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ASYMPTOTIC BEHAVIOR OF (a, k)-REGULARIZED RESOLVENT FAMILIES AT ZERO

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Abstract. This paper is primarily concerned with approximation at 0 of an (a, k)-regularized resolvent family $R(\cdot)$ for Volterra integral equation. We shall consider convergence rates of some kind of local means $Q_m(t), t \geq 0, m \geq 0$, of $R(t)/k(t)$. Some approximation theorems and local ergodic theorems with rates will be deduced from general approximation theorems for regularized approximation processes.

1. INTRODUCTION

Consider the following Volterra equation of convolution type

$$u(t) = \int_0^t a(t-s)Au(s)ds + f(t),\ t \geq 0$$

where A is a closed linear operator on a Banach space X. Let $B(X)$ denote the Banach algebra of all bounded linear operators on X. Let $k \in C[0, \infty), a \in L^1_{loc}$ $([0, \infty))$ be nondecreasing positive functions. A strongly continuous function $R : [0, \infty) \to B(X)$ is called an (a, k)-regularized resolvent family with generator A if it satisfies the conditions:

(R1) $R(0) = k(0)I$;

(R2) $R(t)x \in D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t > 0$;

(R3) $a * R(t)x \in D(A)$ and $R(t)x = k(t)x + Aa * R(t)x$ for all $x \in X$ and $t \geq 0$.

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The notion of (a,k)-regularized resolvent family has been introduced and studied in [6, 7, 8]. It contains integrated solution families ($k(t) = t^\alpha/\Gamma(\alpha+1)$) [10], resolvent families ($k(t) \equiv 1$) [11], integrated semigroups ($a \equiv 1, k(t) = t^\alpha/\Gamma(\alpha+1)$) [5], and integrated cosine functions ($a(t) = t, k(t) = t^\alpha/\Gamma(\alpha+1)$) [12] as special cases.

In this paper, we study approximation properties at 0 of $R(\cdot)$. Denote by a_0 the Dirac measure δ_0 at 0. For $m \geq 0$, let $a_{m+1}(t) = a * a_m(t), t \geq 0$, and let $l_m(0) = 0$ and $l_m(t) = \frac{a_{m+1}*k(t)}{a_m*k(t)}$ for $t > 0$. We define the operator function $Q_m : (0,\infty) \to B(X)$ by

$$Q_m(t)x = \frac{a_m * R(t)x}{a_m * k(t)}$$

for all $x \in X$ and $t > 0$. Note that $Q_0(t) = R(t)/k(t)$ and $Q_1(t) = \int_0^t a(t-s)R(s)ds/\int_0^t a(t-s)k(s)ds$. We shall assume that

$$\|R(t)\| \leq Mk(t) \text{ for all } t > 0. \tag{1.1}$$

Then

$$\begin{aligned}\|Q_m(t)x\| &\leq \frac{1}{a_m * k(t)} \int_0^t a_m(t-s)\|R(s)x\|ds \\ &\leq \frac{M\|x\|}{a_m * k(t)} \int_0^t a_m(t-s)k(s)ds = M\|x\|\end{aligned}$$

for all $x \in X$ and so

$$\|Q_m(t)\| \leq M \text{ for all } t > 0. \tag{1.2}$$

Therefore one can consider the asymptotic behavior of $Q_m(t)$ at zero. Since it can be seen that $\{Q_m(t); t \geq 0\}$ forms a regularized approximation process on $X_1 := \overline{D(A)}$, one can apply general approximation theorems for A-regularized approximation processes (cf. [13]) to deduce results on approximation of $Q_m(t)$. We will do this in Section 3. Before that we shall first recall in Section 2 some needed general results from [13] on approximation of A-regularized approximation processes.

2. Regularized Approximation Processes

In [13], we have obtained general results on the strong and uniform convergence of regularized approximation processes, with emphasis on their optimal and non-optimal convergence rates. This section serves as a brief review of those general results needed in Section 3.

We start with the following definition of a regularized approximation process. In the sequel, we use the notations $D(T)$, $R(T)$, and $N(T)$, for the domain, range, and null space, respectively, of a linear operator T.

