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COMPLETELY CONTINUOUS SUBSPACES OF OPERATOR IDEALS

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Abstract. Ülger, Saksman and Tylli have shown that if X is a reflexive Banach space and $\mathcal{A}$ is a subalgebra of $K(X)$ such that $\mathcal{A}^*$ has the Schur property, then $\mathcal{A}$ is completely continuous. Here by introducing the concept of a strongly completely continuous subspace of an operator ideal, we improve their results. In particular, when X is an l_p- direct sum and Y is an l_q- direct sum of finite-dimensional Banach spaces with $1 < p \leq q < \infty$, we give a characterization of Schur property of the dual $\mathcal{M}^*$ of a closed subspace $\mathcal{M} \subseteq K(X, Y)$ in terms of strong complete continuity of $\mathcal{M}$.

1. Introduction

A Banach space X has the Schur property if every weakly convergent sequence in X converges in norm. There are many Banach spaces with the Schur property. For example the space l^1 of absolutely summable sequences has this property. In 1995, S. W. Brown [1] proved that if $\mathcal{A}$ is a commutative closed subalgebra of the algebra $K(H)$ of all compact operators on a Hilbert space H, that satisfies a very mild condition of density, then the dual $\mathcal{A}^*$ of $\mathcal{A}$ has the Schur property. Following this work of S. W. Brown, A. Ülger [8], characterized all closed subspaces of $K(H)$ such that their duals have the Schur property. He also proved that for a closed subalgebra $\mathcal{A}$ of $K(X)$ of all compact operators on a reflexive Banach space X, the Schur property of $\mathcal{A}^*$ is a sufficient condition for the complete continuity of $\mathcal{A}$ that is, all left and right multiplication operators of elements in $\mathcal{A}$ are compact operators on $\mathcal{A}$. In [7] E. Saksman and H. O. Tylli gave a new proof of this result. Furthermore, if $\mathcal{A}$ is commutative, then it is completely continuous [8]. Here, we introduce the concept of strongly completely continuous subspaces of the space of operator ideals and generalize the results of [7] and [8]. We also obtain

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a characterization of this concept in terms of relative compactness of all its point evaluations related to that subspace. Moreover, when X is either an l_p- or c_0- direct sum of finite-dimensional Banach spaces with $1 < p < \infty$, we show that if $\mathcal{A}$ is a completely continuous subalgebra of $K(X)$ which satisfies a certain density condition, then $\mathcal{A}^*$ has the Schur property.

The notations and terminology concerning Banach spaces are standard. Throughout this article H is a Hilbert space and X, Y, Z and W denote arbitrary Banach spaces. The closed unit ball of a Banach space X is denoted by X_1 and X^* is the dual of X. The duality between X and X^* is denoted by $\langle x, x^* \rangle$ and T^* refers to the adjoint of the operator T. $(\mathcal{U}, A)$ is always a (Banach) operator ideal $\mathcal{U}$ with norm A and its components are denoted by $\mathcal{U}(X, Y)$. For arbitrary Banach spaces X and Y we use $L(X, Y)$ and $K(X, Y)$ for Banach spaces of all bounded and compact linear operators between Banach spaces X and Y, respectively. The abbreviation $\mathcal{U}(X)$ is used for $\mathcal{U}(X, X)$. The projective tensor product of X and Y is denoted by $X\tilde{\otimes}_\pi Y$. We refer the reader to [4] and [5] for undefined terminology.

2. Strongly Completely Continuous Subspaces

For a subspace $\mathcal{M} \subseteq \mathcal{U}(X, Y)$, one can find the point evaluations related to $\mathcal{M}$ by $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ and $\widetilde{\mathcal{M}}_1(y^*) = \{T^*y^* : T \in \mathcal{M}_1\}$, where $x \in X$ and $y^* \in Y^*$. In Section 2 of [6], the authors proved that for many Banach spaces X and Y and a closed subspace $\mathcal{M} \subseteq \mathcal{U}(X, Y)$, if $\mathcal{M}^*$ has the Schur property, then all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively (norm) compact in Y and X^*, respectively. Thus the results of this section, in many cases, extend those in [7] and [8].

We recall that a subalgebra $\mathcal{A}$ of $\mathcal{U}(X)$ is completely continuous if for each $S \in \mathcal{A}$, the left and right multiplications L_S and R_S are compact operators on $\mathcal{A}$, where $L_S(T) = ST$ and $R_S(T) = TS$. We give a refinement of this concept for subspaces of $\mathcal{U}(X, Y)$:

Definition 2.1. A linear subspace $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ is called strongly completely continuous in $K(X, Y)$ (resp., $\mathcal{U}(X, Y)$) if for all Banach spaces W and Z and all compact operators $R : Y \to W$ and $S : Z \to X$, the left and right multiplication operators L_R and R_S as operators from $\mathcal{M}$ into $K(X, W)$ and $K(Z, Y)$ (resp., $\mathcal{U}(X, W)$ and $\mathcal{U}(Z, Y)$) respectively, are compact.

