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ε-ENTROPY THEORETIC ASPECTS OF HOMEOMORPHISM PROBLEMS OF ANALYTIC FUNCTION SPACES

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Abstract. ε-Entropy theoretic solutions to the homeomorphism problem of the entire funcition spaces equipped with the compact open topology and the homeomorphism problem of the analytic function spaces equipped with the norm topology are given.

1. Introduction

In 1957, the theory of ε-entropy was developed by Kolmogorov and Arnold to solve the 13th problem formulated by Hilbert, which is closely related to the superposition representation of continuous functions of several variables by continuous functions of fewer variables (cf. [4, 5]), and in 1964, Vituskin solved the superposition representation problem of continuously differentiable functions of several variables by continuous functions of fewer variables (cf. [8]).

In 1958, Kolmogorov applied this method to the homeomorphism problems of the topological vector spaces consisting of analytic functions defined on bounded open domain (cf. [6]), which was based on the entropy theoretic classification of analytic function spaces obtained by Babenko, Erohin and Vituskin (cf. [2, 3, 7).

In this paper, the method of solving the superposition representation problems of analytic function spaces is applied to the solution to the homeomorphism problems of entire function spaces (cf. [1]). More exactly speaking, for any positive integer n, let $\mathcal{A}_n$ denote the set of all analytic functions which are bounded on the closed unit ball of $\mathbb{C}^n$ and analytic on the open unit ball of $\mathbb{C}^n$. Then, $\mathcal{A}_n$ can be topologized if $\mathcal{A}_n$ is equipped with the norm topology induced from the uniform norm on the closed unit ball. Let $\mathcal{E}_n$ denote the set of all entire functions of n variables. Then, we

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can topologize $\mathcal{E}_n$ if $\mathcal{E}_n$ is equipped with the compact open topology, and moreover, from the point of view of the ε-entropy theory, it is proved that $\mathcal{A}_n$ (resp. $\mathcal{E}_n$) is not homeomorphically homomorophic to $\mathcal{A}_m$ (resp. $\mathcal{E}_m$) if n is not equal to m.

2. ε-ENTROPY OF COMPACT SUBSETS OF ANALYTIC FUNCTION SPACES

Let $\mathcal{K}$ be the set of all compact subsets of $\mathbb{C}^n$ and for any $K \in \mathcal{K}$, let $\|\cdot\|_K$ be the norm on $\mathcal{E}_n$defined as

$$\|f|_K\|_K = \sup_{(z_1,\cdots,z_n)\in K} |f(z_1,\cdots,z_n)|, \quad f \in \mathcal{E}_n.$$

Then, a topology over $\mathcal{E}_n$ is called the compact open topology, which is denoted by τ_n, if its fundamental system of open neighborhoods is equal to $\{\{f \in \mathcal{E}_n; \|f|_K\|_K < \epsilon\}; K \in \mathcal{K}, \epsilon > 0\}$. Let U be the closed unit disc of $\mathbb{C}$ and, for any positive number s that is greater than 1. let sU denote $\{sz; z \in U\}$. For any positive integer n, let $\mathcal{E}_n$ be the set of all entire functions of n variables defined on $\mathbb{C}^n$, and let $\|\cdot\|_{n,s}$ denote the norm defined as

$$\|f|_{(sU)^n}\|_{n,s} = \sup_{z_1,\cdots,z_n\in sU} |f(z_1,\cdots,z_n)|, \quad f \in \mathcal{E}_n.$$

Especially, for any positive number M, $\mathcal{E}_n(s, M)$ denotes as the subset of $\mathcal{E}_n$ defined as

$$\mathcal{E}_n(s, M) = \left\{f \in \mathcal{E}_n; \|f|_{(sU)^n}\|_{n,s} \le M\right\}.$$

Moreover, $d_n(\cdot,\cdot)$ denotes the metric on $\mathcal{E}_n$ defined as

$$d_n(f, g) = \sum_{k=1}^{\infty} \frac{\|f-g\|_{n,k}}{2^k(1+\|f-g\|_{n,k})}, \quad f, g \in \mathcal{E}_n.$$

Then, it can be proved that $(\mathcal{E}_n, \tau_n)$ is homeomorphically homomorphic to $(\mathcal{E}_n, d_n)$ and the metric space $(\mathcal{E}_n, d_n)$ is complete and $\mathcal{E}_n(s, M)$ is a nonempty closed subset of $\mathcal{E}_n$.