Let $e(\alpha)$ be a positive function tending to 0. A net $\{T_\alpha\}$ of bounded linear operators on X is called an *A-regularized approximation process of order $O(e(\alpha))$* on X if it is uniformly bounded, i.e., $\|T_\alpha\| \leq M$ for some $M > 0$ and all α, and satisfies

(A1) there are a (necessarily densely defined) closed linear operator A on X and a uniformly bounded approximation process $\{S_\alpha\}$ on X such that

$$R(S_\alpha) \subset D(A) \text{ and } S_\alpha A \subset AS_\alpha = (e(\alpha))^{-1}(T_\alpha - I) \text{ for all } \alpha.$$

In this case, the process $\{S_\alpha\}$ is called a *regularization process* associated with $\{T_\alpha\}$.

In the following, $\{T_\alpha\}$ denotes an A-regularized approximation process of order $O(e(\alpha))$ with regularization process $\{S_\alpha\}$.

Lemma 2.1. [13]

(i) *$x \in D(A)$ and $y = Ax$ if and only if $y = \lim_\alpha (e(\alpha))^{-1}(T_\alpha - I)x$.*

(ii) *$D(A)$ is dense in X, and $\|T_\alpha x - x\| \to 0$ for all $x \in X$.*

(iii) *If A is bounded, then $\|T_\alpha - I\| = O(e(\alpha)) \to 0$.*

(iv) *$\|T_\alpha - I\| \to 0$ implies $A \in B(X)$ if either $R(T_\alpha) \subset D(A)$ for all α, or S_α and T_α satisfy the following condition:*

(A2) $\|T_\alpha - I\| \to 0$ implies $\|S_\alpha - I\| \to 0$.

A Banach space X is called a *Grothendieck space* if every weakly* convergent sequence in X^* is weakly convergent, and is said to have the *Dunford-Pettis property* if $\langle x_n, x_n^* \rangle \to 0$ whenever $x_n \to 0$ weakly in X and $x_n^* \to 0$ weakly in X^*. The spaces L^∞, H^∞, and $B(S, \Sigma)$ are particular examples of Grothendieck spaces with the Dunford-Pettis property (see [9]). A common phenomenon in such spaces is that strong operator convergence often implies uniform operator convergence. The following is a theorem of this type for regularized approximation processes.

Theorem 2.2. [13] *Let $\{T_\alpha\}$ be an A-regularized approximation process of order $O(e(\alpha))$ on a Grothendieck space X with the Dunford-Pettis property. If $R(T_\alpha) \subset D(A)$ for all α, then $A \in B(X)$ and $\|T_\alpha - I\| = O(e(\alpha))$.*

As usual the rates of convergence will be characterized by means of *K-functional* and *relative completion*, which we recall now.

Definition 2.3. Let X be a Banach space with norm $\|\cdot\|_X$, and Y a submanifold with seminorm $\|\cdot\|_Y$. The *K-functional* is defined by

$$K(t,x) := K(t,x,X,Y,\|\cdot\|_Y) = \inf_{y\in Y}\{\|x-y\|_X + t\|y\|_Y\}.$$

If Y is also a Banach space with norm $\|\cdot\|_Y$, then the *completion of Y relative to X* is defined as

$$\tilde{Y}^X := \{x \in X : \exists\{x_n\} \subset Y \text{ such that } \lim_{n\to\infty}\|x_n - x\|_X = 0 \text{ and } \sup_n \|x_n\|_Y < \infty\}.$$

It is known [1] that $K(t,x)$ is a bounded, continuous, monotone increasing and subadditive function of t for each $x \in X$, and $K(t,x,X,Y,\|\cdot\|_Y) = O(t)$ if and only if $x \in \tilde{Y}^X$. With these terminologies we now state some theorems from [13] on convergence rates. The following is an optimal convergence (saturation) theorem.

