TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 3, pp. 613-629, March 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

(p, q, r)-GENERATIONS OF THE SPORADIC GROUP HN

Ali Reza Ashrafi

Abstract. A finite group G is called (l, m, n)-generated, if it is a quotient group of the triangle group $T(l, m, n) = \langle x, y, z | x^l = y^m = z^n = xyz = 1 \rangle$.

In [16], the question of finding all triples (p, q, r) such that non-abelian finite simple group G is $(p, q, r)-$generated was posed. In this paper we partially answer this question for the sporadic group HN. In fact, we prove that the sporadic group HN is $(p, q, r)-$generated if and only if $(p, q, r) \neq (2, 3, 5)$, where p, q and r are prime divisors of $|HN|$ and $p < q < r$.

1. Introduction

A group G is said to be (l, m, n)-generated if it can be generated by two elements x and y such that $o(x) = l, o(y) = m$ and $o(xy) = n$. In this case G is the quotient of the triangle group $T(l, m, n)$ and for any permutation π of S_3, the group G is also $((l)\pi, (m)\pi, (n)\pi)$-generated. Therefore we may assume that $l \leq m \leq n$. By [4], if the non-abelian simple group G is (l, m, n)-generated, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Hence for a non-abelian finite simple group G and divisors l, m, n of the order of G such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, it is natural to ask if G is a (l, m, n)-generated group. The motivation for this question came from the calculation of the genus of finite simple groups [22]. It can be shown that the problem of finding the genus of a finite simple group can be reduced to one of generations(for details see [19]).

In a series of papers, [12-17] Moori and Ganief established all possible $(p, q, r)-$generations, p, q, r are distinct primes, of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2, and F_{22}. Also, Ashrafi and his co-authors in [2,3] and [7-11], did the same for the sporadic groups Co_1, Th, $O'N$, Ly and He. The motivation for this study is outlined in these papers and the reader is encouraged to consult these papers for background material as well as basic computational techniques.

Received October 9, 2002; accepted May 21, 2004.
Communicated by Pjek-Hwee Lee.
2000 *Mathematics Subject Classification*: Primary 20D08, 20F05.
Key words and phrases: Harada-Norton group, $(p, q, r)-$Generation, Triangle group.

Throughout this paper we use the same notation as in [1, 7, 9, 10]. In particular, $\Delta(G) = \Delta(lX, mY, nZ)$ denotes the structure constant of G for the conjugacy classes lX, mY, nZ, whose value is the cardinality of the set $\Lambda = \{(x, y)|xy = z\}$, where $x \in lX, y \in mY$ and z is a fixed element of the conjugacy class nZ. In Table 2, we list the values $\Delta(pX, qY, rZ)$, p, q and r distinct prime divisors of $|HN|$, using the character table HN. Also, $\Delta^\star(G) = \Delta^\star_G(lX, mY, nZ)$ and $\Sigma(H_1 \cup H_2 \cup \cdots \cup H_r)$ denote the number of pairs $(x, y) \in \Lambda$ such that $G = \langle x, y\rangle$ and $\langle x, y\rangle \subseteq H_i$ (for some $1 \leq i \leq r$), respectively. The number of pairs $(x, y) \in \Lambda$ generating a subgroup H of G will be given by $\Sigma^\star(H)$ and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general conjugacy class of a subgroup H of G with elements of order n will be denoted by nx. Clearly, if $\Delta^\star(G) > 0$, then G is (lX, mY, nZ)-generated and (lX, mY, nZ) is called a generating triple for G. The number of conjugates of a given subgroup H of G containing a fix element z is given by $\chi_{N_G(H)}(z)$, where $\chi_{N_G(H)}$ is the permutation character of G with action on the conjugates of H(cf. [20]). In most cases we will calculate this value from the fusion map from $N_G(H)$ into G stored in GAP, [18].

Let G be a group and nX a conjugacy class of elements of order n in G. Following Woldar [21], the group G is said to be nX-complementary generated if, for any arbitrary non-identity element $x \in G$, there exists a $y \in nX$ such that $G =< x, y >$. The element $y = y(x)$ for which $G =< x, y >$ is called complementary.

Now we discuss techniques that are useful in resolving generation type questions for finite groups. We begin with a result of [5] that, in certain situations, is very effective at establishing non-generations.

Theorem 1.1. *Let G be a finite centerless group and suppose lX, mY and nZ are G-conjugacy classes for which $\Delta^\star(G) = \Delta^\star_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then $\Delta^\star(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

Further useful results that we shall use are:

Lemma 1.2. ([14]). *Let G be a $(2X, sY, tZ)-$generated simple group then G is $(sY, sY, (tZ)^2)-$generated.*

Lemma 1.3. *Let G be a finite simple group and H a maximal subgroup of G containing a fixed element x. Then the number h of conjugates of H containing x is $\chi_H(x)$, where χ_H is the permutation character of G with action on the conjugates of H. In particular,*

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, x_2, \cdots, x_m$ are representatives of the H-conjugacy classes that fuse to the G-conjugacy class of x.

We calculated in Table 3, the value h for suitable conjugacy classes of the group HN.

Lemma 1.4. ([14]). *Let G be a finite group and let l, m and n be integers that are pairwise coprime. Then for any integer t coprime to n, we have*

$$\Delta(lX, mY, nZ) = \Delta(lX, mY, (nZ)^t).$$

Moreover, G is $(lX, mY, nZ)-$generated if and only if G is $(lX, mY, (nZ)^t)-$generated.