It is trivial that the strong complete continuity implies complete continuity in the case of closed subalgebras $\mathcal{A} \subseteq K(X)$. In Example 2.8 we will show that the converse is not true in general. In the following theorem we present a wide class of subspaces of $L(X, Y)$ with strong complete continuity.

Theorem 2.2. *Let $\mathcal{M}$ be a linear subspace of $L(X,Y)$ such that all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact. Then $\mathcal{M}$ is strongly completely continuous in $K(X,Y)$.*

Proof. It is enough to prove that $R\mathcal{M}_1$ and $\mathcal{M}_1 S$ are relatively compact in $K(X,W)$ and $K(Z,Y)$, respectively, where $R: Y \to W$ and $S: Z \to X$ are compact operators.

Let $\mathcal{M}_1(X_1) = \{Tx : T \in \mathcal{M}_1,\ x \in X_1\}$ and let Ω be the norm closure of $R(\mathcal{M}_1(X_1))$ in W. Since R is a compact operator, Ω is a compact subset of W. This shows that the set $\mathcal{F}$ of all restrictions of elements of W_1^* to Ω is an equicontinuous subset of $C(\Omega)$ and by the classical theorem of Ascoli, it is relatively compact in $C(\Omega)$. Now fix an arbitrary sequence $(T_n) \subseteq \mathcal{M}_1$. For each $\varepsilon > 0$ one can find a finite $\varepsilon/3$- net $w_1^*, ..., w_l^*$ for $\mathcal{F}$. Since $\widetilde{\mathcal{M}}_1(R^* w_k^*)$, $1 \le k \le l$, are relatively compact in X^*, the sequence (T_n) has a subsequence, which is denoted again by (T_n), such that for all sufficiently large m, n,

$$\|T_n^*(R^* w_k^*) - T_m^*(R^* w_k^*)\| < \varepsilon/3, \text{ for all } 1 \le k \le l.$$

This shows that for each $x \in X_1,\ w^* \in W_1^*$ and suitable $1 \le k \le l$,

$$|\langle w^*, (RT_n - RT_m)x\rangle| \le |\langle w^* - w_k^*, RT_n x\rangle|$$

$$+|\langle w_k^*, (RT_n - RT_m)x\rangle| + |\langle w_k^* - w^*, RT_m x\rangle|$$

$$\le \varepsilon/3 + \|T_n^*(R^* w_k^*) - T_m^*(R^* w_k^*)\| + \varepsilon/3 < \varepsilon.$$

Thus $\|RT_n - RT_m\| < \varepsilon$, for sufficiently large $m,\ n$ and so $R\mathcal{M}_1$ is relatively compact.

Similarly, since $\widetilde{\mathcal{M}} \subseteq L(Y^*, X^*)$ and S is compact, then Φ, the norm closure of $S^*(\widetilde{\mathcal{M}}_1(Y_1^*))$, is compact in Z^*, hence again by Ascoli's theorem the set $\mathcal{G}$ of all restriction of elements of Z_1 to Φ is a relatively compact subset of $C(\Phi)$. Now, if $(T_n) \subseteq \mathcal{M}_1$ and $\varepsilon > 0$ are given and $z_1, ..., z_l \in Z_1$ is a finite $\varepsilon/3$- net for $\mathcal{G}$, then by a method similar to that of the last paragraph together with the relative compactness of all $\mathcal{M}_1(Sz_k),\ 1 \le k \le l$, we conclude that $|\langle (S^*T_n^* - S^*T_m^*)y^*, z\rangle| < \varepsilon$, for all $z \in Z_1,\ y^* \in Y_1^*$ and all sufficiently large m, n. Hence $\|S^*T_n^* - S^*T_m^*\| < \varepsilon$, for sufficiently large $m,\ n$. This proves that the set $\widetilde{\mathcal{M}_1 S} = S^*\widetilde{\mathcal{M}}_1$ is relatively compact and so $\mathcal{M}_1 S$ is relatively compact. ■

As a corollary, we extend Theorem 2.2 to some class of operator ideals. We recall that an operator ideal $\mathcal{U}$ is closed if its components $\mathcal{U}(X,Y)$ are closed in $L(X,Y)$.