Theorem 2.4. [13] *Let $\{T_\alpha\}$ be an A-regularized approximation process of order $O(e(\alpha))$, and let $D(A)$ be equipped with the graph norm $\|\cdot\|_{D(A)}$. For $x \in X$, we have:*

(i) $\|T_\alpha x - x\| = o(e(\alpha))$ *if and only if $x \in N(A)$, if and only if $T_\alpha x = x$ for all α.*

(ii) *The following are equivalent:*

(a) $\|T_\alpha x - x\| = O(e(\alpha))$*;*

(b) $x \in \widetilde{D(A)}^X$*;*

(c) $x \in D(A)$ *in the case that X is reflexive.*

The next theorem is about non-optimal convergence.

Theorem 2.5. [13] *Let $0 \le e(\alpha) \le f(\alpha) \to 0$. If $K(e(\alpha),x,X,D(A),\|\cdot\|_{D(A)}) = O(f(\alpha))$, then $\|T_\alpha x - x\| = O(f(\alpha))$. The converse statement is also true under the following assumption:*

(A3) $\|S_\alpha x - x\| = O(f(\alpha))$ whenever $\|T_\alpha x - x\| = O(f(\alpha))$.

To consider the sharpness of approximation, we need the following theorem.

Theorem 2.6. [13] *Suppose an A-regularized approximation process $\{T_\alpha\}$ and its regularization process $\{S_\alpha\}$ satisfy condition* (A2)*. Then A is unbounded if and only if for each/some $f(\alpha)$ with $0 \le e(\alpha) < f(\alpha) \to 0$ and $f(\alpha)/e(\alpha) \to \infty$ there exists $x_f \in X$ such that*

$$\|T_\alpha x_f - x_f\| \begin{cases} = O(f(\alpha)); \\ \neq o(f(\alpha)). \end{cases}$$

3. Approximation Properties of Regularized Solution Families

In this section, we apply the general theorems in Section 2 to deduce approximation theorems for regularized solution families. Note that $a_m(t)$ and $a_m * k(t)$ are nondecreasing and positive functions of t. Therefore

$$l_m(t) = \frac{1}{a_m * k(t)} \int_0^t a(t-s)(a_m * k)(s)ds \leq \int_0^t a(s)ds \to 0 \tag{3.1}$$

as $t \to 0$.

Lemma 3.1. *Let $R(\cdot)$ be an (a, k)-regularized resolvent family generated by A such that $\|R(t)\| \leq Mk(t)$ for all $t \geq 0$, and let A_1 be the part of A in $X_1 := \overline{D(A)}$. Then*

$$Q_0(t)D(A) \subset D(A) \;\; and \;\; Q_0(t)Ax = AQ_0(t)x \;\; for \;\; x \in D(A); \tag{3.2}$$

$$Q_{m+1}(t)X \subset D(A) \;\; and \;\; Q_{m+1}(t)A \subset AQ_{m+1}(t) = \frac{1}{l_m(t)}(Q_m(t) - I); \tag{3.3}$$

$$Q_0(t)D(A_1) \subset D(A_1) \;\; and \;\; Q_0(t)A_1x = A_1Q_0(t)x \;\; for \;\; x \in D(A_1); \tag{3.4}$$

$$\begin{aligned} &Q_{m+1}(t)X_1 \subset D(A_1) \;\; and \;\; Q_{m+1}(t)A_1 \subset A_1Q_{m+1}(t)\mid_{X_1} \\ &= \tfrac{1}{l_m(t)}(Q_m(t) - I)\mid_{X_1} \;\; for \;\; all \;\; m \geq 0 \;\; and \;\; t > 0. \end{aligned} \tag{3.5}$$

Proof. Since $Q_0(t) = \frac{1}{k(t)}R(t)$, (3.2) follows from $(R2)$. It implies $Q_0(t)X_1 \subset X_1$. To show (3.4), let $x \in D(A_1)$. Then $x \in D(A), Ax \in X_1$, and $A_1x = Ax$. By (3.2) we have $Q_0(t)x \in D(A)$ and $AQ_0(t)x = Q_0(t)Ax = Q_0(t)A_1x \in Q_0(t)X_1 \subset X_1$, so that $Q_0(t)x \in D(A_1)$ and $A_1Q_0(t)x = AQ_0(t)x = Q_0(t)A_1x$. To show (3.3) for $m \geq 0$, write

