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POWERS OF GENERATORS AND TAYLOR EXPANSIONS OF INTEGRATED SEMIGROUPS OF OPERATORS

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Abstract. Let A be the generator of an n-times integrated semigroup $T(\cdot)$ and let $r \in \mathbb{N}$. We first prove the equivalence of Riemann, Peano, and Taylor operators, which are three different expressions of the r-th power of A_1, the part of A in the closure of the domain $D(A)$ of A. Then we discuss optimal and non-optimal rates of approximation of $T(\cdot)x$ for $x \in D(A_1^{r-1})$, via the $(n+r)$-th Taylor expansion of $T(\cdot)$ in terms of A_1^k, $k = 0, \ldots, r-1$.

1. INTRODUCTION

The well-established theory of C_0-semigroups [7] is a powerful method in studying the first order Cauchy problem:

$$\text{(ACP)} \qquad \begin{cases} \dfrac{du(t)}{dt} = Au(t), \ t \geq 0; \\ u(0) = x. \end{cases}$$

Recently, a generalization of semigroup called n-times integrated semigroup is introduced by W. Arendt [1] and studied by many authors (cf. [1], [2], [8], [9], [10], and [11] etc.). They are useful in solving the first order Cauchy problem, in particular, when A is non-densely defined and generates an integrated semigroup. In [2], the authors showed that C_0-semigroups on a Banach space can induce integrated semigroups on embedded spaces and that, conversely, every integrated semigroup on a Banach space can be "sandwiched" by C_0-semigroups on extrapolation and interpolation spaces. In these cases, the domain of A^n plays an important role.

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On the other hand, we know that $x \in D(A^{n+1})$ is sufficient for the existence and uniqueness of a solution to the problem (ACP).

In studying integer powers A^r of the generator A of a C_0-semigroup $T(\cdot)$, Berens and Butzer [4, Sec. 2.2] introduce three equivalent operators, namely *r-th Taylor, Peano and Riemann operators* and show that they are equal to A^r. They also discussed optimal rate ($O(t^r)$) and non-optimal rate ($O(t^{(r-1+\beta)})$ $(0 < \beta < 1)$) of convergence of the r-th Taylor expansion $\sum_{k=0}^{r-1} \frac{t^k}{k!} A^k x$ of $T(\cdot)x$ for $x \in D(A^{r-1})$. Later more complete results for the case $r = 1$ was also found in [5]. In [6], we generalize the results in [5] to n-times integrated semigroups. In this context, we also refer to [12] for more recent results on regularized approximation processes.

For the more general class of integrated semigroups, the above mentioned two subjects, namely equivalent expressions of integral powers of generators and convergence rates of Taylor expansions of semigroups, seem not found in the literature yet. This paper aims to extend the mentioned known results on C_0-semigroups to n-times integrated semigroups. In Section 2, we characterize the r-th power A_1^r of the part A_1 (in $\overline{D(A)}$) of the generator A of an n-times integrated semigroup $T(\cdot)$ by studying the r-th Taylor, Peano and Riemann operators and showing that they are equivalent to A_1^r. This is a generalization of Berens and Butzer's result (on C_0-semigroups) for integrated semigroups. In Section 3, some optimal and non-optimal rates of approximation of $T(\cdot)$ by its $(n+r)$-th Taylor expansion will be obtained in Theorem 3.3. The special case $n = 0$ of it, Corollary 3.4, is even new for C_0-semigroups. These results generalize and improve the previous results (cf. [4, 5, 6]) on C_0-semigroups, and are new for integrated semigroups.

2. Equivalent Expressions of the R-th Power of Generators

Let X be a Banach space and let $B(X)$ be the Banach algebra of all bounded linear operators on X. First, we recall the definitions of an n-times integrated semigroup and its generator and give definitions of three new operators.

Definition 2.1. [1, 2, 8, 9, 10, 11] Let $n \in \mathbb{N}$. A strongly continuous family $\{T(t); t \geq 0\}$ in $B(X)$ is called an *n-times integrated semigroup* on X, if $T(0) = 0$ and

$$\begin{aligned} T(t)T(s)x &= \frac{1}{(n-1)!}\Big(\int_t^{t+s} (s+t-r)^{n-1} T(r)x\,dr \\ &\quad - \int_0^s (t+s-r)^{n-1} T(r)x\,dr \Big) \end{aligned}$$

for $x \in X$ and $t, s \geq 0$. A semigroup of class C_0, or called C_0-semigroup, is also called a 0-times integrated semigroup. $T(\cdot)$ is said to be *nondegenerate* if $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

The *generator* A of a nondegenerate n-times integrated semigroup $T(\cdot)$ is defined as follows:

$x \in D(A)$ and $Ax = y$ if and only if $T(t)x = \int_0^t T(u)y du + \frac{t^n}{n!}x$ for $t \geq 0$.

It is known [1] that A is a closed linear operator and $T(t)x - \frac{t^n}{n!}x = A\int_0^t T(u)xdu$ for all $x \in X$ and $t \geq 0$. But A need not be densely defined when $n \geq 1$. Let r be a natural number. Repeating substitution using this formula, we obtain

$$
\begin{aligned}
& T(t)x - \sum_{k=0}^{r-1} \frac{t^{n+k}}{(n+k)!} A^k x \\
(2.1) \qquad & = \begin{cases} \dfrac{1}{(r-1)!}\displaystyle\int_0^t (t-u)^{r-1} T(u) A^r x du \text{ for } x \in D(A^r); \\ \dfrac{1}{(r-1)!} A \displaystyle\int_0^t (t-u)^{r-1} T(u) A^{r-1} x du \text{ for } x \in D(A^{r-1}). \end{cases}
\end{aligned}
$$

We will use (2.1) later.