Let $\mathcal{X}$ be a metric space. Then, for any positive number ϵ and for any relatively compact subset $\mathcal{F}$ of $\mathcal{X}$, the ϵ-entropy of $\mathcal{F}$, which is denoted by $S(\mathcal{F}, \epsilon)$, is defined as the base-2 logarithm of the minimum number of the cardinal numbers corresponding to all ϵ-nets of $\mathcal{F}$, and the ϵ-capacity of $\mathcal{F}$, which is denoted by $C(\mathcal{F}, \epsilon)$, is defined as the base-2 logarithm of the maximum number of the cardinal numbers corresponding to all 2ϵ-separated sets of $\mathcal{F}$.

For any positive integer n and for any positive number s that is greater than 1, let $\mathcal{A}_n(s)$ be the set of all complex valued functions of n variables which is continuous on $(sU)^n$ and analytic on the interior of $(sU)^n$. It is known that $(\mathcal{A}_n(s), \|\cdot\|_{n,s})$ is a Banach space. Then, K. I. Babenko [2] and V. D. Erohin [3] had proved that, for any positive number M, the following equality:

$$\lim_{\epsilon\to 0}\frac{S(\{f|_{U^2}; f\in\mathcal{A}_2(s); \|f|_{U^2}\|_{2,1}\le M\},\epsilon)}{\left(\log\frac{1}{\epsilon}\right)^3} = \frac{2}{3!(\log s)^2}$$

holds. After the above equality had been proved, A. G. Vitushkin [5] gave the following generalization:

$$\lim_{\epsilon\to 0}\frac{S(\{f|_{U^n}; f\in\mathcal{A}_n(s); \|f|_{U^n}\|_{n,1}\le M\},\epsilon)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}} = \frac{2}{(n+1)!(\log s)^n}.$$

Here we have the following:

Lemma 1. *For any positive integer n that is greater than 1 and for any two positive numbers r and s, if $r < s$ holds, then the following equalities:*

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{S\left(\overline{\{f|_{(rU)^n}; f\in\mathcal{E}_n(s,M)\}}^{\|\cdot\|_{n,r}},\epsilon\right)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}} &= \lim_{\epsilon\to 0}\frac{S\left(\{f|_{(rU)^n}; f\in\mathcal{E}_n(s,M)\},\epsilon\right)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}}\\ &= \frac{2}{(n+1)!(\log\frac{s}{r})^n}\end{aligned}$$

holds.

Proof. Without loss of generality, we can assume that r is equal to 1. Since the following inclusion:

$$\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\}\subset\{f|_{U^n}; f\in\mathcal{A}_n(s), \|f|_{U^n}\|_{n,1}\le M\}$$

and Babenko-Erohin-Vituskin's theorem shows that the following inequality:

$$\lim_{\epsilon\to 0}\frac{S\left(\overline{\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\}}^{\|\cdot\|_{n,1}},\epsilon\right)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}} \le \frac{2}{(n+1)!(\log s)^n}.$$

Therefore, we have only to prove the converse inequality. Let $N(\epsilon)$ be the positive integer defined as

$$N(\epsilon) = \left[\log\frac{1}{\epsilon}+1\right].$$

Moreover, let $D(\epsilon)$ be the subset of $\mathbb{Z}_+^n$ defined as

$$D(\epsilon) = \left\{(k_1,\cdots,k_n)\in\mathbb{Z}_+^n; \sum_{i=1}^n k_i\le\frac{N(\epsilon)}{\log s}\right\}.$$

Let ϕ be a mapping defined on $D(\epsilon)$ with values in $\mathbb{C}$ satisfying

$$\left|\frac{Re(\phi(k_1,\cdots,k_n))}{2\epsilon}\right| \in \mathbb{Z}_+,\ \left|\frac{Re(\phi(k_1,\cdots,k_n))}{2\epsilon}\right| \le \left[\frac{M/\sqrt{2}}{2^{n+1}\epsilon\prod_{i=1}^n(k_i+1)^2}\prod_{i=1}^n\left(\frac{1}{s}\right)^{k_i}\right],$$

$$\left|\frac{Im(\phi(k_1,\cdots,k_n))}{2\epsilon}\right| \in \mathbb{Z}_+,\ \left|\frac{Im(\phi(k_1,\cdots,k_n))}{2\epsilon}\right| \le \left[\frac{M/\sqrt{2}}{2^{n+1}\epsilon\prod_{i=1}^n(k_i+1)^2}\prod_{i=1}^n\left(\frac{1}{s}\right)^{k_i}\right],$$

where $(k_1,\cdots,k_n)$ is an element belonging to $D(\epsilon)$, and let $\Phi(\epsilon)$ be the set of all mappings satisfying the above conditions. For any $\phi \in \Phi(\epsilon)$, $g_\phi(\cdot)$ denotes the polynomial of n complex variables which is defined as

$$g_\phi(z_1,\cdots,z_n) = \sum_{(k_1,\cdots,k_n)\in D(\epsilon)} \phi(k_1,\cdots,k_n)\prod_{i=1}^n z_i^{k_i}, \quad z_1,\cdots,z_n \in \mathbb{C}.$$