Lemma 1.5. ([4]). *Suppose a and b are permutations of N points such that a has λ_u cycles of length u (for $1 \leq u \leq l$) and b has μ_v cycles of length v (for $1 \leq v \leq m$) and their product ab is an involution having k transpositions and $N - 2k$ fixed points. If a and b generate a transitive group on these N points, then there exists a non-negative integer p such that*

$$k = 2p - 2 + \sum_{1 \leq v \leq m} \lambda_u + \sum_{1 \leq v \leq m} \mu_v.$$

Throughout this paper our notation is standard and taken mainly from $[1, 12, 13]$. *In this paper, we will prove the following theorem:*

Theorem. *The Harada-Norton group HN is $(p, q, r)-$generated if and only if $(p, q, r) \neq (2, 3, 5)$.*

2. $(p, q, r)-$Generations for HN

In this section we obtain all of triples $(p, q, r)-$generations of the group HN. We will use the maximal subgroups of HN listed in the ATLAS extensively, especially those with order divisible by 19. We listed in Table 1, all the maximal subgroups of HN and in Table 3, the fusion maps of these maximal subgroups into HN (obtained from GAP) that will enable us to evaluate $\Delta^\star_{HN}(pX, qY, rZ)$, for prime classes pX, qY and rZ. In this table h denotes the number of conjugates of the maximal subgroup H containing a fixed element z; see Lemma 1.3. For basic properties of the group HN and information on its maximal subgroups the reader is referred to [6]. It is a well known fact that HN has exactly 14 conjugacy classes of maximal subgroups, as listed in Table 1.

2.1. $(2, p, q)-$Generations for HN

If the group HN is $(2, 3, p)-$generated, then by Conder's result [4], $\frac{1}{2}+\frac{1}{3}+\frac{1}{p} < 1$. Thus we only need to consider the cases $p = 7, 11, 19$. Woldar, in [22] determined which sporadic groups other than Fi_{22}, F_{23}, F'_{24}, Th, J_4, B and M are Hurwitz

Table 1. The Maximal Subgroups of HN.

$Group$	$Order$	$Group$	$Order$
A_{12}	$2^9.3^5.5^2.7.11$	$2.HS.2$	$2^{11}.3^2.5^3.7.11$
$U_3(8).3_1$	$2^9.3^5.7.19$	$2^{1+8}.(A_5 \times A_5).2$	$2^{14}.3^2.5^2$
$(D_{10} \times U_3(5)) : 2$	$2^6.3^2.5^4.7$	$5^{1+4} : 2^{1+4}.5.4$	$2^7.5^6$
$2^6.U_4(2)$	$2^{12}.3^4.5$	$(A_6 \times A_6).D_8$	$2^9.3^4.5^2$
$2^3.2^2.2^6.(3 \times L_3(2))$	$2^{14}.3^2.7$	$5^2.5.5^2.4A_5$	$2^4.3.5^6$
$M_{12}.2$	$2^7.3^3.5.11$	$HNM12$	$2^7.3^3.5.11$
$3^4 : 2(A_4 \times A_4).4$	$2^7.3^6$	$3^{1+4} : 4A_5$	$2^4.3^6.5$

groups, i.e. generated by elements x and y with orders $o(x) = 2$, $o(y) = 3$ and $o(xy) = 7$. In fact, G is a Hurwitz group if and only if G is $(2, 3, 7)$-generated. By his result, HN is a Hurwitz group and so HN is $(2, 3, 7)$-generated. For the sake of completeness, we reprove this result by using the character table of HN, see [6].

Lemma 2.1. *The Harada-Norton group HN is $(2X, 3Y, 7A)-$generated, $X, Y \in \{A, B\}$, if and only if $X = Y = B$.*

Proof. Using the algebra constants of HN, Table 2, we can see that $\Delta_{HN}(2A, 3B, 7A) = 0$. Therefore, $\Delta^\star_{HN}(2A,\ 3B, 7A) = 0$ and HN is not $(2A, 3B, 7A)$-generated. On the other hand, by Table 2,

$$\Delta_{HN}(2A, 3A, 7A) = 56\ <\ |C_{HN}(7A)| = 420$$

$$\Delta_{HN}(2B, 3A, 7A) = 35\ <\ |C_{HN}(7A)| = 420.$$

Hence by Theorem 1.1, HN is not $(2A, 3A, 7A)-$ and $(2B, 3A, 7A)-$ generated. Finally, we consider the triple $(2B, 3B, 7A)$. The maximal subgroups of HN, up to isomorphisms, that contain $(2B, 3B, 7A)$-generated subgroups are A_{12}, $U_3(8).3_1$ and $2^3.2^2.2^6.(3 \times L_3(2))$. Using the structure constants, Table 2, we have,

$$\begin{aligned}\Delta(HN)\ &= 2660, \Sigma(A_{12}) = 140, \Sigma(U_3(8).3_1)\\ &= 7\ and\ \ \Sigma(2^3.2^2.2^6.(3 \times L_3(2)) = 0.\end{aligned}$$

Therefore, $\Delta^\star(HN) \geq 2660-1(140)-20(7)-0 > 0$, and so HN is $(2B, 3B, 7A)$-generated. ∎

Lemma 2.2. *The Harada-Norton group HN is $(2X, 3Y, 11A)-$generated, $X, Y \in \{A, B\}$, if and only if $X = Y = B$.*

Proof. Since $\Delta_{HN}(2A, 3A, 11A) = 11 < |C_{HN}(11A)| = 22$, by Theorem 1.1, the Harada-Norton group HN is not $(2A, 3A, 11A)-$generated. Consider the triple

Table 2. The Structure Constants of the Group HN.