$$\begin{aligned} Q_{m+1}(t)x &= \frac{1}{a_{m+1} * k(t)}[a_m * (a * R)](t)x \\ &= \frac{1}{a_{m+1} * k(t)} \int_0^t a_m(t-s)(a * R)(s)xds \end{aligned}$$

for all $x \in X$. Since the integral

$$\begin{aligned} \textstyle\int_0^t Aa_m(t-s)(a * R)(s)xds &= \int_0^t a_m(t-s)A(a * R)(s)xds \\ &= \int_0^t a_m(t-s)[R(s) - k(s)]xds \end{aligned}$$

exists, the closedness of A implies that $[a_m * (a * R)](t)x \in D(A)$ and

$$A[a_m * (a * R)](t)x = \int_0^t Aa_m(t-s)(a * R)(s)xds$$
$$= [a_m * A(a * R)](t)x = a_m * R(t)x - a_m * k(t)x.$$

Hence $Q_{m+1}(t)x \in D(A)$ and

$$AQ_{m+1}(t)x = \frac{1}{a_{m+1} * k(t)}[a_m * R(t)x - a_m * k(t)x] = \frac{1}{l_m(t)}[Q_m(t)x - x]$$

for all $x \in X$. Moreover, if $x \in D(A)$ then by (R2) and (R3) we have

$$AQ_{m+1}(t)x = \frac{1}{a_{m+1} * k(t)}[a_m * A(a * R)](t)x$$
$$= \frac{1}{a_{m+1} * k(t)}[a_m * (a * R)](t)Ax = Q_{m+1}(t)Ax.$$

This shows (3.3). To show (3.5), let $x \in X_1$ and let $\{x_n\} \subset D(A)$ converge to x. (3.3) implies $Q_{m+1}(t)x \in D(A)$. Since A is closed, $AQ_{m+1}(t)$ is bounded, so that $AQ_{m+1}(t)x = \lim_{n\to\infty} AQ_{m+1}(t)x_n = \lim_{n\to\infty} Q_{m+1}(t)Ax_n \in \overline{D(A)} = X_1$. This and (3.3) show that $Q_{m+1}(t)x \in D(A_1)$ and $A_1Q_{m+1}(t)x = AQ_{m+1}(t)x = \frac{1}{l_m(t)}(Q_m(t) - I)x$ for all $x \in X_1$. When $x \in D(A_1)$, we have $x \in D(A)$, $Ax \in X_1$, and $A_1x = Ax$ so that $Q_{m+1}(t)A_1x = Q_{m+1}(t)Ax = AQ_{m+1}(t)x = A_1Q_{m+1}(t)x$. This completes the Proof.

Lemma 3.2. *Let $R(\cdot)$ be an (a, k)-regularized resolvent family with generator A such that $\|R(t)\| \le Mk(t)$ for all $t \ge 0$.*

(i) *For $m \ge 0$, $\|Q_m(t)x - x\| \to 0$ as $t \to 0^+$ if and only if $Q_m(t)x \to x$ weakly as $t \to 0^+$, if and only if there is a sequence $\{t_n\}$ such that $Q_m(t_n)x \to x$ weakly for the case $m \ge 1$, if and only if $x \in X_1$.*

(ii) *If $k(t) \to k(0) \neq 0$ as $t \to 0^+$, then A is densely defined in X.*

Proof.

(i) It follows from (3.1), (1.2), (3.3) that for all $m \ge 0$

$$\|Q_m(t)x - x\| \le l_m(t)\|Q_{m+1}(t)\|\|Ax\| \le l_m(t)M\|Ax\| \to 0$$

as $t \to 0^+$ for all $x \in D(A)$, and hence $Q_m(t)x \to x$ for all $x \in X_1$, by

