}{V(K_2)} \cdot \widetilde{V}_1(I_i(L, K_2))^{\frac{n+1}{i}} \right)^{\frac{i}{n+1}}.
\end{aligned}
$$

So the inequality (3.4) is proved. ∎

Remark. Taking for $K_1 = \ldots = K_{n-1} = B$ and $\lambda = \mu = 1$ in Theorem 3.1, or taking for $i = n-1, \lambda = \mu = 1$ in Theorem 3.2, we can get Theorem 1.1 immediately.

4. Inequalities for the p-Radial Linear Combinations

Let K and L be star bodies in $\mathbb{R}^n$, $p \geq 1, \lambda, \mu \geq 0$. p-radial linear combination $\lambda \cdot K \tilde{+}_p \mu \cdot L$ is a star body whose radial function is given by

$$
\rho(\lambda \cdot K \tilde{+}_p \mu \cdot L, u)^p = \lambda \rho(K, u)^p + \mu \rho(L, u)^p. \tag{4.1}
$$

Note that " $\cdot$ " rather than " $\cdot_p$ " is written for radial scalar multiplication. Obviously, radial and Minkowski scalar multiplications are related by $\lambda \cdot K = \lambda^{1/p} K$.

For p-radial linear combination, we still will prove two results more general than Theorem 1.2.

Theorem 4.1. *Let $C, K, L \in \varphi_o^n, n \geq 2, \lambda, \mu \geq 0$, $i \in \mathbb{R}^+$.*

(i) *If $1 \leq p \leq i$, then*

$$
V(I_i(C, \lambda \cdot K \tilde{+}_p \mu \cdot L))^{\frac{p}{in}} \leq \lambda V(I_i(C, K))^{\frac{p}{in}} + \mu V(I_i(C, L))^{\frac{p}{in}}. \tag{4.2}
$$

(ii) If $p \geq max\{ni, 1\}$, then

$$V(I_i(C, \lambda \cdot K \widetilde{+}_p \mu \cdot L))^{\frac{p}{in}} \geq \lambda V(I_i(C, K))^{\frac{p}{in}} + \mu V(I_i(C, L))^{\frac{p}{in}}. \tag{4.3}$$

Equalities of the above two inequalities hold if and only if L is a dilatate of K.

Theorem 4.2. *Let $K,\ L, C \in \varphi_o^n$, $n \geq 2$, $i \in \mathbb{R}^+$.*

(i) If $1 \leq p \leq min\{n-1, (n-1)i\}$, then

$$\widetilde{V}_i(C, I(K \tilde{+}_p L))^{\frac{p}{(n-1)i}} \leq \widetilde{V}_i(C, IK)^{\frac{p}{(n-1)i}} + \widetilde{V}_i(C, IL)^{\frac{p}{(n-1)i}}. \tag{4.4}$$

(ii) If $p \geq max\{n-1, (n-1)i\}$, where $i > 0$, then

$$\widetilde{V}_i(C, I(K \tilde{+}_p L))^{\frac{p}{(n-1)i}} \geq \widetilde{V}_i(C, IK)^{\frac{p}{(n-1)i}} + \widetilde{V}_i(C, IL)^{\frac{p}{(n-1)i}}. \tag{4.5}$$

Equalities of the above two inequalities hold if and only if L is a dilatate of K.

To prove Theorem 4.1 and Theorem 4.2, we establish the following two lemmas.

Lemma 4.3. *Let $K,\ L,\ C \in \varphi_o^n, n \geq 2$, and $i \in \mathbb{R}^+, \lambda, \mu \geq 0$. If $1 \leq p \leq i$, then*

$$\rho(I_i(C, \lambda \cdot K \tilde{+}_p \mu \cdot L), \cdot)^{\frac{p}{i}} \leq \lambda \rho(I_i(C, K), \cdot)^{\frac{p}{i}} + \mu \rho(I_i(C, L), \cdot)^{\frac{p}{i}}. \tag{4.6}$$

If $p \geq max\{i, 1\}$, then the reverse inequality of the inequality (4.6)holds. Equality holds when $p \neq i$ if and only if L is a dilatate of K.

Proof. For $u \in S^{n-1}$, since $1 \leq p \leq i$, by (2.7), (4.1), Minkowski integral inequality ($\frac{i}{p} \geq 1$) and (2.7) again, we have

$$\begin{aligned}
\rho(I_i(C, \lambda \cdot K \tilde{+}_p \mu \cdot L), u) &= \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho(\lambda \cdot K \widetilde{+}_p \mu \cdot L, v)^i \rho(C, v)^{n-i-1} dv \\
&\leq [\lambda \left(\frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho(K, v)^i \rho(C, v)^{n-i-1} dv \right)^{\frac{p}{i}} \\
&\quad + \mu \left(\frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho(L, v)^i \rho(C, v)^{n-i-1} dv \right)^{\frac{p}{i}}]^{\frac{i}{p}} \\
&= \left(\lambda \rho(I_i(C, K), u)^{\frac{p}{i}} + \mu \rho(I_i(C, L), u)^{\frac{p}{i}} \right)^{\frac{i}{p}}.
\end{aligned}$$

So, when $1 \leq p \leq i$, the inequality of Lemma 4.2 is proved. By the same way, we can prove the reverse inequality of (4.4) holds for $p \geq max\{i, 1\}$. ∎

Taking for $i = n-1,\ \lambda = \mu = 1$ in Lemma 4.3, we have

Lemma 4.4. *Let* $K, L \in \varphi_o^n, n \geq 2$, *and* $1 \leq p \leq n-1$, *then*

$$\rho(I(K\tilde{+}_p L), \cdot)^{\frac{p}{n-1}} \leq \rho(IK, \cdot)^{\frac{p}{n-1}} + \rho(IL, \cdot)^{\frac{p}{n-1}}. \tag{4.7}$$

If $p \geq n-1$, *then the reverse inequality of the inequality (4.7)holds. Equality holds when* $p \neq n-1$ *if and only if* L *is a dilatate of* K.

