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BILEVEL OPTIMIZATION PROBLEMS IN TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study the bilevel optimization problems in topological spaces without linear structure. For these problems, we establish two models which are different in the feasible region setting of lower-level problems. Some new existence results are obtained in rather weak conditions. These theorems improve and generalize the corresponding results of bilevel optimization in the literatures.

1. Introduction

We shall focus on the bilevel optimization problems with topological spaces setting in this paper. Optimization problems are encountered in a wide variety of domains, such as experts and their guesses (as in the case of expert evaluation analysis), members of a group and their votes (voting models), the various indicators of quality of a system and their values (decision making based on many criteria), or the starting characteristics as well as the partitions of objectives into classes that they generate (classification problems). Generally speaking, hierarchical optimization problems or multilevel optimization problems play an important role in the optimalization decision problems, including the engineering and experimental natural science, regional planning, management, and economics problems (see [5-9]). The bilevel (or called the Stackelberg) optimization problem is the first step to study the hierarchical optimization problems. Many management problems in company or organization are the outstanding examples. For example, we would like to choose a ”nice” car from the market in accordance with some criteria such as prices, safety, comfortableness, oil-savedness, etc. via multi-person decision behaviors. Thus we face a bilevel optimization problem with the social utility function in the objective of leader level and criteria in the objective of lower level.

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2. Generalized Stackelberg Problems

The famous Stackelberg games play an important role in economy, design of mechanical structures, transpotations, resource allocation, decomposition reformulations of large-scale mathematical programming, minimax mathematical programming and decision science. This problem is one special type of bilevel optimization problems, that have been introduced to optimization area in the seventies of the 20th century (see [2, 5-7, 9]). The framework of bilevel optimization problems is that there are both leader-level and lower-level problems, where each problem has its own objective function but interactions between the two problems. The problem framework can be described as the following

$$\text{(GSP)} \begin{cases} \min & f(x,y) \\ \text{s.t.} & y \in Sol(x),\ x \in X, \end{cases}$$

where the decision variable of leader level is $x \in X$, the decision variable of lower level is $y \in Y$, $f : X \times Y \to \mathbf{R}$ is a leader-level objective function, $Sol : X \to 2^Y$ is a set-valued mapping such that for each $x \in X$, $Sol(x)$ is the solution/reaction set of lower-level problem for a given x, characterized by

$$Sol(x) = \{y \in Y : g(x,y) \le g(x,z),\ \forall\ z \in \Omega(x)\},$$

where $g : X \times Y \to \mathbf{R}$ is a lower-level objective function, and $\Omega : X \to 2^Y$. That is, the lower-level problem for a given x, is formulated as

$$\begin{cases} \min & g(x,y) \\ \text{s.t.} & y \in \Omega(x). \end{cases}$$

In fact, the lower-level problem is parametrized by x. Variational inequalities (equilibrium problems, or complementary problems) are powerful tools for studying optimization problems, especially in existence aspects, because some kinds of existence or optimal conditions obtained by classical optimization techniques are difficult to verify in practice (see [8-10]). Moreover, if the lower-level problem happens to be a convex program, it can be formulated by a variational inequality or a generalized equation under suitable conditions (see [6, 8]).

Based on the idea above, the problem GSP, whose lower-level problem characterized by a variational inequality, complementarity problem, or equilibrium problem, is called the generalized Stackelberg problem or mathematical program with equilibrium constraints (MPEC for short), see [6, 9]. Lignola and Morgan [3,4] introduced a more wide bilevel problem than MPEC, that contains many kinds of variational problems, such as quasi-variational inequality, generalized variational inequality, generalized quasi-variational inequality, etc., in the lower-level problem and is called the bilevel variational problem (BVP for short). In [4] the lower-level problrms were characterized by the equilibrium problems, such as the Nash and generalized Nash equilibrium problems. Patriksson and Wynter [8] introduced the stochastic version of generalized Stackelberg problems.

3. Main Results

Let X and Y be topological spaces and $\gamma : X \to 2^Y$ be a set-valued mapping. For $B \subseteq Y$, let $\gamma^+(B) = \{x \in X : \gamma(x) \subseteq B\}$. The mapping γ is said to be upper semicontinuous at $x \in X$ if for all open set $V \subseteq Y$ with $x \in \gamma^+(V)$, there is a neighborhood $U(x)$ of x such that $U(x) \subseteq \gamma^+(V)$. The graph of the set-valued mapping, denoted by Gr(γ), is the set $\{(x, y) \in X \times Y : y \in \gamma(x)\}$. It is easy to see that if the set-valued mapping γ is upper semicontinuous with compact values on X, then the graph of γ is closed.