We say $A \in I_n$ if A generates an n-times integrated semigroup $T(\cdot)$ which satisfies $||T(t)|| = O(t^n)$ $(t \to 0^+)$. It is known [9] that if A satisfies the Hille-Yosida condition, then $A \in I_1$. The I_n classes will play an important role in this paper. The part of A in $\overline{D(A)}$ will be denoted by A_1. A_1 is densely defined in $\overline{D(A)}$ (cf. [6, Lemma 3.5]).

Proposition 2.2. *Let $A \in I_n$ generate an n-times integrated semigroup $T(\cdot)$. Then*

(i) $y := \lim_{t\to 0^+} \frac{n!}{t^n} T(t)x$ *exists if and only if* $x \in \overline{D(A)}$. *Moreover, in this case, we must have* $y = x$ *and* $\lim_{t\to 0^+} \frac{(n+1)!}{t^{n+1}} \int_0^t T(u)xdu = x$.

(ii) *Let* $x \in \overline{D(A)}$. *Then* $z := \lim_{t\to 0^+} \frac{n+1}{t}(\frac{n!}{t^n}T(t)x - x)$ *exists if and only if* $x \in D(A_1)$. *In this case, we have*

(a) $\lim_{t\to 0^+} \frac{n+1}{t}(\frac{(n+1)!}{t^n}T(t)x - x) = A_1 x$;

(b) $\lim_{t\to 0^+} \frac{n+2}{t}(\frac{(n+1)!}{t^{n+1}}\int_0^t T(u)xdu - x) = A_1 x$.

Proof.

(i) Suppose $y := \lim_{t\to 0^+} \frac{n!}{t^n} T(t)x$ exists. Then

$$\|\frac{(n+1)!}{t^{n+1}}\int_0^t T(u)xdu - y\| \leq \frac{n+1}{t^{n+1}}\int_0^t s^n \|\frac{n!}{s^n}T(s)x - y\| ds \to 0$$

as $t \to 0$. This implies $\frac{n!}{t^n}\int_0^t T(u)xdu \to 0$. Since, from the definition of A, we have $\frac{n!T(t)x}{t^n} - x = \frac{n!}{t^n}A\int_0^t T(u)xdu$, by the closedness of A, we see that $y = x$. Since $T(t)x = \lim_{h\to 0^+}\frac{1}{h}\int_t^{t+h} T(s)xds \in \overline{D(A)}$ for all $x \in X$ and $t \geq 0$, x also belong to $\overline{D(A)}$. The converse statement is proved in Theorem 3.3 of [6].

(ii) Since (i) implies $\lim_{t\to 0^+}\frac{(n+1)!}{t^{n+1}}\int_0^t T(u)xdu = x$, if $z := \lim_{t\to 0^+}\frac{n+1}{t}(\frac{n!}{t^n}T(t)x - x)$ exists, then, by (2.1) (for $r = 1$) and the closedness of A, we have that $x \in D(A)$ and $Ax = z \in \overline{D(A)}$, so that $x \in D(A_1)$ and $A_1x = z$. The converse and (a) and (b) are proved in Lemma 3.5 of [6].

In [4, Sec. 2.2], P. L. Butzer and H. Berens introduced three equivalent representations of the r-th power of generator of a C_0-semigroup, namely, the r-th Taylor operator, the r-th Peano operator, and the r-th Riemann operator. In this section, we extend their definitions to the case of n-times integrated semigroups and prove that they are identical to the r-th power of A_1.

Definition 2.3. Let $r \in \mathbb{N}$ and for each $t > 0$ let $B_t^{(r)}$ be the operator given by

$$B_t^{(r)}x = \frac{(n+r)!}{t^{n+r}}[T(t)x - \sum_{k=0}^{r-1}\frac{t^{n+k}}{(n+k)!}A^k x],\ x \in D(A^{r-1}).$$

The *r-th Taylor operator* $B^{(r)}$ of $T(\cdot)$ is defined by

$$B^{(r)}x = \lim_{t\to 0^+} B_t^{(r)}x$$

whenever the limit exists.

Definition 2.4. Let $r \in \mathbb{N}$. For an $x \in X$, if there exist elements $g_{k,r} \in X$, $k = 0, 1, \ldots, r-1$ such that

$$P_t^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})x = \frac{(n+r)!}{t^{n+r}}[T(t)x - \sum_{k=0}^{r-1}\frac{t^{n+k}}{(n+k)!}g_{k,r}]$$

converges as $t \to 0^+$, then we write $x \in D(P^{(r)})$ and use the notation $P^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})x$ to denote the limit. By the definition, it is easy to see that $D(P^{(r)})$ is a linear manifold in X. It will be seen that $g_{0,r}, g_{1,r}, \ldots, g_{r-1,r}$ and the corresponding limit $P^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})x$ turn out to be uniquely determined by x. So we can denote $P^{(r)}x = P^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})x$. $P^{(r)}$ is called the *r-th Peano operator*.

Lemma 2.5. *Let $A \in I_n$ generate an n-times integrated semigroup $T(\cdot)$ and let $r \in \mathbb{N}$. Suppose $x \in D(P^{(r)})$, say $P^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})x$ exists. Then we have:*

(a) $g_{0,r} = x$ *and* $x \in \overline{D(A)}$.