pX	$\Delta(2A,3A,pX)$	$\Delta(2A,3B,pX)$	$\Delta(2B,3A,pX)$	$\Delta(2B,3B,pX)$
$7A$	56	0	35	2660
$11A$	0	44	176	2156
$19A$	0	57	95	2565
pX	$\Delta(2A,5A,pX)$	$\Delta(2A,5B,pX)$	$\Delta(2A,5C,pX)$	$\Delta(2A,5D,pX)$
$7A$	56	0	0	0
$11A$	0	11	44	44
$19A$	0	0	57	57
pX	$\Delta(2A,5E,pX)$	$\Delta(2B,5A,pX)$	$\Delta(2B,5B,pX)$	$\Delta(2B,5C,pX)$
$7A$	2772	35	0	7980
$11A$	1452	11	220	5610
$19A$	513	95	95	4275
pX	$\Delta(2B,5D,pX)$	$\Delta(2B,5E,pX)$	$\Delta(2A,7A,pX)$	$\Delta(2B,7A,pX)$
$7A$	7980	27090	-	-
$11A$	5610	33495	4620	171237
$19A$	4275	28500	3781	178030
pX	$\Delta(2A,11A,pX)$	$\Delta(2B,11A,pX)$	$\Delta(3A,5A,pX)$	$\Delta(3A,5B,pX)$
$7A$	-	-	4830	546
$11A$	-	-	682	2167
$19A$	70110	3365755	760	893
pX	$\Delta(3A,5C,pX)$	$\Delta(3A,5D,pX)$	$\Delta(3A,5E,pX)$	$\Delta(3B,5A,pX)$
$7A$	9240	9240	435960	20440
$11A$	25630	25630	293645	19624
$19A$	31920	31920	197505	14839
pX	$\Delta(3B,5B,pX)$	$\Delta(3B,5C,pX)$	$\Delta(3B,5D,pX)$	$\Delta(3B,5E,pX)$
$7A$	27720	504840	504840	3987060
$11A$	15444	582560	582560	3509220
$19A$	18772	624340	624340	3743760
pX	$\Delta(3A,7A,pX)$	$\Delta(3B,7A,pX)$	$\Delta(3A,11A,pX)$	$\Delta(3B,11A,pX)$
$11A$	1331451	22766700	-	-
$19A$	1197323	22293612	22797093	425584952
pX	$\Delta(5A,7A,pX)$	$\Delta(5B,7A,pX)$	$\Delta(5C,7A,pX)$	$\Delta(5D,7A,pX)$
$11A$	1144132	1369445	41058490	41058490
$19A$	1033923	1305319	43409110	43409110
pX	$\Delta(5E,7A,pX)$	$\Delta(5A,11A,pX)$	$\Delta(5B,11A,pX)$	$\Delta(5C,11A,pX)$
$11A$	267104035	-	-	-
$19A$	260179065	19699732	24822075	827373050
pX	$\Delta(5D,11A,pX)$	$\Delta(5E,11A,pX)$	$\Delta(7A,11A,pX)$	
$19A$	827373050	4964188425	29548731391	

$(2A, 3B, 11A)$. By Table 2, $\Delta_{HN}(2A, 3B, 11A) = 44$ and by Table 3, $U_3(8).3_1$ is the only maximal subgroup of HN with non-empty intersection with all the conjugacy classes in this triple. Our calculation give $\Sigma(U_3(8).3_1) = 44$. But, $\Delta^\star(HN) \leq 44 - 44 = 0$ and we conclude that HN is not $(2A, 3B, 11A)$-generated.

We show that $(2B, 3A, 11A)$ is not a generating triple for HN. To do this, we consider the action of HN on the cosets of A_{12}. It is clear that this action is transitive. If χ denotes the permutation character of this action then $\chi = 1_{A_{12}}^{HN}$ and we have:

$$\begin{aligned}\chi \;=\; & 1a + 133a + 133b + 760a + 3344a + 8910a + 16929a + \\ & 35112a + 35112b + 267520a + 365750a + 406296a,\end{aligned}$$

Table 3. The Partial Fusion Maps into HN.

A_{12}-class	2a	2b	2c	3a	3b	3c	3d	5a	5b	7a
$\rightarrow HN$	2A	2A	2B	3A	3A	3A	3B	5A	5E	7A
h										1
A_{12}-class	11a	11b								
$\rightarrow HN$	11A	11A								
h	4	4								
$2.HS.2$-class	2a	2b	2c	2d	2e	3a	5a	5b	5c	7a
$\rightarrow HN$	2A	2A	2B	2A	2B	3A	5B	5A	5E	7A
h										15
$2.HS.2$-class	11a									
$\rightarrow HN$	11A									
h	1									
$U_3(8).3_1$-class	2a	3a	3b	3c	3d	3e	3f	3g	3h	3i
$\rightarrow HN$	2B	3A	3A	3B	3A	3A	3B	3B	3B	3B
$U_3(8).3_1$-class	7a	19a	19a							
$\rightarrow HN$	7A	19A	19B							
h	20	1	1							
$(D_{10} \times U_3(5)) : 2$-class	2a	2b	2c	3a	5a	5b	5c	5d	5e	5f
$\rightarrow HN$	2A	2A	2B	3A	5B	5A	5E	5A	5A	5E
$(D_{10} \times U_3(5)) : 2$-class	5g	5h	7a							
$\rightarrow HN$	5C	5D	7A							
h			6							
$2^3.2^2.2^6.(3 \times L_3(2))$-class	2a	2b	2c	2d	2e	3a	3b	3c	3d	3e
$\rightarrow HN$	2B	2A	2B	2A	2B	3A	3A	3A	3B	3B
$2^3.2^2.2^6.(3 \times L_3(2))$-class	7a	7b								
$\rightarrow HN$	7A	7A								
h	10	10								
$M_{12}.2$-class	2a	2b	2c	3a	3b	5a	11a			
$\rightarrow HN$	2A	2B	2B	3B	3A	5E	11A			
h							2			
$HNM12$-class	2a	2b	2c	3a	3b	5a	11a			
$\rightarrow HN$	2A	2B	2B	3B	3A	5E	11A			
h							2			

in which, na denotes the first irreducible character with degree n, in the character table of HN, see [6]. Now for $g \in HN$, the value of $\chi(g)$ is the number of cosets of HN fixed by g. Suppose $N = |HN : A_{12}|$. Then we have:

$$\lambda_3 = \frac{N - 645}{3} = 379785$$
$$\mu_{11} = \frac{N - 4}{11} = 103636$$
$$k = \frac{N - 800}{2} = 569600.$$

Therefore, by Lemma 1.5, $p = \frac{86181}{2}$ must be integer, a contradiction. Thus, $(2B, 3A, 11A)$ is not a generating triple for HN.