If $(z_1,\cdots,z_n) \in sU$ holds, then we have

$$\begin{aligned}|g_\phi(z_1,\cdots,z_n)| &\le \sum_{(k_1,\cdots,k_n)\in D(\epsilon)} |\phi(k_1,\cdots,k_n)|\prod_{i=1}^n s^{k_i} \\ &\le \sum_{(k_1,\cdots,k_n)\in D(\epsilon)} \frac{M}{2^n\prod_{i=1}^n(k_i+1)^2} \\ &\le \frac{M}{2^n}\left(1+\sum_{k=1}^\infty\frac{1}{k(k+1)}\right)^n \\ &\le M.\end{aligned}$$

Therefore, g_ϕ is an element belonging to $\mathcal{E}_n(s,M)$. Let ϕ_1 and ϕ_2 be two elements belonging to $\Phi(\epsilon)$. Then, there exists an element $(k_1,\cdots,k_n) \in D(\epsilon)$ satisfying

$$|\phi_1(k_1,\cdots,k_n) - \phi_2(k_1,\cdots,k_n)| \ge 2\epsilon.$$

This inequality implies that

$$\|g_{\phi_1}|_{U^n} - g_{\phi_2}|_{U^n}\|_{n,1} \ge 2\epsilon$$

holds, and $\{g_\phi|_{U^n};\phi\in\Phi(\epsilon)\}$ is a 2ϵ-separated set of $\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\}$. Let $\Delta(\epsilon)$ be the subset of $\mathbb{R}_+^n$ defined as

$$\Delta(\epsilon) = \left\{(x_1,\cdots,x_n)\in\mathbb{R}_+^n; \sum_{i=1}^n x_i \le \frac{N(\epsilon)}{\log s}\right\}.$$

If $(k_1,\cdots,k_n)$ is an element belonging to $D(\epsilon)$, then, for any sufficiently small ϵ, we obtain the following inequalities:

$$\prod_{i=1}^{n}(k_i+1)^2 \le \left(\frac{2}{\log s}\right)^{2n} N(\epsilon)^{2n},$$

$$card(D(\epsilon)) \le \int\cdots\int_{\Delta(\epsilon)} dx_1\cdots dx_n = \frac{N(\epsilon)^n}{n!(\log s)^n}$$

and

$$\sum_{(k_1,\cdots,k_n)\in D(\epsilon)} \log\left(\prod_{i=1}^{n}\frac{1}{s^{k_i}}\right) = -\sum_{(k_1,\cdots,k_n)\in D(\epsilon)} \log s \sum_{i=1}^{n} k_i \ge -\int\cdots\int_{\Delta(\epsilon)} \log s \sum_{i=1}^{n}(x_i+1)\, dx_1\cdots dx_n \ge \frac{-nN(\epsilon)^{n+1}}{(n+1)!(\log s)^n} + \mathcal{O}\left(N(\epsilon)^n\right).$$

Therefore, a lower bound of $C(\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\},\epsilon)$ can be estimated as follows:

$$\begin{aligned} C(\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\},\epsilon) &\ge \log card(\Phi(\epsilon)) \\ &\ge \log \prod_{(k_1,\cdots,k_n)\in D(\epsilon)} \left(\frac{2M}{2^{n+1}\epsilon\prod_{i=1}^{n}(k_i+1)^2}\prod_{i=1}^{n}\frac{1}{s^{k_i}}+1\right)^2 \\ &\ge \frac{2N(\epsilon)^n}{n!(\log s)^n}\log\frac{1}{\epsilon} - \frac{2nN(\epsilon)^{n+1}}{(n+1)!(\log s)^n} \\ &\quad +\mathcal{O}\left(\left(\log\frac{1}{\epsilon}\right)^n \log\log\frac{1}{\epsilon}\right). \end{aligned}$$