(b) *If* $r = 1$, *then* $x \in D(A_1)$ *and* $P^{(1)}x = A_1 x$.

(c) *If* $r \geq 2$, *then* $x \in D(P^{(k)})$ *and* $P^{(k)}(g_{0,r}, g_{1,r}, \ldots, g_{k-1,r})x = g_{k,r}$ *for each* $k = 1, 2, \ldots, r-1$. *In particular,* $g_{1,r} = P^{(1)}x = A_1 x$.

Proof.

(a) Since $\lim\limits_{t\to 0^+} \frac{t^r}{(n+r)\cdots(n+1)} P_t^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})x = 0$, it is easy to see that $\lim\limits_{t\to 0^+} \frac{n!T(t)x}{t^n} = g_{0,r}$. Then it follows from (i) of Proposition 2.2 that $g_{0,r} = x$.

(b) has been proved in (ii) of Proposition 2.2.

(c) Since $P_t^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})$ is convergent as $t \to 0^+$, we have

$$\begin{aligned}
&||P_t^{(r-1)}(g_{0,r}, g_{1,r}, \ldots, g_{r-2,r})x - g_{r-1,r}|| \\
&= ||\frac{(n+r-1)!}{t^{n+r-1}}\{T(t)x - \sum_{k=0}^{r-1} \frac{t^{n+k}}{(n+k)!} g_{k,r}\}|| \\
&= \frac{t}{(n+r)} ||P_t^{(r)}(g_{0,r}, g_{1,r}, \ldots, g_{r-1,r})x|| \to 0 \text{ as } t \to 0^+.
\end{aligned}$$

So, $x \in D(P^{(r-1)})$ and $P^{(r-1)}(g_{0,r}, g_{1,r}, \ldots, g_{r-2,r})x = g_{r-1,r}$. Repeating this argument, we have $x \in D(P^{(k)})$ and $P^{(k)}(g_{0,r}, \ldots, g_{k-1,r})x = g_{k,r}$ for all $k = 1, 2, \ldots, r-1$.

Definition 2.6. Let $r \in \mathbb{N}$ and for each $t > 0$ let operator $C_t^{(r)}x$ be given by

$$C_t^{(r)}x = \frac{(n+1)^r}{t^r}[\frac{n!T(t)}{t^n} - I]^r x, \ x \in X.$$

We define the operator $C^{(r)}$ by

$$C^{(r)}x = \lim_{t\to 0^+} C_t^{(r)}x$$

whenever the limit exists. We call $C^{(r)}$ the r*-th Riemann operator*.

For need in the proof of the main theorem (Theorem 2.10) of this section, we first prove the following three lemmas.

Lemma 2.7. *Let* $A \in I_n$ *generate an* n*-times integrated semigroup* $T(\cdot)$ *and let* $r \in \mathbb{N}$. *Then we have*

(a) $D(B^{(r)}) \subset \overline{D(A)}$.

(b) $D(P^{(r)}) \subset \overline{D(A)}$.

(c) $D(C^{(r)}) \subset \overline{D(A)}$.

Proof.

(a) Let $x \in D(B^{(r)})$. Then $\lim\limits_{t\to 0^+} \frac{t^r}{(n+r)\cdots(n+1)} B_t^{(r)} x = 0$ and it implies that $\lim\limits_{t\to 0^+} \frac{n!T(t)x}{t^n} = x$. By Proposition 2.2(i), $x \in \overline{D(A)}$.

(b) is proved in Lemma 2.5.

(c) Let $x \in D(C^{(r)})$. Then $\lim\limits_{t\to 0}[\frac{n!T(t)}{t^n} - I]^r x = 0$. If $r = 1$, then the result is true by Proposition 2.2(i). For $r \geq 2$, using the Binomial Theorem, we have that $0 = \lim\limits_{t\to 0^+}(\frac{n!T(t)}{t^n} - I)^r x = \lim\limits_{t\to 0^+} \sum\limits_{i=1}^{r} (-1)^{r-i} \frac{r!}{i!(r-i)!} (\frac{n!T(t)}{t^n})^i x - (-1)^r x$. Since $R(T(t)) \subset \overline{D(A)}$ for each $t \geq 0$, it follows that $x \in \overline{D(A)}$.

Lemma 2.8. *Let $A \in I_n$ generate an n-times integrated semigroup $T(\cdot)$ and let $r \in \mathbb{N}$. If $x \in \overline{D(A)}$, then for each $t_k \geq 0$, $k = 1, 3, \ldots, r$*

$$
\begin{aligned}
&[T(t_r) - \frac{t_r^n}{n!}I][T(t_{r-1}) - \frac{t_{r-1}^n}{n!}I]\cdots[T(t_1) - \frac{t_1^n}{n!}I]x \\
&= \lim_{t\to 0^+} \int_0^{t_r} T(u_r) \int_0^{t_{r-1}} T(u_{r-1}) \cdots \int_0^{t_1} T(u_1) C_t^{(r)} x du_1 du_2 \cdots du_r.
\end{aligned}
$$