Let X and Y be topological spaces, Ω_1 and Ω_2 be nonempty closed subsets of X and Y, respectively. Consider the following problem

$$\text{(HP)}\ \ \min\,\{F(x, y) : (x, y) \in \mathrm{Gr}(S)\},$$

where $S : \Omega_1 \to 2^{\Omega_2}$ is a set-valued mapping and F is a bifunction defined on $\Omega_1 \times \Omega_2$. We note that all problems in the Section 1 can be reduced to (HP).

We now state and prove the following existence result for the problem (HP).

Theorem 1. *Let $\Omega_1$and Ω_2 be nonempty subsets of topological spaces X and Y, respectively, such that Ω_1 is compact. Suppose that*

(i) $S : \Omega_1 \to 2^{\Omega_2}$ *is an upper semicontinuous set-valued mapping with nonempty compact values,*

(ii) $F : \Omega_1 \times \Omega_2 \to [-\infty, \infty]$ *is lower semicontinuous with lower bounds on Gr(S).*

Then the problem (HP) has at least one solution.

Proof. By Aubin and Ekeland [1, Proposition 3.1.11], $S(\Omega_1)$ is compact in Y. Hence Gr$(S) = \Omega_1 \times S(\Omega_1)$ is compact in $\Omega_1 \times \Omega_2$. Since F is lower semicontinuous with lower bounds on Gr(S), let

$$\min\nolimits_{(x,y)\in \mathrm{Gr}(S)} F(x, y) = \gamma.$$

Then there exists a net $\{(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda} \subset$Gr$(S)$ such that $F(x_\lambda, y_\lambda) \to \gamma$. Since Gr$(S)$ is compact, there exists a subnet $\{(x_\alpha, y_\alpha)\}$ of $\{(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ such that $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y}) \in$Gr$(S)$. By the lower semicontinuity of F, we get that

$$F(\bar{x}, \bar{y}) \le \lim\inf_\alpha F(x_\alpha, y_\alpha) = \gamma.$$

Therefore we have

$$F(\bar{x}, \bar{y}) = \min_{(x,y)\in \mathrm{Gr}(S)} F(x, y).$$

This completes the proof. ■

Remark 1. If F is upper semicontinuous with upper bounds, then $-F$ is lower semicontinuous with lower bounds. By the above proof, there exists $(\bar{x}, \bar{y}) \in$Gr(S) such that

$$-F(\bar{x}, \bar{y}) = \min_{(x,y)\in \mathrm{Gr}(S)}(-F(x, y)).$$

Consequently, $F(\bar{x}, \bar{y}) = \max\{F(x, y) : (x, y) \in \mathrm{Gr}(S)\}$.

Let $F : \Omega_1 \times \Omega_2 \to$**R** be a leader-level objective function, $f : \Omega_1 \times \Omega_2 \to$**R** a lower-level objective function. We can also consider the following bilevel problems:

1. Let BP denote the bilevel program, formulated in terms of the optimization form as follows

$$(\mathrm{BP}) \begin{cases} \min & F(x, y) \\ \text{s.t.} & y \in Sol(x), \end{cases}$$

where $Sol : \Omega_1 \to 2^{\Omega_2}$ is a set-valued mapping such that for each $x \in \Omega_1$, $Sol(x)$ is the solution set of the lower-level problem characterized by

$$Sol(x) = \{z^* \in \Omega(x) : f(x, z^*) \le f(x, z),\ \forall\ z \in \Omega(x)\},$$

where $\Omega : \Omega_1 \to 2^{\Omega_2}$ is a constraint mapping.

2. Let BPOV denote the bilevel programming with its optimal-value response, formulated in terms of the optimization form as follows

$$(\mathrm{BPOV})\ \min\ \{F(x, v(x)) : x \in \Omega_1\},$$

where $v(x)$ denotes the optimal value function of lower-level problem characterized by

$$v(x) = \min\ \{f(x, y) : y \in \Omega(x)\},$$

where $\Omega : \Omega_1 \to 2^{\Omega_2}$ is a constraint mapping.

Note that problem (BPOV) is included in problem (BP) (see [8, 9]) and has been studied in Shimizu et al. [9]. It is interesting to observe that Mean-Variance-VaR Portfolio Problems and Generalized Stackelberg Problems can be reduced to (BP). Hence we will focus on the discussion of (BP).