Finally, we consider the triple $(2B, 3B, 11A)$. The maximal subgroups of HN that may contain $(2B, 3B, 11A)$-generated proper subgroups are isomorphic to A_{12}, $M_{12}.2$ and $HNM12$. We calculate that $\Delta(HN) = 2156$, $\Sigma(A_{12}) = 220$ and $\Sigma(M_{12}.2) = \Sigma(HNM12) = 11$. Thus, $\Delta^\star(HN) \geq \Delta(HN) - 4.\Sigma(A_{12}) - 2.\Sigma(M_{12}.2) - 2.\Sigma(HNM12) > 0$, and so HN is $(2B, 3B, 11A)$-generated. This completes the proof. ■

Lemma 2.3. *The Harada-Norton group HN is $(2X, 3Y, 19Z)-$generated, for every $X, Y, Z \in \{A, B\}$.*

Proof. By Table 3, there is no maximal subgroup of HN that contains $(2A, 3X, 19A)-$generated proper subgroups. Therefore, $\Delta^\star_{HN}(2A, 3X, 19A) = \Delta_{HN}(2A, 3X, 19A) > 0$. Thus, HN is $(2A, 3X, 19A)-$generated. We now consider the triple $(2B, 3A, 19A)$. Amongst the maximal subgroups of HN with order divisible by 19, the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to $U_3(8).3_1$. Our calculations give, $\Delta^\star(HN) = \Delta(HN) = 95 > 0$, proving the generation of HN by this triple. Next, we consider the triple $(2B, 3B, 19A)$, then by Table 3, the maximal subgroups of HN, up to isomorphisms, that contain $(2B, 3B, 19A)$-generated subgroups are $U_3(8).3_1$. We calculate that $\Delta(HN) = 2565$ and $\Sigma(U_3(8).3_1) = 57$. Using Table 2, we have, $\Delta^\star(HN) \geq 2565 - 1(57) > 0$, and so HN is $(2B, 3B, 19A)$-generated. Thus $(2X, 3Y, 19A)$ is a generating triple for the group HN, $X, Y \in \{A, B\}$.

Finally, since $(19A)^2 = 19B$ and $(19B)^2 = 19A$ [6], we can apply Lemma 1.4, to prove that the Harada-Norton group HN is $(2X, 3Y, 19B)-$generated, $X, Y \in \{A, B\}$, proving the lemma. ■

Lemma 2.4. *The Harada-Norton group HN is $(2X, 5Y, 11A)-$generated, $X, Y \in \{A, B\}$, if and only if $X = B$ and $Y = C$ or D.*

Proof. Using algebra constants of HN, Table 2, $\Delta_{HN}(2A,\ 5B,\ 7A) = \Delta_{HN}(2A,\ 5C,\ 7A) = \Delta_{HN}(2A,\ 5D,\ 7A) = 0$. Thus HN is not $(2A, 5B, 7A)-$,

$(2A,\ 5C,\ 7A)-$ and $(2A,\ 5D,\ 7A)-$generated. On the other hand, $\Delta_{HN}(2A, 5A, 7A) = 56 < 420 = |C_{HN}(7A)|$. Hence, by Theorem 1.1, the sporadic group HN is not $(2A, 5A, 7A)-$generated. We now consider the triple $(2A,\ 5E,\ 7A)$. Using the permutation character of the group HN on the cosets of A_{12}, computed in Lemma 2.2, we can see that

$$\begin{aligned} \lambda_5 &= \frac{N-50}{5} &&= 227990 \\ \mu_7 &= \frac{N-1}{7} &&= 162857 \\ k &= \frac{N-8800}{2} &&= 565600. \end{aligned}$$

Therefore by Lemma 1.5, $p = \frac{174755}{2}$ must be integer, which is a contradiction. Thus, $(2A, 5E, 7A)$ is not a generating triple for HN. Also, we can apply a similar method to show that the group HN is not $(2B, 5E, 7A)-$generated.

Using Table 2, we calculate that $\Delta_{HN}(2B, 5B, 7A) = 0$, so HN is not $(2B, 5B, 7A)-$generated. Also, $\Delta_{HN}(2B, 5A, 7A) = 35 < 420 = |C_{HN}(7A)|$. Hence, by Theorem 1.1, the group HN is not $(2B, 5A, 7A)-$generated. We now consider the triple $(2A, 5C, 7A)$. The maximal subgroups of HN that may contain $(2B, 5C, 7A)$-generated proper subgroups are isomorphic to $(D_{10} \times U_3(5)) : 2$. We calculate that $\Delta(HN) = 7980$ and $\Sigma((D_{10} \times U_3(5)) : 2) = 0$. Thus, $\Delta^{\star}(HN) = \Delta(HN) = 7980 > 0$, and so HN is $(2B, 5C, 7A)$-generated. But $(5C)^2 = 5D$, so by Lemma 1.4, HN is $(2B, 5D, 7A)-$generated, as desired. ∎

Lemma 2.5. *The Harada-Norton group HN is $(2X, 5Y, 11A)-$generated, $X \in \{A, B\}$ and $Y \in \{A, B, C, D, E\}$, if and only if $X = Y = B$ or $Y \in \{C, D\}$.*

Proof. Since $\Delta_{HN}(2A, 5A, 11A) = 0$, the group HN is not $(2A, 5A, 11A)-$generated. Consider the triples $(2A, 5B, 11A)$ and $(2B, 5A, 11A)$. By the algebra constants of the group HN, Table 2,

$$\Delta_{HN}(2A, 5B, 11A) = \Delta_{HN}(2B, 5A, 11A) = 11 < 22 = |C_{HN}(11A)|,$$

and by Theorem 1.1, these are not generating triples for HN. On the other hand, there is no maximal subgroup of HN with non-empty intersection with all the conjugacy classes in triples $(2X, 5Y, 11A)$, $X \in \{A, B\}$ and $Y \in \{C, D\}$. Thus, these are generating triples for HN.