Proof. By the definition of A and $C_t^{(r)}$, we see that for $x \in X$

$$
\begin{aligned}
&\int_0^{t_r} T(u_r) \int_0^{t_{r-1}} T(u_{r-1}) \cdots \int_0^{t_1} T(u_1) C_t^{(r)} x du_1 du_2 \cdots du_r \\
&= \left(\frac{(n+1)!}{t^{n+1}}\right)^r A^r \int_0^{t_r} T(u_r) \int_0^{t_{r-1}} T(u_{r-1}) \cdots \int_0^{t_1} T(u_1) \int_0^t T(s_r) \int_0^t T(s_{r-1}) \\
&\quad \cdots \int_0^t T(s_1) x ds_1 ds_2 \cdots ds_r du_1 du_2 \cdots du_r \\
&= \left(\tfrac{(n+1)!}{t^{n+1}}\right)^r \textstyle\int_0^t T(s_r)[T(t_r) - \frac{t_r^n}{n!}I] \int_0^t T(s_{r-1})[T(t_{r-1}) - \frac{t_{r-1}^n}{n!}I] \\
&\quad \cdots \int_0^t T(s_1)[T(t_1) - \frac{t_1^n}{n!}I] x ds_1 ds_2 \cdots ds_r.
\end{aligned}
$$

If $x \in \overline{D(A)}$, then $[T(t_r) - \frac{t_r^n}{n!}I][T(t_{r-1}) - \frac{t_{r-1}^n}{n!}I]\cdots[T(t_1) - \frac{t_1^n}{n!}I]x \in \overline{D(A)}$. Thus, by taking $t \to 0^+$ and using (i) of Proposition 2.2, one can easily deduce the asserted equality.

Lemma 2.9. *Let $A \in I_n$ generate an n-times integrated semigroup $T(\cdot)$ and let $r \in \mathbb{N}$. If x and y belong to $\overline{D(A)}$ and satisfy*

$$\begin{aligned}&[T(t_r) - \frac{t_r^n}{n!}I][T(t_{r-1}) - \frac{t_{r-1}^n}{n!}I]\cdots[T(t_1) - \frac{t_1^n}{n!}I]x \\ &= \int_0^{t_r} T(u_r)\int_0^{t_{r-1}} T(u_{r-1})\cdots\int_0^{t_1} T(u_1)ydu_1du_2\cdots du_r\end{aligned} \tag{2.2}$$

for any $t_k \geq 0$, $k = 1,\ 2\cdots r$. Then $x \in D(A_1^r)$ and $A_1^r x = y$.

Proof. We will show it by induction. For $r = 1$, the assertion is true by the definition of generators of integrated semigroups. Now, assume that $r \geq 2$ and the assertion holds for $r - 1$. Suppose (2.2) holds for $x, y \in \overline{D(A)}$. Since $f_1(t_1) := (T(t_1) - \frac{t_1^n}{n!})x$ and $g_1(t_1) := \int_0^{t_1} T(u)ydu$ lie in $\overline{D(A)}$, by the induction assumption, we have that

$$f_1(t_1) \in D(A_1^{r-1}) \text{ and } A_1^{r-1}f_1(t_1) = g_1(t_1) \text{ for all } t_1 > 0. \tag{2.3}$$

We next show that $x \in D(A_1)$. Since $f_1(t_1) \in D(A^{r-1})$, from (2.1) we get

$$\begin{aligned}[T(t) - \frac{t^n}{n!}]f_1(t_1) = &\sum_{k=1}^{r-2}\frac{t^{n+k}}{(n+k)!}A^k f_1(t_1) \\ &+\frac{1}{(r-2)!}\int_0^t (t-u)^{r-2}T(u)g_1(t_1)du.\end{aligned} \tag{2.4}$$

Now, integrating both sides of (2.4) relative to t_1 over $[0, s]$ and dividing them by t^{n+1}, we get

$$\begin{aligned}&\frac{1}{t^{n+1}}\int_0^s [T(t) - \frac{t^n}{n!}I]f_1(t_1)dt_1 \\ &= \frac{1}{t^{n+1}}[\frac{t^{n+1}}{(n+1)!}\int_0^s Af_1(t_1)dt_1 + \sum_{k=2}^{r-2}\frac{t^{n+k}}{(n+k)!}A^k\int_0^s f_1(t_1)]dt_1 \\ &+\frac{1}{t^{n+1}}\cdot\frac{1}{(r-2)!}\int_0^t (t-u)^{r-2}T(u)\int_0^s g_1(t_1)dt_1du.\end{aligned}$$

When letting $t \to 0^+$, the first term converges to $\frac{1}{(n+1)!}\int_0^s Af_1(t_1)dt_1$, the second term converges to 0. Since $\frac{1}{(n+1)!}\int_0^s g_1(t_1)dt_1 \in \overline{D(A)}$, by (i) of Proposition 2.2, the third term converges to $\frac{1}{(n+1)!}\int_0^s g_1(t_1)dt_1$ when $r = 2$ and 0 when $r > 2$. Hence

$$\frac{1}{t^{n+1}}\int_0^s [T(t) - \frac{t^n}{n!}I]f_1(t_1)dt_1 \text{ converges as } t \to 0^+. \tag{2.5}$$

Since $x \in \overline{D(A)}$ and $A \in I_n$, by Proposition 2.2(i),

$$\frac{1}{t^{n+1}}\int_0^t T(t_1)x dt_1 \text{ converges to } \frac{1}{(n+1)!}x \text{ as } t \to 0^+. \tag{2.6}$$