Corollary 2.3. *Let $\mathcal{U}$ be a closed operator ideal and $\mathcal{M}$ be a linear subspace of $\mathcal{U}(X,Y)$ such that all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact. Then $\mathcal{M}$ is strongly completely continuous in $\mathcal{U}(X,Y)$.*

Proof. We first note that by the definition of operator ideal, L_R and R_S are operators into $\mathcal{U}(X,W)$ and $\mathcal{U}(Z,Y)$, respectively. Now as $\mathcal{M}$ is a linear subspace of $L(X,Y)$, by Theorem 2.2, $R\mathcal{M}_1$ and $\mathcal{M}_1 S$ are relatively compact in $K(X,W)$ and $K(Z,Y)$, respectively. But $\mathcal{U}(X,W)$ and $\mathcal{U}(Z,Y)$ are closed in $L(X,W)$ and $L(Z,Y)$ respectively and therefore the proof is completed. ∎

Now we will prove that the converse of the above result is also valid in every operator ideal $\mathcal{U}$.

Theorem 2.4. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{U}(X,Y)$ such that for some Banach spaces W and Z, the operators $L_R : \mathcal{M} \to \mathcal{U}(X,W)$ and $R_S : \mathcal{M} \to \mathcal{U}(Z,Y)$ are compact for all finite-rank operators $R : Y \to W$ and $S : Z \to X$. Then all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact.*

Proof. We only prove the relative compactness of $\mathcal{M}_1(x)$. The proof of the relative compactness of $\widetilde{\mathcal{M}}_1(y^*)$ is the same. Let $x \in X$ be arbitrary. Fix a normalized element $z \in Z$ and choose a normalized element $z^* \in Z^*$ such that $z^*(z) = 1$. If we set $S = z^* \otimes x$, then $S(z) = x$ and by assumption $\mathcal{M}_1 S$ is relatively compact in $\mathcal{U}(Z,Y)$. So $\mathcal{M}_1(x) = (\mathcal{M}_1 S)(z)$ is relatively compact in Y. ∎

From Corollary 2.3 and Theorem 2.4, we deduce the following result.

Corollary 2.5. *Let $\mathcal{U}$ be a closed operator ideal and $\mathcal{M}$ be a linear subspace of $\mathcal{U}(X,Y)$. Then the following assertions are equivalent:*

(a) *All of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}}_1(y^*)$ are relatively compact in Y and X^* respectively.*

(b) *$\mathcal{M}$ is strongly completely continuous in $\mathcal{U}(X,Y)$.*

(c) *$\mathcal{M}$ is strongly completely continuous in $K(X,Y)$.*

(d) *For some Banach spaces W and Z, the operators $L_R : \mathcal{M} \to \mathcal{U}(X,W)$ and $R_S : \mathcal{M} \to \mathcal{U}(Z,Y)$ (or into $K(X,W)$ and $K(Z,Y)$) are compact for all finite-rank operators $R : Y \to W$ and $S : Z \to X$.*

Remark. In the case when X is an l_p- direct sum and Y is an l_q- direct sum of finite-dimensional Banach spaces with $1 < p \leq q < \infty$, and $\mathcal{M}$ is a closed subspace of $K(X,Y)$, then by Theorem 2.3 (or Theorem 2.5) and Corollary 3.5 of

[6], the assertions of this corollary are also equivalent to the Schur property of $\mathcal{M}^*$. Thus the above Corollary refines Corollary 4, Theorem 4 and Theorem 6 of [7].

The following Theorem extends Theorem 5 of [8] to a not necessarily reflexive Banach space X.

Theorem 2.6. *Let X be an arbitrary Banach space. Then every commutative subalgebra $\mathcal{A}$ of $K(X)$ is completely continuous.*

Proof. Let $\Omega = X_1^{**}$ endowed with the relative weak*- topology of X^{**}. Since for each $T \in K(X)$, $T^{**}(X^{**}) \subseteq X$, we can embed $K(X)$ isometrically into the Banach space $C(\Omega, X)$ of all continuous X- valued functions on Ω with the sup norm. So it is enough to show that, for each fixed $S \in \mathcal{A}_1$, the set $\mathcal{A}_1 S = \{TS : T \in \mathcal{A}_1\}$, when identified by its image in $C(\Omega, X)$, is a relatively compact subset of $C(\Omega, X)$. This is straightforward by the vector-valued version of Ascoli's theorem. As $\mathcal{A}$ is commutative, by the compactness of S, the set $\mathcal{A}_1 S(x^{**}) = S^{**}(\{T^{**}x^{**} : T \in \mathcal{A}_1\})$ is relatively compact in X for all $x^{**} \in X^{**}$. Also by the compactness of S , the restricted operator $S^{**}|_{\Omega} : \Omega \to X$ is weak*-norm continuous and so for each $\varepsilon > 0$ there exists a weak* neighborhood V of 0 in Ω such that $||S^{**}x^{**}|| < \varepsilon$ for all $x^{**} \in V$. This shows that $\mathcal{A}_1 S$ is equicontinuous on Ω, because $||(TS)^{**}x^{**}|| < \varepsilon$ for all $T \in \mathcal{A}_1$ and all $x^{**} \in V$. ■