We show that $(2A, 5E, 11A)$ is not a generating triple for HN. To do this, we consider the action of HN on the cosets of maximal subgroup $5^{1+4} : 2^{1+4}.5.4$. Since this action is transitive, if χ denotes the permutation character of the action then $\chi = 1_{5^{1+4}:2^{1+4}.5.4}^{HN}$ and we have:

$$\begin{aligned}\chi \;=\; & 1a + 2.8910a + 16929a + 65835a + 65835b + 69255a + 69255b \\ & +214016a + 267520a + 2.365750a + 2.653125a + 656250a + 656250b \\ & +718200a + 718200b + 4.1185030a + 2.1354320a + 1361920a \\ & +1625184a + 2031480a + 3.2375000a + 2.2407680a + 4.2661120a \\ & +3.2784375a + 2.2985984a + 3200000a + 3.3424256a + 2.3878280a \\ & +4156250a + 2.4561920a + 3.4809375a + 5103000a + 5103000b \\ & +2.5332635a + 2.5878125a.\end{aligned}$$

Therefore by Lemma 1.5, the equation $68246640 = 2p - 2 + 27303087 + 12410496$ has an integer solution, which is a contradiction. Thus, $(2A,\ 5E,\ 11A)$ is not a generating triple for HN. A similar argument shows that HN is not $(2A,\ 5E,\ 11A)-$generated. Finally, $\Delta_{HN}(2B, 5B, 11A) = 220$, $2.HS.2$ is the only maximal subgroup of HN with a non-empty intersection with the conjugacy classes $2B, 5B, 11A$, and $\Sigma(2.HS.2) = 0$. Hence, HN is $(2B, 5B, 11A)-$generated. This completes the proof. ∎

Lemma 2.6. *The group HN is $(2X, 5Y, 19Z)-$generated, $X, Z \in \{A, B\}$ and $Y \in \{A, B, C, D, E\}$, if and only if $(X, Y) \neq (A, A), (A, B)$.*

Proof. By the character table of HN [6], we can see that there is no maximal subgroup of HN which its order is divisible by 5×19. On the other hand, if $(X, Y) \neq (A, A), (A, B)$ then $\Delta_{HN}(2X, 5Y, 19Z) \neq 0$, proving the lemma. ∎

Lemma 2.7. *The group HN is $(2X, 7A, 11A)-$, $(2X, 7A, 19Y)-$ and $(2X, 11A, 19Y)-$generated, where $X, Y \in \{A, B\}$.*

Proof. Using Table 2, we can see that $\Delta_{HN}(2X, 11A, 19Y) > 0$, where $X, Y \in \{A, B\}$. On the other hand, there is no maximal subgroup with order divisible by 11×19, so $\Delta^{\star}_{HN}(2X, 11A, 19Y) = \Delta_{HN}(2X, 11A, 19Y) > 0$. Therefore, the group HN is $(2X, 11A, 19Y)-$generated. We now claim that HN is $(2A, 7A, 11A)-$generated. To do this, the only maximal subgroup of HN, up to isomorphisms, with non-empty intersection with any conjugacy class in above triples are A_{12} and $2.HS.2$. By Tables 2 and 3, we calculate that $\Delta_{HN}(2A, 7A, 11A) = 4620$,, $\Sigma(A_{12}) = 22$ and $\Sigma(2.HS.2) = 396$. Thus, $\Delta^{\star}_{HN}(2A, 7A, 11A) \geq 4620 - 4(22) - 396 > 0$ and HN is $(2A, 7A, 11A)-$generated. Next, we show that HN is $(2B, 7A, 11A)-$ generated. To see this, the only maximal subgroups of HN that may contain $(2B, 7A, 11A)-$generated subgroups, are isomorphic to A_{12} and $2.HS.2$. We easily calculate the structure constant $\Delta_{HN}(2B, 7A, 11A) = 171237$,

$\Sigma(2.HS.2) = 429$ and $\Sigma(A_{12}) = 110$. Therefore, $\Delta^\star_{HN} > 0$ and the group HN is $(2B, 7A, 11A)-$generated.

Finally, we find the $(2X, 7A, 19Y)-$generations of the sporadic group HN. To do this, by the character table of HN, it is enough to assume that $Y = A$. If $X = A$ then by Table 1 and 3, there is no maximal subgroup of HN that contains $(2A, 7A, 19A)-$generated proper subgroups. Therefore, $\Delta^\star_{HN}(2A, 7A, 19A) = \Delta_{HN}(2A, 7A, 19A) > 0$, and so the group HN is $(2A, 7A, 19A)-$generated. Also, for the case $X = B$, amongst the maximal subgroups of HN with order divisible by 19, the only maximal subgroups with non-empty intersection with any conjugacy class in this triple are isomorphic to $U_3(8).3_1$. On the other hand, by Tables 2 and 3, $\Delta_{HN}(2B, 7A, 19A) = 178030$ and $\Sigma(U_3(8).3_1) = 513$. Thus, $\Delta^\star_{HN}(2B, 7A, 19A) \geq 178030 - 1(513) > 0$ and HN is $(2B, 7A, 19A)$. This completes the proof. ∎

We now summarize the above results in the following theorem.

Theorem 2.8. *The Harada-Norton group HN is (2,p,q)-generated for all $p, q \in \{3, 5, 7, 11, 19\}$ with $p < q$.*

Proof. The proof follows from Lemmas 2.1-2.7 and the fact that the triangular group $T(2, 3, 5) \cong A_5$. ∎

2.2. $(3, p, q)$-Generations for HN.

We consider triples $(3, p, q)$, in which p, q are primes and $q > p \geq 5$. The next lemma which proves the $(3A, 5A, 7A)-$ generation of HN is critical and done by Thomas Breuer. In the end of the paper, we include a GAP program which we need it in the proof of Lemma 2.9. Also, the algorithm of the program seems to be useful for similar generation type problems. This program is also written by Thomas Breuer and the author wishes to express here his gratitude to him.