Since we can write

$$\begin{aligned}
&(T(s)-\frac{s^n}{n!}I)\frac{1}{t^{n+1}}\int_0^t T(t_1)xdt_1 - \frac{s^{n+1}}{n!(n+1)!}\cdot\frac{1}{t}[\frac{n!T(t)}{t^n}-I]x\\
&= \frac{1}{t^{n+1}}\int_0^s A\int_0^t T(u)T(t_1)xdudt_1 - \frac{s^{n+1}}{n!(n+1)!}\cdot\frac{1}{t}[\frac{n!T(t)}{t^n}-I]x\\
&= \frac{1}{t^{n+1}}\int_0^s [T(t)-\frac{t^n}{n!}I]T(t_1)xdt_1 - \frac{1}{t^{n+1}}\int_0^s [T(t)-\frac{t^n}{n!}I]\frac{t_1^n}{n!}xdt_1\\
&= \frac{1}{t^{n+1}}\int_0^s [T(t)-\frac{t^n}{n!}I]f_1(t_1)dt_1,
\end{aligned}$$

from (2.5) and (2.6) it follows that

$$\frac{1}{t}(\frac{n!T(t)}{t^n}-I)x \text{ is convergent to some } z \in X \text{ as } t \to 0^+. \tag{2.7}$$

Thus, $(\frac{n!T(t)}{t^n}-I)x \to 0$ as $t \to 0$, so that, by Proposition 2.2(i), we see that $x \in \overline{D(A)}$, which implies $z \in \overline{D(A)}$. Moreover, since $\frac{1}{t}(\frac{n!T(t)}{t^n}x - x) = A\frac{n!}{t^{n+1}}\int_0^t T(u)xdu$, by (2.6), (2.7), and the closedness of A, we know that $x \in D(A)$ and $Ax = (n+1)z \in \overline{D(A)}$, i.e., $x \in D(A_1)$.

It follows from (a) in (ii) and (i) of Proposition 2.2 that $\frac{(n+1)!}{t_1^{n+1}}f_1(t_1) \to A_1x$ and $\frac{(n+1)!}{t_1^{n+1}}g_1(t_1) \to \widetilde{y}$ as $t_1 \to 0^+$. These facts together with (2.3) and the closedness of A_1^{r-1} imply that $A_1x \in D(A_1^{r-1})$ and $A_1^{r-1}A_1x = y$. Hence $x \in D(A_1^r)$ and $A_1^r x = A_1^{r-1}A_1x = y$.

The following theorem is the main theorem of this section. It generalizes Theorem 2.2.13 of [4] from C_0-semigroups to n-times integrated semigroups.

Theorem 2.10. *Let $A \in I_n$ generate an n-times integrated semigroup $T(\cdot)$ and let $r \in \mathbb{N}$. The following statements are equivalent*

(a) $x \in D(A_1^r)$,

(b) $x \in D(B^{(r)})$,

(c) $x \in D(P^{(r)})$,

(d) $x \in D(C^{(r)})$.

Moreover, we have $A_1^r x = B^{(r)}x = P^{(r)}x = C^{(r)}x$ *for each* $x \in D(A_1^r)$.

Proof. We separate the proof into the following implications "(a) $\Leftrightarrow$ (b)", "(b) $\Leftrightarrow$ (c)" and "(a) $\Leftrightarrow$ (d)".

(a) $\Rightarrow$ (b). Let $x \in D(A_1^r)$. By Definition 2.3 and (2.1) it follows that $B_t^{(r)}x = \frac{(n+r)!}{(r-1)!t^{n+r}} \int_0^t (t-u)^{r-1} T(u) A_1^r x du$. So, we have

$$\begin{aligned}\lim_{t\to 0^+} ||B_t^{(r)}x - A_1^r x|| &= \lim_{t\to 0^+} ||\tfrac{(n+r)!}{(r-1)!t^{n+r}} \textstyle\int_0^t (t-u)^{r-1}(T(u) - \frac{u^n}{n!})A_1^r x du|| \\ &\le \frac{1}{(r-1)!} \overline{\lim_{t\to 0^+}} \sup_{0\le u\le t} ||(\frac{n!T(u)}{u^n} - I)A_1^r x|| = 0\end{aligned}$$

by the fact that $A_1^r x \in \overline{D(A)}$ and Proposition 2.2(i). Hence, $x \in D(B^{(r)})$ and $B^{(r)}x = A_1^r x$.

(b) $\Rightarrow$ (a). Let $x \in D(B^{(r)})$. By Lemma 2.7, we know that $x \in \overline{D(A)}$ and so $\lim\limits_{t\to 0^+} \frac{(n+1)!}{t^{n+1}} \int_0^t T(u) x du = x$, by Proposition 2.2(i). If $r = 1$, then $B^{(1)}x = \lim\limits_{t\to 0} \frac{(n+1)!}{t^{n+1}} (T(t)x - \frac{t^n}{n!}x) \in \overline{D(A)}$. Using (2.1) for the special case $r = 1$ and the fact that A is a closed operator, we have that $x \in D(A)$ and $Ax = B^{(1)}x \in \overline{D(A)}$, i.e., $x \in D(A_1)$ and $A_1 x = B^{(1)}x$. This shows "(b) $\Rightarrow$ (a)" for the case $r = 1$.

To prove the assertion by induction, we assume $r \ge 2$ and that "(b) $\Rightarrow$ (a)" holds for $r-1$. By the definition of $B^{(r)}$, we have $x \in D(A^{r-1}) \subset D(A^{r-2})$ and $B_t^{(r)}x = \frac{n+r}{t}(B_t^{(r-1)}x - A^{r-1}x)$, so that

$$\lim_{t\to 0^+} B_t^{(r-1)}x = \lim_{t\to 0^+} \frac{t}{n+r} B_t^{(r)}x + A^{r-1}x = 0 \cdot B^{(r)}x + A^{r-1}x = A^{r-1}x.$$

Hence $x \in D(B^{(r-1)})$ and $B^{(r-1)}x = A^{r-1}x$. By the induction assumption, we have $x \in D(A_1^{r-1})$ and $A_1^{r-1}x = B^{(r-1)}x = \lim\limits_{w\to 0^+} B_w^{(r-1)}x$.