In the rest of this article we always assume that $\mathcal{A}$ is a subalgebra of $\mathcal{U}(X)$ such that $span\mathcal{A}(X)$ and $span\widetilde{\mathcal{A}}(X^*)$ are dense in X and X^*, respectively, where $\widetilde{\mathcal{A}} = \{S^* : S \in \mathcal{A}\}$. In this case, we say that $\mathcal{A}$ satisfies the density condition. We conclude the article by proving a result similar to Theorem 1.1 of [1] and Theorem 7 of [8] for some reflexive and nonreflexive Banach spaces. We prove that the same conclusion is valid for closed subalgebras of $K(X)$ where X is either an l_p- or c_0-direct sum of finite-dimensional Banach spaces with $1 < p < \infty$.

In the following we obtain a refinement of Theorem 7 of [8]. Let us recall that when $(\mathcal{U}, A)$ is an operator ideal, then $\|T\| \leq A(T)$ for each $T \in \mathcal{U}(X)$.

Theorem 2.7. *Let X be an arbitrary Banach space and $\mathcal{A}$ be a completely continuous subalgebra of $\mathcal{U}(X)$ that satisfies the density condition. Then all of the point evaluations $\mathcal{A}_1(x)$ and $\widetilde{\mathcal{A}}_1(x^*)$ are relatively compact.*

Proof. We follow the technique given for the proof of Theorem 7 of [8]. Let $x \in X$ and $\varepsilon > 0$ be given. By the density condition of $\mathcal{A}$, there exists an element $y = \lambda_1 T_1(x_1) + ... + \lambda_n T_n(x_n)$ in X such that $\|x - y\| < \varepsilon$. As $\mathcal{A}$ is completely continuous, the last remark shows that, each of the sets $\lambda_1(\mathcal{A}_1 T_1)(x_1), \lambda_2(\mathcal{A}_1 T_2)(x_2), ..., \lambda_n(\mathcal{A}_1 T_n)(x_n)$ is relatively compact in X. It follows that the set $K_\varepsilon = \lambda_1(\mathcal{A}_1 T_1)(x_1) + \lambda_2(\mathcal{A}_1 T_2)(x_2) + ... + \lambda_n(\mathcal{A}_1 T_n)(x_n)$ is also relatively compact in X and $\mathcal{A}_1(x) \subseteq \mathcal{A}_1(x-y)+\mathcal{A}_1(y) \subseteq \varepsilon X_1 + K_\varepsilon$. Hence $\mathcal{A}_1(x)$ is a relatively compact subset of X. Similarly, using $(TS)^* = S^*T^*$, the density assumption and the relative compactness of $T\mathcal{A}_1$, we can show as above that for each x^* in X^*, the set $\widetilde{\mathcal{A}_1}(x^*)$ is relatively compact in X^*. ■

In the following example, we will show that the density condition of Brown and Ülger is essential in our Theorem 2.7. Moreover, this example provides a closed commutative subalgebra which is completely continuous but not strongly completely continuous.

Example 2.8. *Let (e_n) be the standard orthonormal basis in l^2. Put $H_1 = [e_{2n} : n = 1, 2, ...]$ and $H_2 = [e_{2n+1} : n = 0, 1, 2, ...]$.*

$$\text{Take } \mathcal{A} = \{\begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix} : U \in K(H_1, H_2)\} \subset K(l^2).$$

Here $ST = 0$ whenever $S, T \in \mathcal{A}$, so that $\mathcal{A} \subset K(l^2)$ is a closed commutative subalgebra, which is trivially completely continuous ($L_S = R_S = 0$ for $S \in \mathcal{A}$). The closed linear span $[Sx : S \in \mathcal{A}, x \in l^2]$ equals H_2, so that the density condition fails. Moreover, $e_{2n+1} \in \mathcal{A}_1(e_1)$ for $n = 0, 1, ...$, so that $\mathcal{A}_1(e_1)$ is not relatively compact in l^2. Therefore the complete continuity of $\mathcal{A} \subseteq K(l^2)$ does not imply the relative compactness of the point evaluations in the absence of the density condition. On the other hand, Corollary 2.5 implies that $\mathcal{A}$ is not strongly completely continuous in $K(l^2)$. Thus, the strong complete continuity is a strictly stronger notion than the complete continuity for closed subalgebras $\mathcal{A} \subseteq K(l^2)$.