Lemma 2.9. (Thomas Breuer) *The group HN is $(3A, 5A, 7A)-$generated.*

Proof. We will find a generating $(3A, 5A, 7A)$ triple for HN. Using the AtlasRep package of GAP, we can get a permutation representation for the group HN and compute the conjugacy classes of this permutation representation. We will use this fact that no proper subgroup of HN contains elements of the orders 11 and 19. By the character table of HN [6], if a and b are elements of orders 21 and 35, respectively, then $a^7 \in 3A$ and $b^7 \in 5A$. Therefore, it is enough to find elements a and b of orders 21 and 35, respectively, such that $x = a^7b^7$ has order 7 and $HN = \langle a^7, b^7 \rangle$.

To do this, we look at the orders of 100 pseudo random elements and check whether the elements generate the whole group. Finally, a GAP program shows that there is an $(3A, 5A, 7A)-$generation triple for HN. ■

Lemma 2.10. *The group* HN *is* $(3X, 5Y, 7A)-$*generated,* $X, Y \in \{A, B\}$*, if and only if* $(X, Y) \neq (A, B)$.

Proof. By Lemma 2.9, HN is $(3A, 5A, 7A)-$generated. Thus, it is enough to investigate the case $(X, Y) \neq (A, A)$. We first assume that $X = A$. For $Y = B$, the maximal subgroup $2.HS.2$ intersects the triple $(3A, 5B, 7A)$. Moreover, by Table 2 and 3, $\Delta_{HN}(3A, 5B, 7A) = 546$ and $\Sigma(2.HS.2) = 143$. Thus,

$$\Delta_{HN}(3A, 5B, 7A) - 143 = 403 < 420 = |C_{HN}(7A)|.$$

Hence by Theorem 1.1, $(3A, 5B, 7A)$ is not a generating triple for HN. Suppose $Y \in \{C, D\}$. In this case $\Delta_{HN}(3A, 5Y, 7A) = 9240$ and $(D_{10} \times U_3(5)) : 2$ is the unique maximal subgroups of HN, up to isomorphisms, with non-empty intersection with each of the classes $3A, 5Y$ and $7A$. However, $\Sigma((D_{10} \times U_3(5)) : 2) = 0$, proving the $(3A, 5C, 7A)-$ and $(3A, 5D, 7A)-$ generation of HN. We now complete the case $X = A$. To do this, we assume that $Y = E$. From the list of maximal subgroups of HN, Table 1, we observe that, up to isomorphisms, A_{12}, $2.HS.2$ and $(D_{10} \times U_3(5)) : 2$ are the only maximal subgroups of HN that admit $(3A, 5E, 7A)-$generated subgroups. From the structure constant we calculate $\Delta_{HN}(3A, 5E, 7A) = 435960$, $\Sigma(A_{12}) = 3780$, $\Sigma(2.HS.2) = 3500$ and $\Sigma((D_{10} \times U_3(5)) : 2) = 280$. Thus, $\Delta^{\star}_{HN}(3A, 5E, 7A) \geq \Delta_{HN}(3A, 5E, 7A) - 1(3780) - 15(3500) - 6(280) > 0$. This shows that the group HN is $(3A, 5E, 7A)$-generated.

Next we suppose that $X = B$. If $Y \in \{C, D\}$ then there is no maximal subgroups which intersects the conjugacy classes $3B, 5Y$ and $7A$. Since by Table 2, $\Delta_{HN}(3B, 5Y, 7A) \neq 0$, HN is $(3B, 5Y, 7A)-$generated, for $Y \in \{C, D\}$. Suppose $Y \in \{A, E\}$. The only maximal subgroups of HN with non-empty intersection with the conjugacy classes in this triple is, up to isomorphisms, A_{12}. We calculate that $\Delta_{HN}(3B, 5Y, 7A) - \Sigma(A_{12}) > 0$. Hence the group HN is $(3A, 5Y, 7A)-$generated, for $Y \in \{A, E\}$. Finally, for the case of $Y = B$, $\Delta_{HN}(3B, 5B, 7A) = 27720$ and there is no maximal subgroups of HN that contains $(3B, 5B, 7A)-$generated proper subgroups. Therefore, $\Delta^{\star}_{HN}(3B, 5B, 7A) = \Delta_{HN}(3B, 5B, 7A) = 27720$, and so HN is $(3B, 5B, 7A)$-generated. This completes the proof. ■

Lemma 2.11. *The group* HN *is* $(3X, pA, qZ)-$*generated, for* $5 < p < q$.

Proof. For the case $p = 11$, we can see that $\Delta_{HN}(3B, 5B, 7A) > 0$ and there is no maximal subgroups of HN that contains $(3X, 11A, 19Z)-$generated proper subgroups, $Z \in \{A, B\}$. Therefore, $\Delta^{\star}_{HN}(3X, 11A, 19Z)$ = $\Delta_{HN}(\ 3X, 11A, 19Z) > 0$, and so HN is $(3X, 11A, 19Z)$-generated, for $Z \in \{A, B\}$. We now assume that $p = 7$ and $q = 17$. In this case, $U_3(8).3_1$ is the unique maximal subgroups of HN, up to isomorphisms, with non-empty intersection with each of the classes $3X, 11A$ and $19Z$, $X, Z \in \{A, B\}$. If $X = A$ then $\Delta_{HN}(3A, 11A, 19Z) = 1197323$ and we calculate, $\Sigma(U_3(8).3_1) = 2052$ and $\Delta^{\star}_{HN}(3A, 11A, 19Z) \geq 1197323 - 2052 > 0$, $Z \in \{A, B\}$. Thus, HN is $(3A, 11A, 19A)-$ and $(3A, 11A, 19B)-$generated. If $X = B$ then $\Delta_{HN}(3B, 11A, 19Z) = 22293612$ and we calculate, $\Sigma(U_3(8).3_1) = 19494$ and $\Delta^{\star}_{HN}(3B, 11A, 19Z) \geq 22293612 - 19494 > 0$, $Z \in \{A, B\}$. Thus, HN is $(3B, 11A, 19A)-$ and $(3B, 11A, 19B)-$generated.