Next, by the definitions of $B_t^{(r)}x$ and $B_t^{(r-1)}x$, we have

$$\begin{aligned}&\int_0^s T(u) B_t^{(r)} x du \\ &= \frac{(n+r)!}{t^{n+r}} \{ \int_0^s T(u)[T(t) - \frac{t^n}{n!}]x du - \sum_{k=1}^{r-1} \frac{t^{n+k}}{(n+k)!} \int_0^s T(u) A^k x du \} \\ &= \frac{(n+r)!}{t^{n+r}} \{ \int_0^s T(u) A \int_0^t T(w) x dw du - (T(s) - \frac{s^n}{n!}) \sum_{k=1}^{r-1} \frac{t^{n+k}}{(n+k)!} A^{k-1} x \}\end{aligned}$$

$$= \frac{(n+r)!}{t^{n+r}}[T(s)-\frac{s^n}{n!}]\int_0^t [T(w)-\sum_{k=0}^{r-2}\frac{w^{n+k}}{(n+k)!}A^k]xdw$$
$$= \frac{(n+r)}{t^{n+r}}[T(s)-\frac{s^n}{n!}]\int_0^t w^{n+r-1}B_w^{(r-1)}xdw.$$

for $s > 0$, and so

$$||\int_0^s T(u)B_t^{(r)}xdu - [T(s)-\frac{s^n}{n!}]A_1^{r-1}x||$$
$$= ||\frac{(n+r)}{t^{n+r}}[T(s)-\frac{s^n}{n!}]\int_0^t w^{n+r-1}[B_w^{(r-1)}x - A_1^{r-1}x]dw|| \to 0$$

as $t \to 0^+$. Hence, $(T(s)-\frac{s^n}{n!})A_1^{r-1}x = \int_0^s T(u)B^{(r)}xdu$, which implies $A_1^{r-1}x \in D(A)$ and $A(A_1^{r-1}x) = B^{(r)}x$. This and the fact that $T(t)x \in \overline{D(A)}$ for all $t > 0$ imply that $B_t^{(r)}x \in \overline{D(A)}$ for all $t > 0$, and hence $B^{(r)}x \in \overline{D(A)}$. It follows that $A_1^{r-1}x \in D(A_1)$ and $A_1^r x = A(A_1^{r-1}x) = B^{(r)}x$.

(b) $\Rightarrow$ (c). It is obvious by taking $g_{k,r} = A^k x$ for $k = 1, 2, \cdots, r-1$.

(c) $\Rightarrow$ (b). Let $x \in D(P^{(r)})$. Then there exist $g_{j,r} (j = 0, 1, 2, \ldots, r-1)$ such that $P_t^{(r)}(g_{0,r}, g_{1,r} \cdots g_{r-1,r})x = \frac{(n+r)!}{t^{n+r}}\{T(t)x - \sum_{k=0}^{r-1}\frac{t^{n+k}}{(n+k)!}g_{k,r}\}$ is convergent as $t \to 0^+$. By Lemma 2.5, we know that $x \in D(P^{(k)})$ and $P^{(k)}(g_{0,r}, g_{1,r} \cdots g_{k-1,r})x = g_{k,r}$ for $k = 1, \ldots, r-1$. To show $x \in D(B^{(r)})$, by comparing Definitions 2.3 and 2.4, it suffices to show that $g_{k,r} = A_1^k x$ for $k = 1, 2, \ldots, r-1$. If $k = 1$, by Lemma 2.5, we know that $P^{(1)}x = A_1 x$, $g_{0,r} = x$ and $g_{1,r} = A_1 x$. Moreover,

$$\begin{aligned} P_t^{(2)}(g_{0,r}, g_{1,r})x &= \frac{(2+n)!}{t^{n+2}}\{T(t)x - \frac{t^n}{n!}g_{0,r} - \frac{t^{n+1}}{(n+1)!}g_{1,r}\} \\ &= \frac{(2+n)!}{t^{n+2}}\{T(t)x - \frac{t^n}{n!}x - \frac{t^{n+1}}{(n+1)!}A_1 x\} \\ &= B_t^{(2)}x. \end{aligned}$$

So, $\lim_{t\to 0^+} B_t^{(2)}x = \lim_{t\to 0^+} P_t^{(2)}(g_{0,r}, g_{1,r})x = P^{(2)}(g_{0,r}, g_{1,r})x = g_{2,r}$ so that $x \in D(B^{(2)})$ and $B^{(2)}x = g_{2,r}$. Then by "$(b) \Rightarrow (a)$" for $r = 2$, we have $x \in D(A_1^2)$ and $A_1^2 x = B^{(2)}x = g_{2,r}$. Then by induction, we complete the proof that $g_{k,r} = A_1^k x$ for $k = 1, 2, \ldots, r-1$. This shows that $\{g_{k,r}; k = 1, 2, \ldots, r-1\}$ is uniquely determined by x if $x \in D(P^{(r)})$.

(a) $\Rightarrow$ (d). Define the linear operator D_t from X into X by $D_t = \frac{n!}{t^n}\int_0^t T(s)ds$ for each $t > 0$. Then there exists $M > 0$ such that $||D_t|| \le Mt$ as $t \to 0^+$.