(1.2). Conversely, from the estimate,

$$
\begin{aligned}
&| < Q_{m+1}(t)x - x, x^* > | \\
&= \frac{1}{a_{m+1} * k(t)} \left| \left\langle \int_0^t a(t-s)(a_m * R(s)x) ds \right.\right. \\
&\qquad \left.\left. - \int_0^t a(t-s)(a_m * k)(s) x ds, x^* \right\rangle \right| \qquad (3.6) \\
&\le \frac{1}{a_{m+1} * k(t)} \int_0^t a(t-s)(a_m * k)(s) | < Q_m(s)x - x, x^* > | ds \\
&\le \sup\{| < Q_m(s)x - x, x^* > |; 0 \le s \le t\}, x \in X, x^* \in X^*,
\end{aligned}
$$

one sees that if $Q_m(t)x \to x$ weakly, then $Q_{m+1}(t)x \to x$ weakly, which and the fact that $R(Q_{m+1}(t)) \subset D(A)$ show that $x \in X_1$. When $m \ge 1, R(Q_m(t_n)) \subset D(A)$, and so $x = w - \lim Q_m(t_n)x \in X_1$.

(ii) When $k(t) \to k(0) \ne 0$ as $t \to 0^+$, since $Q_0(t) = R(t)/k(t) \to I$ strongly as $t \to 0^+$, (3.6) implies that

$$\|Q_1(t)x - x\| \le \sup\{\|Q_0(s)x - x\|; 0 \le s \le t\} \to 0$$

for all $x \in X$. Then we have $X_1 = X$, by the fact that $Q_1(t)X \subset D(A)$. That is, A is densely defined.

Thus, from (3.2), (3.5) and Lemma 3.2, we see that X_1 is invariant under $Q_m(t)$ for each $m \ge 0$, and $\{T_t := Q_m(t) \mid_{X_1}\}$ is an A_1-regularized approximation process on X_1 with the regularization process $\{S_t := Q_{m+1}(t) \mid_{X_1}\}$ and with the optimal order $O(l_m(t))(t \to 0^+)$. In particular , $D(A_1)$ is dense in X_1. Moreover, by Lemma 3.1 we have $T_t D(A_1) \subset D(A_1)$ if $m = 0$ and $R(T_t) \subset D(A_1)$ if $m \ge 1$.

Lemma 3.3. *The above pair $(\{T_t\}, \{S_t\})$ satisfies $(A2)$. If $l_m(t)$ is nondecreasing for t near 0, then (A3) with $f(t) = (l_m(t))^\beta (0 < \beta \le 1))$ also holds.*

Proof. From (3.6) one can see that $\|S_t - I\|_{X_1} \le \sup\{\|T_s - I\|_{X_1}; 0 \le s \le t\}$, which shows (A2). Moreover, if $\|T_t x - x\| \le M(k_m(t))^\beta$ for all $t \in [0, 1]$, then $\|S_t x - x\| \le M \sup\{(l_m(s))^\beta; 0 \le s \le t\} \le M(l_m(t))^\beta$ for all $t \in [0, 1]$, showing (A3).

From Lemmas 2.1 and 3.3 and Theorem 2.2 we deduce the following uniform convergence theorem.

Theorem 3.4. *Let $R(\cdot)$ be an (a, k)-regularized resolvent family with generator A such that $\|R(t)\| \le Mk(t)$ for all $t \ge 0$.*

(i) *For* $m \geq 0$, $\|Q_m(t) - I\| \to 0$ *as* $t \to 0^+$ *if and only if* $A \in B(X)$. *In this case,* $\|Q_m(t) - I\| = O(l_m(t))(t \to 0^+)$.

(ii) *When* X_1 *is a Grothendieck space with the Dunford-Pettis property,* A *must be bounded on* X, *and consequently* $\|R(t) - k(t)I\| = O(a * k(t))(t \to 0^+)$.

Proof.

(i) This follows from Lemmas 2.1 and 3.3.

(ii) Applying Theorem 2.2 to $\{T_t := Q_1(t) \mid_{X_1}\}$ yields that A_1 is bounded on X_1, so that $\|Q_1(t) \mid_{X_1} - I|_{X_1}\| \leq l_1(t)\|A_1\|\|Q_2(t)\| \leq l_1(t)\|A_1\|\ M \to 0$ as $t \to 0^+$. Hence $Q_1(t) \mid_{X_1}$ is invertible on X_1 for small t. Then by (3.3) we have $X_1 = R(Q_1(t) \mid_{X_1}) \subset R(Q_1(t)) \subset D(A)$, which shows that $D(A)$ is closed and A is bounded. Due to Lemma 3.3, (iii) and (iv) of Lemma 2.1 together imply that $A \in B(X)$. By (i), $\|Q_m(t) - I\| = O(l_m(t))(t \to 0^+)$, and in particular, $\|R(t) - k(t)I\| = O(a * k(t))(t \to 0^+)$.