Proof of Theorem 4.1. By (2.4), we have

$$V(I_i(C, \lambda \cdot K \widetilde{+}_p \mu \cdot L)) = \frac{1}{n}\int_{S^{n-1}} \left(\rho(I_i(C, \lambda \cdot K \widetilde{+}_p \mu \cdot L), u)^{\frac{p}{i}}\right)^{\frac{in}{p}} du.$$

For $1 \leq p \leq i$, applying (4.6), Minkowski integral inequality ($\frac{in}{p} > 1$) and (2.4), we infer that

$$\begin{aligned}
&V(I_i(C, \lambda \cdot K \widetilde{+}_p \mu \cdot L)) \\
&\leq \frac{1}{n}\int_{S^{n-1}} \left[\lambda\rho(I_i(C,K),u)^{\frac{p}{i}} + \mu\rho(I_i(C,L),u)^{\frac{p}{i}}\right]^{\frac{in}{p}} du \\
&\leq \left[\lambda V(I_i(C,K))^{\frac{p}{in}} + \mu V(I_i(C,L))^{\frac{p}{in}}\right]^{\frac{in}{p}}.
\end{aligned}$$

Then, the inequality (4.2) is proved.

In the same way, we can get the reverse inequality for $p \geq max\{ni, 1\}$.

From the equality conditions of Minkowski integral inequality, Theorem 4.1 holds with equality if and only if $\rho(I_i(C,K), \cdot)$ and $\rho(I_i(C,L), \cdot)$ are proportional. And by the equality condition of Lemma 4.3, the equality condition of Theorem 4.1 is that L is a dilatate of K . ■

Proof of Theorem 4.2. When $1 \leq p \leq min\{n-1, (n-1)i\}$, we have

$$\begin{aligned}
&\left(\widetilde{V}_i(C, I(K\tilde{+}_p L))\right)^{\frac{p}{(n-1)i}} \\
&= \left(\frac{1}{n}\int_{S^{n-1}} \rho_{I(K\tilde{+}_p L)}(u)^i \rho_C(u)^{n-i} dS(u)\right)^{\frac{p}{(n-1)i}} \\
&\leq \left(\frac{1}{n}\int_{S^{n-1}} (\rho_{IK}(u)^{\frac{p}{n-1}} + \rho_{IL}(u)^{\frac{p}{n-1}})^{\frac{(n-1)i}{p}} \rho_C(u)^{n-i} dS(u)\right)^{\frac{p}{(n-1)i}} \\
&\leq \widetilde{V}_i(C, IK)^{\frac{p}{(n-1)i}} + \widetilde{V}_i(C, IL)^{\frac{p}{(n-1)i}}.
\end{aligned}$$

In the same way, we can get the reverse inequality for $p \geq max\{n-1, (n-1)i\}$.

From the equality conditions of Minkowski integral inequality and Lemma 4.4, the equality condition of Theorem 4.2 is that L is a dilatate of K. $\square$

Remark. Taking for $\lambda = \mu = 1, i = n-1$ in Theorem 4.1, or taking for $i = n$ in Theorem 4.2, we can get Theorem 1.2 immediately.

Taking for $C = B, \lambda = \mu = 1$ in Theorem 4.1 and in Theorem 4.2 , we have two corollaries as follows.

Corollary 4.3. *Let $K, L \in \varphi_o^n$, $i \in \mathbb{R}^+$. If $1 \le p \le i$, then*

$$V(I_i(K \widetilde{+}_p L))^{\frac{p}{in}} \le V(I_i K)^{\frac{p}{in}} + V(I_i L)^{\frac{p}{in}} \tag{4.8.}$$

The reverse inequality holds when $p \ge max\{in, 1\}$. Equality holds if and only if L is a dilatate of K.

Corollary 4.4. *Let $K, L \in \varphi_o^n$, $i \in \mathbb{R}^+$. If $1 \le p \le min\{n-1, (n-1)i\}$, then*

$$\widetilde{W}_{n-i}(I(K \tilde{+}_p L))^{\frac{p}{(n-1)i}} \le \widetilde{W}_{n-i}(IK)^{\frac{p}{(n-1)i}} + \widetilde{W}_{n-i}(IL)^{\frac{p}{(n-1)i}}. \tag{4.9}$$

If $p \ge max\{n-1, (n-1)i\}$, then the reverse inequality of (4.9) holds. Equality holds if and only if L is a dilatate of K.

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Shufeng Yuan
Department of Mathematics,
Shangyu College,
Shaoxing University,
Zhejiang 312367,
P. R. China.
E-mail: yuanshufeng2003@163.com

Jun Yuan and Gangsong Leng
Department of Mathematics,
Shanghai University,
Shanghai 200444,
P. R. China.