From (i) of Proposition 2.2 we see that $\frac{(n+1)D_t x}{t}$ strongly converges to x for each $x \in \overline{D(A)}$. It implies that $\frac{(n+1)D_t A_1^k x}{t}$ strongly converges to $A_1^k x$ as $t \to 0^+$ for each $k = 1, 2, \ldots, r$ and each $x \in D(A_1^r)$. So, for each $x \in D(A_1^r)$, we have

$$
\begin{aligned}
& ||\frac{(n+1)^r}{t^r}(\frac{n!T(t)}{t^n} - I)^r x - A_1^r x|| \\
&= ||\frac{(n+1)^r}{t^r} D_t^r A_1^r x - A_1^r x|| \\
&= ||\frac{(n+1)^r}{t^r} D_t^r A_1^r x - (\frac{n+1}{t})^{r-1} D_t^{r-1} A_1^r x \\
&\quad +(\frac{n+1}{t})^{r-1} D_t^{r-1} A_1^r x - (\frac{n+1}{t})^{r-2} D_t^{r-2} A_1^r x \\
&\quad +(\frac{n+1}{t})^{r-2} D_t^{r-2} A_1^r x + \cdots + \frac{n+1}{t} D_t A_1^r x - A_1^r x|| \\
&\le ||(\frac{n+1}{t})^{r-1} D_t^{r-1} [\frac{n+1}{t} D_t A_1^r x - A_1^r x]|| \\
&\quad +||(\frac{n+1}{t})^{r-2} D_t^{r-2} [\frac{n+1}{t} D_t A_1^r x - A_1^r x]|| \\
&\quad + \cdots + ||\frac{n+1}{t} D_t A_1^r x - A_1^r x|| \\
&\le \sum_{k=0}^{r-1} (n+1)^k M^k ||\frac{n+1}{t} D_t A_1^r x - A_1^r x|| \to 0 \text{ as } t \to 0^+.
\end{aligned}
$$

So, $x \in D(C^{(r)})$ and $C^{(r)} x = A_1^r x$.

(d) $\Rightarrow$(a). Suppose that $x \in D(C^{(r)})$ and $C^{(r)} x = y$. By Lemma 2.7, we know that $x \in \overline{D(A)}$. Then by Lemma 2.8 we have that

$$
\begin{aligned}
& [T(t_r) - \frac{t_r^n}{n!} I][T(t_{r-1}) - \frac{t_{r-1}^n}{n!} I] \cdots [T(t_1) - \frac{t_1^n}{n!} I] x \\
& - \int_0^{t_r} T(u_r) \int_0^{t_{r-1}} T(u_{r-1}) \cdots \int_0^{t_1} T(u_1) y du_1 \cdots du_r \\
= & \lim_{t \to 0} \int_0^{t_r} T(u_r) \int_0^{t_{r-1}} T(u_{r-1}) \cdots \int_0^{t_1} T(u_1)(C_t^{(r)} x - y) du_1 \cdots du_r = 0
\end{aligned}
$$

for any arbitrarily fixed $t_k > 0$, $k = 1, 2 \cdots r$. Hence, by Lemma 2.9, it follows that $x \in D(A_1^r)$ and $A_1^r x = y = C^{(r)} x$.

3. Approximation Via $(n+r)$-th Taylor Expansion

In this section, we consider approximation of n-times integrated semigroups via their $(n+r)$-th Taylor expansions. First, we recall the definition of a K-functional.

Definition 3.1. Let X be a Banach space with norm $||\cdot||_X$ and Y be a submanifold with seminorm $||\cdot||_Y$. The K-*functional* is defined by

$$K(t,x) := K(t,x,X,Y,\|\cdot\|_Y) = \inf_{y\in Y}\{||x-y||_X + t||y||_Y\}.$$

It is well known that $K(t,x)$ is a bounded, continuous, monotonically increasing and subadditive function of t for each $x \in X$ [3].

The following approximation theorem can be found in [6, Theorem 3.10] and [12, Theorem 4.8].

Proposition 3.2. *Let $A \in I_n$ generate an n-times integrated semigroup $T(\cdot)$. Then the following statements are equivalent for $0 < \beta \le 1$ and $x \in \overline{D(A)}$*

(a) $||\frac{n!}{t^n}T(t)x - x|| = O(t^\beta)$ $(t \to 0^+)$,

(b) $K(t,x,X,D(A),||\cdot||_{D(A)}) = O(t^\beta)$ $(t \to 0^+)$.

The main result of this section is the following approximation theorem which treats both the optimal ($\beta = 1$) and non-optimal ($0 < \beta < 1$) rates of convergence.

Theorem 3.3. *Let $A \in I_n$ generate an n-times integrated semigroup $T(\cdot)$ and let $r \ge 1$ be a natural number. Then the following statements are equivalent for $0 < \beta \le 1$ and $x \in D(A_1^{r-1})$.*

(a)

$$||[T(t) - \sum_{j=n}^{n+r-1} \frac{t^j}{j!}A_1^{j-n}]x|| = O(t^{(r-1+n+\beta)})\ (t \to 0^+).$$

(b) *There exist $g_{k,r} \in X$ for $k = 0,1,\ldots,r-1$ such that*

$$||T(t)x - \sum_{j=n}^{n+r-1} \frac{t^j}{j!}g_{j-n,r}|| = O(t^{(r-1+n+\beta)})\ (t \to 0^+).$$

(c) $K(t,A^{r-1}x,X,D(A),||\cdot||_{D(A)}) = O(t^\beta)$ $(t \to 0^+)$.