Finally, we assume that $p = 7$ and $q = 11$. Our main proof will consider two separate cases. We first assume that $X = A$. Amongst the maximal subgroups of HN with order divisible by $3 \times 7 \times 11$, the only subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to A_{12} and $2.HS.2$. We can see that $\Delta_{HN}(3A, 7A, 11A) = 1331451$, $\Sigma(A_{12}) = 1176$ and $\Sigma(2.HS.2) = 8899$. Our calculations give, $\Delta^{\star}_{HN}(3A, 7A, 11A) \geq \Delta_{HN}(3A, 7A, 11A) - 4(1176) - 1(8899) > 0$. Thus, HN is $(3A, 7A, 11A)-$generated. Next we assume that $X = B$. In this case, $\Delta_{HN}(3B, 7A, 11A) = 22766700$ and the only maximal subgroups with non-empty intersection with any conjugacy classes in this triple are isomorphic to A_{12}. We calculate that $\Sigma(A_{12}) = 2200$. Our calculations give, $\Delta^{\star}_{HN}(3B, 7A, 11A) \geq \Delta_{HN}(3B, 7A, 11A) - 4(2200) > 0$. Therefore, HN is $(3B, 7A, 11A)$-generated and the proof is complete. ■

Theorem 2.12. *The Harada-Norton group HN is (3X,pY,qZ)-generated for all $p, q \in \{5, 7, 11, 19\}$ with $p < q$.*

Proof. The proof is straightforward and follows from Lemmas 2.9, 2.10 and 2.11. ■

2.3. (p, q, r)-Generations for $HN, p > 3$.

We consider triples (p, q, r), in which $p \geq 5$ and p, q and r are prime numbers. We deal separately with each case in the following two lemmas.

Lemma 2.13. *The group HN is $(5X, 7A, 19Y)-$, $(5X, 11A, 19Y)-$ and $(7A, 11A, 19Y)-$generated, for $X \in \{A, B, C, D, E\}$ and $Y \in \{A, B\}$.*

Proof. By Table 3, there is no maximal subgroup of HN that contains $(5X, 7A, 19Y)-$, $(5X, 11A, 19Y)-$ and $(7A, 11A, 19Y)-$generated proper subgroups, for $X \in \{A, B, C, D, E\}$ and $Y \in \{A, B\}$. Therefore by Table 2,

$$\begin{aligned}\Delta^{\star}_{HN}(5X, 7A, 19Y) &= \Delta_{HN}(5X, 7A, 19Y) > 0\\ \Delta^{\star}_{HN}(5X, 11A, 19Y) &= \Delta_{HN}(5X, 11A, 19Y) > 0\\ \Delta^{\star}_{HN}(7A, 11A, 19Y) &= \Delta_{HN}(7A, 11A, 19Y) > 0.\end{aligned}$$

Thus, HN is $(5X, 7A, 19Y)-$, $(5X, 11A, 19Y)-$ and $(7A, 11A, 19Y)-$generated, as desired. ■

Lemma 2.14. *The group HN is $(5X, 7A, 11A)-$generated, for $X \in \{A, B, C, D, E\}$.*

Proof. If $X \in \{C, D\}$ then by Table 1 and 3, there is no maximal subgroup of HN that contains $(5X, 7A, 11A)-$generated proper subgroups. Therefore, $\Delta^{\star}_{HN}(5X, 7A, 11A) = \Delta_{HN}(5X, 7A, 11A) > 0$, and so the group HN is $(5C, 7A, 11A)-$ and $(5D, 7A, 11A)-$generated. If $Y \in \{A, E\}$ then the only maximal subgroups of HN that may contain $(5Y, 7A, 11A)$-generated proper subgroups are isomorphic to A_{12} and $2.HS.2$. Using a similar argument as in above, we can see that in any case

$$\Delta^{\star}_{HN}(5Y, 7A, 11A) \geq \Delta_{HN}(5Y, 7A, 11A) - 4\Sigma(A_{12}) - \Sigma(2.HS.2) > 0.$$

Hence HN is $(5A, 7A, 11A)-$ and $(5E, 7A, 11A)-$generated. Finally, by Table 3, $2.HS.2$ is the unique maximal subgroups of HN, up to isomorphisms, with non-empty intersection with each of the classes $5B, 7A$ and $11A$. Our calculations give $\Delta_{HN}(5B, 7A, 11A) = 1369445$ and $\Sigma(2.HS.2) = 6332$. Thus, $\Delta^{\star}_{HN}(5B, 7A, 11A) \geq \Delta_{HN}(5B, 7A, 11A) - \Sigma(2.HS.2) > 0$, proving the $(5B, 7A, 11A)$-generation of HN. This concludes the lemma. ■

Theorem 2.15. *The Harada-Norton group HN is (pX,qY,rZ)-generated for all prime numbers p, q and r with $3 < p < q$.*

Proof. The proof is straightforward and follows from Lemmas 2.13 and 2.14. ■

We now summarize the above results in the following theorem.