From Theorems 2.4, 2.5, 2.6 and Lemma 3.3 we can deduce the next theorem.

Theorem 3.5. *Let* $R(\cdot)$ *be as assumed in Theorem 3.4 and let* $m \geq 0$, $0 < \beta \leq 1$, *and* $x \in X_1 = \overline{D(A)}$.

(i) $\|Q_m(t)x - x\| = o(l_m(t))(t \to 0^+)$ *if and only if* $x \in N(A_1) = N(A)$.

(ii) $\|Q_m(t)x - x\| = O(l_m(t))(t \to 0^+)$ *if and only if* $x \in \widetilde{D(A_1)}^{X_1}$ $(= D(A_1)$, *if* X *is reflexive).*

(iii) *If* $K(l_m(t), x, X_1, D(A_1), \|.\|_{D(A_1)}) = O((l_m(t))^\beta)(t \to 0^+)$, *then* $\|Q_m(t)x - x\| = O((l_m(t))^\beta)(t \to 0^+)$. *The converse is also true if* $l_m(t)$ *is nondecreasing for* t *near* 0.

(iv) A *is unbounded if and only if for some/each* $0 < \beta < 1$ *and* $m \geq 0$ *there exist* $x^*_{m,\beta} \in X_1 = \overline{D(A)}$ *such that*

$$\|Q_m(t)x^*_{m,\beta} - x^*_{m,\beta}\| \begin{cases} = O((l_m(t))^\beta) \\ \neq o((l_m(t))^\beta) \end{cases} \quad (t \to 0^+).$$

Next, we assume that the nondecreasing positive functions $a, k \in L^1_{loc}([0, \infty))$ are Laplace transformable, i.e., there is $\omega \geq 0$ such that $\hat{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt < \infty$ and $\hat{k}(\lambda) < \infty$ for all $\lambda > \omega$. Then it is easy to see that $\hat{a}(\lambda) \to 0$ as $\lambda \to \infty$.

Lemma 3.6. *Suppose* $\hat{a}(\lambda) < \infty$ *for all* $\lambda > \omega$, *and let* $R(\cdot)$ *be an* (a, k)*-regularized resolvent family with generator* A *such that* $\|R(t)\| \leq Mk(t)$ *for all* $t \geq 0$. *Then* $(\hat{a}(\lambda))^{-1} \in \rho(A), ((\hat{a}(\lambda))^{-1} - A)^{-1} = \hat{k}(\lambda)^{-1}\hat{a}(\lambda)\hat{R}(\lambda)$, *and* $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}\| \leq M$ *for all* $\lambda > \omega$.

Proof. Under the assumption (1.1) we can take Laplace transform of the equation in (R3) to obtain

$$\hat{R}(\lambda)x = \begin{cases} \hat{k}(\lambda)x + \hat{a}(\lambda)\hat{R}(\lambda)Ax, & x \in D(A), \\ \hat{k}(\lambda)x + A\hat{a}(\lambda)\hat{R}(\lambda)x, & x \in X, \end{cases}$$

for all $\lambda > \omega$. Thus

$$\hat{k}(\lambda)^{-1}\hat{a}(\lambda)\hat{R}(\lambda)((\hat{a}(\lambda))^{-1} - A) \subset ((\hat{a}(\lambda))^{-1} - A)\hat{k}(\lambda)^{-1}\hat{a}(\lambda)\hat{R}(\lambda) = I,$$

that is, $(\hat{a}(\lambda))^{-1} \in \rho(A)$ and $((\hat{a}(\lambda))^{-1} - A)^{-1} = \hat{k}(\lambda)^{-1}\hat{a}(\lambda)\hat{R}(\lambda)$ for $\lambda > \omega$. Moreover, (1.1) implies

$$\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1}-A)^{-1}\| = \|\hat{k}(\lambda)^{-1}\hat{R}(\lambda)\| = \|\hat{k}(\lambda)^{-1}\int_0^\infty e^{-\lambda t}R(t)dt\| \le M.$$