The following notions were introduced by Tian and Zhou [10]. Let X and Y be two topological spaces and $G : X \to 2^Y$ be a set-valued mapping. A function $f : X \times Y \to$**R**$\cup \{\pm\infty\}$ is said to be transfer upper (respectively, lower) continuous in (x, y) with respect to G if, for every $(x, y) \in X \times Y$ with $y \in G(x)$, $f(x, z) > f(x, y)$ (respectively, $f(x, z) < f(x, y)$) for some $z \in G(x)$ implies that there exists a point $z' \in Y$ and a neighborhood $N(x, y)$ of (x, y) such that for any $(x', y') \in N(x, y)$ with $y' \in G(x')$, $f(x', z') > f(x', y')$ (respectively, $f(x', z') < f(x', y')$) and $z' \in G(x')$.

Lemma 1. (Generalized Maximum Theorem [10]) *Let X and Y be two topological spaces. Let the set-valued mapping $M : X \to 2^Y$ be defined by*

$$M(x) = \{y \in G(x) : g(x, y) = \sup_{z\in G(x)} g(x, z)\} \ \forall \ x \in X.$$

Suppose that

(i) $G : X \to 2^Y$ *is a set-valued mapping with nonempty compact values and closed graph,*

(ii) $g : X \times Y \to [-\infty, \infty]$ *is transfer upper continuous in (x, y) with respect to G.*

Then $M(x)$ is nonempty compact-valued and has the closed graph. If, in addition, G is upper semicontinuous, then M is also upper semicontinuous.

From Theorem 1 and Lemma 1, we have the following existence result.

Theorem 2. *Let $\Omega_1$and Ω_2 be nonempty subsets of topological spaces X and Y, respectively, such that Ω_1 is compact. Suppose that*

(i) $\Omega : \Omega_1 \to 2^{\Omega_2}$ *is an upper semicontinuous set-valued mapping with compact values such that $\Omega(x) \neq \emptyset$ for each $x \in \Omega_1$,*

(ii) $f : \Omega_1 \times \Omega_2 \to [-\infty, \infty]$ *is transfer lower continuous in (x, y) with respect to Ω,*

(iii) $F : \Omega_1 \times \Omega_2 \to [-\infty, \infty]$ *is lower semicontinuous with lower bounds on Gr(Sol), where $Sol : \Omega_1 \to 2^{\Omega_2}$ is a set-valued mapping defined by, for each $x \in \Omega_1$*

$$Sol(x) = \{y^* \in \Omega(x) : f(x, y^*) \leq f(x, y), \ \forall \ y \in \Omega(x)\}$$

Then the problem (BP) has a solution.

Proof. Let $g(x,y) = -f(x,y)$ for $(x,y) \in \Omega_1 \times \Omega_2$. Then

$$Sol(x) = \{y^* \in \Omega(x) : g(x,y^*) = \max_{y\in\Omega(x)} g(x,y)\}.$$

Since f is transfer lower semicontinuous in (x,y) with respect to Ω. By Lemma 1, the set-valued mapping Sol is upper semicontinuous with nonempty compact values. The conclusion of Theorem 2 now follows from Theorem 1. ■

Theorem 3. *Let Ω_1 and Ω_2 be nonempty subsets of topological spaces X and Y, respectively, such that Ω_1 is compact. Suppose that*

(i) $\Omega : \Omega_1 \to 2^{\Omega_2}$ *is an upper semicontinuous set-valued mapping with nonempty compact values,*

(ii) $f : \Omega_1 \times \Omega_2 \to [-\infty, \infty]$ *is transfer upper continuous in (x,y) with respect to Ω,*

(iii) $F : \Omega_1 \times \Omega_2 \to [-\infty, \infty]$ *is upper semicontinuous with upper bounds on Gr(S), where $S : \Omega_1 \to 2^{\Omega_2}$ is a set-valued mapping defined by, for each $x \in \Omega_1$*

$$S(x) = \{y^* \in \Omega(x) : f(x,y^*) \geq f(x,y),\ \forall\ y \in \Omega(x)\}.$$

Then there exists $(x^*, y^*) \in \mathrm{Gr}(S)$ such that

$$F(x^*, y^*) = \max_{(x,y)\in\mathrm{Gr}(S)} F(x,y).$$

Proof. The result follows from Remark 1 and Lemma 1. ■

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