Theorem 2.16. *The Harada-Norton group HN is $(p, q, r)-$generated if and only if $(p, q, r) \neq (2, 3, 5)$.*

Proof. The proof is follows from Theorem 2.8, 2.12 and 2.15. ■

A GAP Program for Constructing a $(3A, 5A, 7A)-$Generating Triple for HN

```
gap> # How to find a generating (3A, 5A, 7) triple for HN.
gap> # Later we will use that no proper subgroup of HN contains elements
gap> # of the orders 11 and 19.
gap>
gap> hntbl:= CharacterTable( "HN" );;
gap> maxes:= List( Maxes( hntbl ), CharacterTable );;
gap> Filtered( maxes, t -> Size( t ) mod ( 11 * 19 ) = 0 );;
gap>
gap> # Get a permutation representation of HN.
gap>
gap> RequirePackage( "atlasrep");;
gap> gens:= OneAtlasGeneratingSet( "HN" );;
gap> hn:= Group( gens.generators );;
gap> SetSize( hn, Size( hntbl ) );
gap>
gap> # Find elements in the classes 3A and 5A.
gap> # We could also use the straight line program for computing conjugacy
gap> # class representatives.
gap>
gap> # Any element of order 21 powers to a 3A element.
gap> repeat
> a:= PseudoRandom( hn );
> until Order( a ) = 21;
gap> a := a^7;;
gap>
gap> # Any element of order 35 powers to a 5A element.
gap> repeat
> b:= PseudoRandom( hn );
> until Order( b ) = 35;
gap> b := b^7;;
gap>
gap> repeat
>
> Print( "Try to find a generating (3A,5A,7) triple",\n" );
>
```

```
> # Conjugate the 5A element until the product has order 7.
> repeat
> b:= b^PseudoRandom( hn );
> until Order( a * b ) = 7;
>
> # Check whether the elements generate the whole group.
> # For that, we look at the orders of 100 pseudo random elements.
>
> u:= SubgroupNC( hn, [a, b] );;
> found11:= false;;found19:= false;;
> > for i in [1..100] do
> ord:= Order( PseudoRandom( u ) );
> if ord mod 11 = 0 then
> found11:= true; fi;
> > if ord mod 19 = 0 then
> found19:= true; fi;
> if found11 and found19 then
> Print("a and b generate HN, i = ",i,"\n");
> break; fi; od; fi; od;
   > until found11 and found19;
Try to find a generating (3A,5A,7) triple.
a and b generate HN, i = 4.
gap>
gap> # How long did the computations run?
gap> # The workspace was about 200m.
gap> Runtime();
81710
gap>
gap> # If we do not believe the character theoretic argument then
gap> # we may compute the order of the group.
gap> # Random methods suffice, but still we should allow GAP to get
gap> # sufficient workspace for that (command line option -o 700m).
gap>
gap> StabChainOptions( u ).random:= 100;;
gap> Size( u ) = Size( g );
true
```

Acknowledgement

The author wishes to thanks Dr. Thomas Breuer for writing a program to prove Lemma 2.10.

References

1. M. Aschbacher, *Sporadic Group*, Cambridge University Press, U.K., 1997.
2. A. R. Ashrafi, Generating Pairs for the Thompson Group Th, *SUT J. Math.*, **39(1)** (2003), 41-54.
3. A. R. Ashrafi, Generating Pairs for the Held Group He, *J. Appl. Math.* $\&$ *Computing*, **10** (2002), 167-174.
4. M. D. E. Conder, Some results on quotients of triangle groups, *Bull. Austral. Math. Soc.*, **30** (1984), 73-90.
5. M. D. E. Conder, R. A. Wilson and A. J. Wolder, The symmetric genus of sporadic groups, *Proc. Amer. Math. Soc.*, **116** (1992), 653-663.
6. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford Univ. Press (Clarendon), Oxford, 1985.
7. M. R. Darafsheh and A. R. Ashrafi, Generating Pairs for the Sporadic Group Ru, *J. Appl. Math.* $\&$ *Computing*, **12(1-2)** (2003), 143-154.
8. M. R. Darafsheh and A. R. Ashrafi, $(2, p, q)$-Generation of the Conway group Co_1, *Kumamoto J. Math.*, **13** (2000), 1-20.
9. M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, (p, q, r)-Generations and nX-Complementary Generations of the Sporadic Group Ly, *Kumamoto J. Math.*, **16** (2003), 13-25.
10. M.R. Darafsheh ; A.R. Ashrafi and G.A. Moghani, (p, q, r)-Generations of the Conway group Co_1, for odd p, Kumamoto J. Math., **14** (2001), 1-20.
11. M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, (p, q, r)-Generations of the Sporadic group $O'N$, Groups St. Andrews 2001 in: *Oxford. Vol. I, 101-109, London Math. Soc. Lecture Note Ser.*, Cambridge Univ. Press, Cambridge, 2003, p. 304.
12. S. Ganief and J. Moori, (p, q, r)-Generations of the smallest Conway group Co_3, *J. Algebra*, **188** (1997), 516-530.
13. S. Ganief and J. Moori, Generating pairs for the Conway groups Co_2 and Co_3, *J. Group Theory*, **1** (1998), 237-256.
14. S. Ganief and J. Moori, 2-Generations of the Forth Janko Group J_4, *J. Algebra*, **212** (1999), 305-322.
15. S. Ganief and J. Moori, (p, q, r)-Generations and nX-complementary generations of the sporadic groups HS and McL, J. Algebra **188** (1997), 531-546.

16. J. Moori, (p, q, r)-Generations for the Janko groups J_1 and J_2, *Nova J. Algebra and Geometry*, **2(3)** (1993), 277-285.

17. J. Moori, $(2, 3, p)$-Generations for the Fischer group F_{22}, *Comm. Algebra*, **2(11)** (1994), 4597-4610.

18. M. Schonert et al., *GAP, Groups, Algorithms and Programming*, Lehrstuhl De fur Mathematik, RWTH, Aachen, 1992.

19. A. J. Woldar, Representing M_{11}, M_{12}, M_{22} and M_{23} on surfaces of least genus, *Comm. Algebra*, **18** (1990), 15-86.

20. A. J. Woldar, Sporadic simple groups which are Hurwitz, *J. Algebra*, **144** (1991), 443-450.

21. A. J. Woldar, $3/2$-Generation of the Sporadic simple groups, *Comm. Algebra*, **22(2)** (1994), 675-685.

22. A. J. Woldar, On Hurwitz generation and genus actions of sporadic groups, *Illinois Math. J.*, **33(3)** (1989), 416-437.

Ali Reza Ashrafi
Department of Mathematics,
Faculty of Science,
University of Kashan,
Kashan, Iran
E-mail: ashrafi@kashanu.ac.ir