As a consequence of the above result we establish that if $\mathcal{A}$ is a subalgebra of $K(X)$, then the converse of Theorem 2.7 is also valid.

Corollary 2.9. *Let X be a Banach space and $\mathcal{A} \subseteq K(X)$ be a closed subalgebra that satisfies the density condition. Then $\mathcal{A}$ is completely continuous if and only if all of the point evaluations $\mathcal{A}_1(x)$ and $\widetilde{\mathcal{A}}_1(x^*)$ are relatively compact.*

Proof. The sufficiency condition deduces from Corollary 2.5, because every strongly completely continuous subalgebra of $K(X)$ is completely continuous. The necessity condition is a direct consequence of Theorem 2.7. ■

Remark. When $\mathcal{A}$ is a commutative subalgebra of $K(X)$ that satisfies the density condition, Theorem 2.6 implies that $\mathcal{A}$ is completely continuous. Now from Corollary 2.9 we deduce that all point evaluations $\mathcal{A}_1(x)$ and $\widetilde{\mathcal{A}}_1(x^*)$ are relatively compact. In particular, when X is either an l_p- or c_0- direct sum of finite-dimensional Banach spaces with $1 < p < \infty$, Corollaries 3.5 and 3.6 of [6] imply the Schur property of $\mathcal{A}^*$. This improves the main theorem of [1] for a subalgebra $\mathcal{A}$ of $K(X)$ for some Banach space X instead of $K(H)$. We also remark that when X is an l_p- direct sum of finite-dimensional Banach spaces and $\mathcal{A}$ is a commutative closed subalgebra of $K(X)$, then one can prove the relative compactness of all point evaluations related to $\mathcal{A}$ by the same methods as for Lemmas 1.2 and 1.4 of [1]. In fact, since X has the RNP and the approximation property, by Proposition 16.7 of [5], $\mathcal{N}(X^*)^* = L(X) = K(X)^{**}$, where $\mathcal{N}(X^*)$ is the operator ideal of all nuclear operators on X^*, which is essential in the proof of these lemmas.

The following corollary is an improvement of Theorem 6 of [7] and Theorem 7 of [8] for a subalgebra of $K(X)$ for some Banach space X instead of $K(H)$.

Corollary 2.10. *Let X be either an l_p- or c_0- direct sum of finite dimensional Banach spaces with $1 < p < \infty$. If $\mathcal{A}$ is a completely continuous subalgebra of $K(X)$ that satisfies the density condition, then $\mathcal{A}^*$ has the Schur property.*

Proof. By Corollary 2.9, all point evaluations are relatively compact. Now an appeal to Corollaries 3.5 and 3.6 of [6] completes the proof. ∎

We conclude this paper by an application of our results for the class of all compact operators on special Banach spaces which improves Theorem 6 of [7] and Theorem 7 of [8].

Corollary 2.11. *Let X be an l_p- direct sum of finite-dimensional Banach spaces with $1 < p < \infty$. Let $\mathcal{A}$ be a closed subalgebra of $K(X)$ that satisfies the above density condition. Then the following assertions are equivalent:*

- (a) *$\mathcal{A}$ has the Dunford- Pettis property.*
- (b) *$\mathcal{A}^*$ has the Schur property.*
- (c) *$\mathcal{A}$ is completely continuous.*
- (d) *All of the point evaluations $\mathcal{A}_1(x)$ and $\widetilde{\mathcal{A}}_1(x^*)$ are relatively compact in X and X^* respectively.*
- (e) *$\mathcal{A}$ is strongly completely continuous in $K(X)$.*
- (f) *For some Banach spaces Y and Z, the operators $L_R : \mathcal{A} \to K(X, Y)$ and $R_S : \mathcal{A} \to K(Z, X)$ are compact for all finite-rank operators $R : X \to Y$ and $S : Z \to X$.*

Proof. Since by Corollary 1.12 of [2], $K(X)$ contains no copy of l_1, the statements (a) and (b) are equivalent by [3]. (b) implies (c) by Proposition 6 of [8]. (c) implies (d) by Theorem 2.7. (d) implies (b) by Corollary 3.5 of [6] and finally, by Corollary 2.5, (d), (e) and (f) are equivalent. ∎

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