Thus A is a generalized Hille-Yosida operator. Since

$$((\hat{a}(\lambda))^{-1} - A_1)^{-1}A_1 \subset A_1((\hat{a}(\lambda))^{-1} - A_1)^{-1} = (\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A_1)^{-1} - I,$$

$\{(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A_1)^{-1}\}$ is an A_1-regularized approximation process of order $O(\hat{a}(\lambda))(\lambda \to \infty)$ on X_1, having itself as a regularization process. Then we can deduce the following local Abelian ergodic theorem, which follows from the general results in Section 2.

Theorem 3.7. *Let $a \in L^1_{loc}([0,\infty))$ be nondecreasing, positive, and Laplace transformable, and let $R(\cdot)$ be an (a,k)-regularized resolvent family with generator A such that $\|R(t)\| \le Mk(t)$ for all $t \ge 0$.*

(i) $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x - x\| \to 0$ *as* $\lambda \to \infty$ *if and only if* $x \in X_1$.

(ii) $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1} - I\| \to 0$ *as* $\lambda \to \infty$ *if and only if* $A \in B(X)$. *In this case,* $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1} - I\| = O(\hat{a}(\lambda))(\lambda \to \infty)$.

(iii) *For* $x \in X_1$, $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x - x\| = o(\hat{a}(\lambda))(\lambda \to \infty)$ *if and only if* $x \in N(A)$.

(iv) *For* $0 < \beta \le 1$ *and* $x \in X_1$, $\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1} - A)^{-1}x - x\| = O((\hat{a}(\lambda)^\beta)$ $(\lambda \to \infty)$ *if and only if* $K(t,x,X,D(A),\|.\|_{D(A)}) = O(t^\beta)(t \to 0^+)$, *if and only if* $x \in \widetilde{D(A_1)}^{X_1}$ *in the case that* $\beta = 1$, *if and only if* $x \in D(A_1)$ *in the case that* $\beta = 1$ *and* X *is reflexive.*

(v) *A is unbounded if and only if for each* $0 < \beta < 1$ *there exists* $x^*_\beta \in X_1$ *such that*

$$\|(\hat{a}(\lambda))^{-1}((\hat{a}(\lambda))^{-1}-A)^{-1}x^*_\beta - x^*_\beta\| \begin{cases} = O((\hat{a}(\lambda)^\beta) \\ \neq o((\hat{a}(\lambda)^\beta) \end{cases} \quad (\lambda \to \infty).$$

If one takes $k(t) = j_r(t) := \frac{t^r}{\Gamma(r+1)}$, $r \geq 0$, then $l_0(t) = \frac{a*j_r(t)}{j_r(t)}$, $l_1(t) = \frac{a*a*j_r(t)}{a*j_r(t)}$, $Q_0 = \frac{R(t)}{j_r(t)}$, and $Q_1 = \frac{a*R(t)}{a*j_r(t)}$. In this case, $R(t)$ become an r-times integrated resolvent family with generator A. Then a combination of applications of Theorems 3.4 and 3.5 to $Q_0(t)$ and $Q_1(t)$ and of Theorem 3.7 leads to the following approximation and local ergodic theorem.

Lemma 3.8. *Let $T(\cdot)$ be an r-times integrated resolvent family with generator A and satisfying $\|T(t)\| \leq M\frac{t^r}{\Gamma(r+1)}$, $r > 0$, for all $t \geq 0$.*

(i) $\|(\Gamma(r+1)/t^r)T(t)x - x\| \to 0$ *as* $t \to 0^+$ *if and only if* $\|\frac{a*T(t)}{(a*t^r/\Gamma(r+1))}x - x\| \to 0$ *as* $t \to 0^+$, *if and only if* $\|\lambda(\lambda - A)^{-1}x - x\| \to 0$ *as* $\lambda \to \infty$, *if and only if* $x \in X_1$.