These results imply that the following inequality:

$$\liminf_{\epsilon\to 0}\frac{C(\{f|_{U^n}; f\in\mathcal{E}_n(s,M)\},\epsilon)}{\left(\log\frac{1}{\epsilon}\right)^{n+1}} \ge \frac{2}{(n+1)!(\log s)^n}$$

holds, and therefore, we can conclude the proof. ■

3. The Homeomorphism Problem of Entire Function Spaces

Let s be any positive number that is greater than 1. Then, in this section, for any positive number M, $\mathcal{G}_n(s,M)$ denotes the subset of $\mathcal{E}_n$ defined as

$$\mathcal{G}_n(s,M) = \left\{f\in\mathcal{E}_n; \|f|_{(sU)^n}\|_{n,s} < M\right\}.$$

Here, we can prove the following

Theorem 4. *For any two positive integers m and n, There does not exist any homeomorphic homomorphism on $(\mathcal{E}_n, \tau_n)$ with values in $(\mathcal{E}_m, \tau_m)$, if m is not equal to n.*

Proof. Assume that $m < n$ holds and there exists a homeomorphic homomorphism $h_{n,m}$ on $\mathcal{E}_n$ with values in $\mathcal{E}_m$. Then, there exist two positive numbers r and L satisfying

$$\mathcal{G}_m(r, L) \subset h_{n,m}(\mathcal{G}_n(1, 1)).$$

because $h_{n,m}(\mathcal{G}_n(1, 1))$ is an open subset of $\mathcal{E}_m$. Since $h_{n,m}^{-1}(\mathcal{G}_m(r+1, L))$ is also an open subset of $\mathcal{E}_n$, there exist two positive numbers s and M satisfying

$$h_{n,m}(\mathcal{G}_n(s, M)) \subset \mathcal{G}_m(r+1, L).$$

For any sufficiently small positive number α, there exists a positive number ϵ_α satisfying the following three inequalities:

$$\begin{aligned} S\left(\left\{g|_{U^m}; g \in \mathcal{G}_m(\frac{r+1}{r}, L)\right\}, \epsilon\right) &\le \left(\frac{2}{(m+1)!(\log(\frac{r+1}{r}))^m} + \alpha\right)\left(\log\frac{1}{\epsilon}\right)^{m+1} \\ &< \left(\frac{2}{(n+1)!(\log(s+1))^n} - \alpha\right)\left(\log\frac{1}{\epsilon}\right)^{n+1} \\ &\le S\left(\{g|_{U^n}; g \in \mathcal{G}_n(s, M)\}, \epsilon\right) \end{aligned}$$

where ϵ is any positive number that is less than ϵ_α. Then, there exists a positive integer ℓ_ϵ satisfying the next estimation from above:

$$\log \ell_\epsilon \le \left(\frac{2}{(m+1)!\left(\log\left(\frac{r+1}{r}\right)\right)^m} + \alpha\right)\left(\log\frac{1}{\epsilon}\right)^{m+1}$$

and the following inclusion:

$$\left\{f|_{(rU)^m}; f \in \mathcal{G}_m(r+1, L)\right\} \subset \bigcup_{i=1}^{\ell_\epsilon} f_m^i|_{(rU)^m} + \epsilon\left\{f|_{(rU)^m}; f \in \mathcal{G}_m(r, L)\right\},$$

where $\{f_m^i; 1 \le i \le \ell_\epsilon\}$ is a finite subset of $\mathcal{E}_m(r+1, L)$. It follows from the unicity theorem that the following inclusion:

$$\mathcal{G}_m(r+1, L) \subset \bigcup_{i=1}^{\ell_\epsilon} f_m^i + \epsilon\mathcal{G}_m(r, L),$$

holds. Therefore, we have

$$h_{n,m}^{-1}\left(\mathcal{G}_m(r+1, L)\right) \subset \bigcup_{i=1}^{\ell_\epsilon} h_{n,m}^{-1}(f_m^i) + \epsilon h_{n,m}^{-1}\left(\mathcal{G}_n(r, L)\right),$$

Since the following two inclusions:

$$h_{n,m}\left(\mathcal{G}_n(s,M)\right) \subset \mathcal{G}_m\left(r+1,L\right)$$

and

$$\mathcal{G}_m(r,L) \subset h_{n,m}\left(\mathcal{G}_n(1,1)\right)$$

hold, we obtain

$$\mathcal{G}_n(s,M) \subset \bigcup_{i=1}^{\ell_\epsilon} h_{n,m}^{-1}(f_m^i) + \epsilon\mathcal{G}_n\left(1,1\right).$$