Proof. First, we write

$$\sum_{j=n}^{n+r-1} \frac{t^j}{j!}A_1^{j-n} = \sum_{k=0}^{r-1} \frac{t^{n+k}}{(n+k)!}A_1^k$$

and

$$\sum_{j=n}^{n+r-1} \frac{t^j}{j!} g_{j-n,r} = \sum_{k=0}^{r-1} \frac{t^{n+k}}{(n+k)!} g_{k,r}.$$

(a) $\Rightarrow$ (c). The case $r = 1$ is true by Proposition 3.2. To prove it for $r \geq 2$, let $y_t := \frac{(n+r)!}{t^{n+r}} \int_0^t (T(u) - \sum\limits_{k=0}^{r-2} \frac{u^{n+k}}{(n+k)!} A^k) x du$. Then $y_t \in D(A)$ and

$$Ay_t = \frac{(n+r)!}{t^{n+r}} [(T(t) - \sum_{k=0}^{r-1} \frac{t^{n+k}}{(n+k)!} A^k] x.$$

Now using (a) we have

$$t||Ay_t|| = ||\frac{(n+r)!}{t^{n+r-1}} [(T(t) - \sum_{k=0}^{r-1} \frac{t^{n+k}}{(n+k)!} A_1^k] x|| = O(t^\beta) \ (t \to 0^+)$$

and

$$\begin{aligned}
&||A^{r-1}x - y_t|| = ||A_1^{r-1}x - y_t|| \\
&= \left\| \frac{(n+r)!}{t^{n+r}} \int_0^t \left(T(u) - \sum_{k=0}^{r-1} \frac{u^{n+k}}{(n+k)!} A_1^k \right) x du \right\| \\
&\leq \frac{(n+r)!}{t^{n+r}} \int_0^t \left\| \left(T(u) - \sum_{k=0}^{r-1} \frac{u^{n+k}}{(n+k)!} A_1^k \right) x \right\| du = O(t^\beta) \ (t \to 0^+).
\end{aligned}$$

Hence (c) holds by the definition of K-functional.

(c) $\Rightarrow$ (a). The case $r = 1$ is obvious by Proposition 3.2. For $r \geq 2$, since $A_1^{r-1} x \in \overline{D(A)}$, by (2.1) and Proposition 3.2(applied to $A_1^{r-1} x$), we see that (c) implies

$$\begin{aligned}
&||\frac{(n+r)!}{t^{n+r-1}} [T(t) - \sum_{k=0}^{r-1} \frac{t^{n+k}}{(n+k)!} A_1^k] x|| \\
&= ||\frac{(n+r)!}{t^{n+r-1}} \cdot \frac{1}{(r-2)!} \int_0^t (t-u)^{r-2} A \int_0^u T(s) A_1^{r-1} x ds du \\
&= ||\frac{(n+r)!}{t^{n+r-1}} \cdot \frac{1}{(r-2)!} \int_0^t (t-u)^{r-2} \frac{u^n}{n!} (\frac{n! T(u)}{u^n} A_1^{r-1} x - A_1^{r-1} x) du|| \\
&= O(t^\beta) \ (t \to 0^+),
\end{aligned}$$

which is the same as (a).

(a) $\Rightarrow$ (b). It is obvious.

(b) $\Rightarrow$ (a). Since (b) implies

$$\begin{aligned}
&\lim_{t\to 0^+} ||P_t^{(r-1)}(g_{0,r}, g_{1,r}\cdots g_{r-2,r})x - g_{r-1,r}|| \\
&= \lim_{t\to 0^+} ||\frac{(n+r-1)!}{t^{r-1+n}}(T(t)x - \sum_{k=0}^{r-2}\frac{t^{n+k}}{(n+k)!}g_{k,r}) - g_{r-1,r}|| \\
&= \lim_{t\to 0^+} ||\frac{(n+r-1)!}{t^{r-1+n}}[T(t)x - \sum_{k=0}^{r-1}\frac{t^{n+k}}{(n+k)!}g_{k,r}]|| \\
&= 0,
\end{aligned}$$

we have $P^{(r-1)}(g_{0,r}, g_{1,r}\cdots g_{r-1,r})x = g_{r-1,r}$. From the proof of ((c) $\Rightarrow$ (b)) in Theorem 2.10, we see that $g_{0,r} = x$ and $g_{k,r} = A_1^k x$ for $k = 1, \ldots, r-1$. It follows that (a) is true.

For the special case $n = 0$ we obtain the following corollary which is new.

Corollary 3.4. *Let A generate a C_0-semigroup $T(\cdot)$ and let $r \geq 1$ be a natural number. Then the following statements are equivalent for $0 < \beta \leq 1$ and $x \in D(A^{r-1})$.*

(a)

$$||[T(t) - \sum_{k=0}^{r-1}\frac{t^k}{k!}A^k]x|| = O(t^{(r-1+\beta)})\ (t \to 0^+).$$

(b) *There exist $g_{k,r} \in X$, $k = 0, 1, \ldots, r-1$, such that*

$$||T(t)x - \sum_{k=0}^{r-1}\frac{t^k}{k!}g_{k,r}|| = O(t^{(r-1+\beta)})\ (t \to 0^+).$$

(c) $K(t, A^{r-1}x, X, D(A), ||\cdot||_{D(A)}) = O(t^\beta)\ (t \to 0^+)$.

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