(ii) $\|(\Gamma(r+1)/t^r)T(t) - I\| \to 0$ *as* $t \to 0^+$, *if and only if* $\|\frac{a*T(t)}{(a*t^r/\Gamma(r+1))} - I\| \to 0$ *as* $t \to 0^+$, *if and only if* $\|\lambda(\lambda - A)^{-1} - I\| \to 0$ *as* $\lambda \to \infty$, *if and only if* $A \in B(X)$. *In this case,* $\|\frac{\Gamma(r+1)}{t^r}T(t) - I\| = O(\frac{a*(t^r/\Gamma(r+1))}{t^r/\Gamma(r+1)})(t \to 0^+)$, *if and only if* $\|\frac{a*T(t)}{(a*t^r/\Gamma(r+1))} - I\| = O(\frac{a*a*(t^r/\Gamma(r+1))}{a*(t^r/\Gamma(r+1))})(t \to 0^+)$, *if and only if* $\|\lambda(\lambda - A)^{-1} - I\| = O(\lambda^{-1})(\lambda \to \infty)$.

(iii) *For* $x \in X_1$, $\|(\Gamma(r+1)/t^r)T(t)x - x\| = o(\frac{a*(t^r/\Gamma(r+1))}{t^r/\Gamma(r+1)})(t \to 0^+)$, *if and only if* $\|\frac{a*T(t)}{(a*t^r/\Gamma(r+1))}x - x\| = o(\frac{a*a*(t^r/\Gamma(r+1))}{a*(t^r/\Gamma(r+1))})(t \to 0^+)$, *if and only if* $\|\lambda(\lambda - A)^{-1}x - x\| = o(\lambda^{-1})(\lambda \to \infty)$, *if and only if* $x \in N(A_1) = N(A)$.

(iv) *For* $0 < \beta \leq 1$ *and* $x \in X_1$, *the following are equivalent:*

- (a) $\|\frac{\Gamma(r+1)}{t^r}T(t)x - x\| = O((\frac{a*(t^r/\Gamma(r+1))}{t^r/\Gamma(r+1)})^\beta)(t \to 0^+)$;
- (b) $\|\frac{a*T(t)}{(a*t^r/\Gamma(r+1))}x - x\| = O((\frac{a*a*(t^r/\Gamma(r+1))}{a*(t^r/\Gamma(r+1))})^\beta)(t \to 0^+)$;
- (c) $\|\lambda(\lambda - A)^{-1}x - x\| = O(\lambda^{-\beta})(\lambda \to \infty)$;
- (d) $K(\frac{a*(t^r/\Gamma(r+1))}{t^r/\Gamma(r+1)}, x, X, D(A), \|.\|_D(A)) = O((\frac{a*(t^r/\Gamma(r+1))}{t^r/\Gamma(r+1)})^\beta)(t \to 0^+)$;
- (e) $x \in \widetilde{D(A_1)}^{X_1}$ *in the case that* $\beta = 1$;
- (f) $x \in D(A_1)$ *in the case that* $\beta = 1$ *and* X *is reflexive.*

(v) A *is unbounded if and only if for some(each)* $0 < \beta < 1$ *there exist* $x^*_{1,\beta}$, $x^*_{2,\beta}$, $x^*_{3,\beta} \in X_1 = \overline{D(A)}$ *such that*

$$\|(\Gamma(r+1)/t^r)T(t)x^*_{1,\beta} - x^*_{1,\beta}\| \begin{cases} = O((\frac{a*t^r}{t^r})^\beta) \\ \neq o((\frac{a*t^r}{t^r})^\beta) \end{cases} \quad (t \to 0^+),$$

$$\left\|\frac{a*T(t)}{(a*t^r/\Gamma(r+1))}x^*_{2,\beta} - x^*_{2,\beta}\right\| \begin{cases} = O((\frac{a*t^r}{t^r})^\beta) \\ \neq o((\frac{a*t^r}{t^r})^\beta) \end{cases} \quad (t \to 0^+),$$

and

$$\|\lambda(\lambda - A)^{-1}x^*_{3,\beta} - x^*_{3,\beta}\| \begin{cases} = O(\lambda^{-\beta}) \\ \neq o(\lambda^{-\beta}) \end{cases} \quad (\lambda \to \infty).$$

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