Therefore the following inclusion:

$$\{g|_{U^n}; f \in \mathcal{G}_n(s,M)\} \subset \bigcup_{i=1}^{\ell_\epsilon} h_{n,m}^{-1}(f_m^i)|_{U^n} + \{g|_{U^n}; g \in \mathcal{G}_n\left(1,\epsilon\right)\}$$

holds. This inclusion assures that there exists a finite open ϵ-covering of $\{g|_{U^n}; g \in \mathcal{G}_n(s,M)\}$ whose cardinal number is equal to ℓ_ϵ. Therefore, this fact implies that ℓ_ϵ satisfies the next estimation from below:

$$\log \ell_\epsilon \geq \left(\frac{2}{(n+1)!\,(\log(s+1))^n} - \alpha\right)\left(\log\frac{1}{\epsilon}\right)^{n+1}.$$

Actually, these two estimates contradict each other. ■

4. The Homeomorphism Problem of Banach Spaces Consisting of Analytic Functions of Several Variables

In this section, for any positive integer n, $\mathcal{A}_n(1)$ is abbreviated to $\mathcal{A}_n$ and $\|\cdot\|_{n,1}$ is also abbreviated to $\|\cdot\|_{n,1}$ for simplicity. Then, we can prove the following

Theorem 5. *For any two positive integers m and n, if m is not equal to n, there does not exist any norm-preserving homomorphism on $(\mathcal{A}_m, \|\cdot\|_m)$ with values in $(\mathcal{A}_n, \|\cdot\|_n)$ satisfying the condition that, for any element $f \in \mathcal{E}_m$, the image of $f|_{U^m}$ under the homomorphism has the analytic extension whose domain is $\mathbb{C}^n$.*

Proof. Assume that $m < n$ holds and that there exists a norm-preserving homomorphism $h_{m,n}$ on $\mathcal{A}_m$ with values in $\mathcal{A}_n$. Then, we have

$$\{f|_{U^n}; f \in \mathcal{G}_n(1,1)\} \subset h_{m,n}(\{f|_{U^m}\mathcal{G}_m(1,1)\}),$$

because $h_{m,n}$ is norm-preserving. Therefore, for any positive number ε, we can obtain,

$$S\left(\{f|_{U^n}; f\in\mathcal{G}_n(1,1)\},\varepsilon\right)\le S\left(\{f|_{U^m}; f\in\mathcal{G}_m(1,1)\},\varepsilon\right).$$

Actually, for any sufficiently small positive number α, Babenko-Erohin-Vituskin's inequalities and Lemma 1 assure that there exists a positive number ε_α satisfying

$$\begin{aligned} S\left(\{f|_{U^m}; f\in\mathcal{G}_m(1,1)\},\varepsilon\right) &\le S(\{f|_{U^m}; f\in\mathcal{A}_m(2); \|f|_{U^m}\|_m\le 1\},\epsilon)\\ &\le \left(\frac{2}{(m+1)!(\log 2)^m}+\alpha\right)\left(\log\frac{1}{\varepsilon}\right)^{m+1}, 0<\varepsilon<\varepsilon_\alpha,\end{aligned}$$

and

$$\begin{aligned}\left(\frac{2}{(n+1)!(\log 2)^n}-\alpha\right)\left(\log\frac{1}{\varepsilon}\right)^{n+1} &\le S\left(\left\{f|_{U^n}; f\in\mathcal{E}_n\left(2,\frac{1}{2}\right)\right\},\varepsilon\right)\\ &\le S\left(\{f|_{U^n}; f\in\mathcal{G}_n(1,1)\},\varepsilon\right),\ 0<\varepsilon<\varepsilon_\alpha.\end{aligned}$$

These inequalities show that Babenko-Erohin-Vituskin's inequalities and Lemma 1 contradict the condition that $h_{m,n}$ is norm-preserving. ∎

Remark 1. It can be proved that, for any two positive integers m and n, there does not exist any homeomorphic homomorphism on $(\mathcal{A}_n,\|\cdot\|_n)$ with values in $(\mathcal{A}_m,\|\cdot\|_m)$, if m is not equal to n, because the condition that a homomorphism $h_{m,n}$ is homeomorphic is equivalent to the condition that there exists two positive numbers a and b satisfying

$$a\|f\|_m\le\|h(f)\|_n\le b\|f\|_m,\quad f\in\mathcal{A}_n.$$

Remark 2. For any positive integer n, It is known that $(\mathcal{A}_n,\|\cdot\|_n)$ is a Banach space. Therefore, Banach's open mapping theorem and Theorem 5 assure that there does not exists any bijective continuous mapping on $(\mathcal{A}_n,\|\cdot\|_n)$ with values in $(\mathcal{A}_m,\|\cdot\|_m)$ if n is not equal to m.

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