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ISHIKAWA ITERATION WITH ERRORS FOR APPROXIMATING FIXED POINTS OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS OF BROWDER-PETRYSHYN TYPE

L. C. Zeng*, G. M. Lee and N. C. Wong**

Abstract. Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a strictly pseudocontractive mapping in the sense of F. E. Browder and W. V. Petryshyn [1]. Let $\{u_n\}$ be a bounded sequence in K and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be real sequences in [0,1] satisfying some restrictions. Let $\{x_n\}$ be the bounded sequence in K generated from any given $x_1 \in K$ by the Ishikawa iteration method with errors: $y_n = (1 - \beta_n)x_n + \beta_n T x_n, x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n, n \geq 1$. It is shown in this paper that if T is compact or demicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

1. INTRODUCTION

Let E be a real Banach space with norm $\|\cdot\|$ and dual E^*. Let $\langle\cdot,\cdot\rangle$ denote the generalized duality pairing between E and E^*, and let $J_q : E \to 2^{E^*} (q > 1)$ denote the generalized duality mapping defined as the following: for each $x \in E$,

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q = \|x\|\|f\|\}.$$

In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is well known (see [6]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$, and that if E^* is strictly convex then J_q is single-valued. In the sequel we shall denote the single-valued generalized duality mapping by j_q.

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Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be strictly pseudocontractive [1] if for all $x, y \in D(T)$, there exist $\lambda > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \lambda\|x-y-(Tx-Ty)\|^2. \tag{1.1}$$

Remark 1.1. Without loss of generality we may assume $\lambda \in (0,1)$. If I denotes the identity operator, then (1.1) can be rewritten in the form

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \lambda\|(I-T)x-(I-T)y\|^2. \tag{1.2}$$

In Hilbert space, (1.1) (and hence (1.2)) is equivalent to the following inequality:

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \; k = (1-\lambda) < 1.$$

Definition 1.2. A mapping T with domain $D(T)$ and range $R(T)$ in E is called

(i) compact if for any bounded sequence $\{x_n\}$ in $D(T)$, there exists a strongly convergent subsequence of $\{Tx_n\}$;

(ii) demicompact if for any bounded sequence $\{x_n\}$ in $D(T)$, whenever $\{x_n - Tx_n\}$ is strongly convergent, there exists a strongly convergent subsequence of $\{x_n\}$.

In 1974, Rhoades [4] proved the following strong convergence theorem using the Mann iteration method.

Theorem 1.1. *Let H be a real Hilbert space and K be a nonempty compact convex subset of H. Let $T : K \to K$ be a strictly pseudocontractive mapping and let $\{\alpha_n\}$ be a real sequence satisfying the following conditions:* (i) $\alpha_0 = 1$; (ii) $0 < \alpha_n < 1, \; \forall n \geq 1$; (iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; (iv) $\lim_{n\to\infty}\alpha_n = \alpha < 1$.
Then the sequence $\{x_n\}$ generated from an arbitrary $x_0 \in K$ by the Mann iteration method

$$x_{n+1} = (1-\alpha_n)x_n + \alpha T x_n, \; \forall n \geq 1,$$

converges strongly to a fixed point of T.

Recently, Osilike and Udomene [3] improved, unified and developed Theorem 1.1 and Browder and Petryshyn's corresponding result [1] in the following aspects: (1) Hilbert spaces are extended to the setting of $q-$uniformly smooth Banach spaces. (2) The Mann iteration method is extended to the case of Ishikawa iteration method.

Theorem 1.2. [3, Corollary 2] *Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E, $T : K \to K$*

be a demicompact strictly pseudocontractive mapping with a nonempty fixed-point set, i.e.,$F(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $0 < a \leq \alpha_n^{q-1} \leq b < (q\lambda^{q-1}/c_q)(1-\beta_n),\ \forall n \geq 1$ *and for some constants* $a, b \in (0,1)$;

(ii) $\sum_{n=1}^{\infty} \beta_n^{\tau} < \infty$, *where* $\tau = \min\{1, (q-1)\}$.

Then the sequence $\{x_n\}$ *generated from an arbitrary* $x_1 \in K$ *by the Ishikawa iteration method*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,\ n \geq 1. \end{cases}$$

converges strongly to a fixed point of T.

Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex (not necessarily bounded) subset of E, and $T: K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K,$\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in [0,1] satisfying certain restrictions. Let $\{x_n\}$ be the bounded sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration method with errors

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n,\ n \geq 1. \end{cases}$$

It is shown in this paper that if T is compact or demicompact then $\{x_n\}$ converges strongly to a fixed point of T. Our result improves, extends and develops Osilike and Udomene [3, Corollary 2] in the following aspects: (1) The Ishikawa iteration method is extended to the case of Ishikawa iteration method with errors. (2) The stronger condition (ii) in [3, Corollary 2] is removed and replaced by a weaker condition which is convenient to verify. In addition, our result also improves and generalizes corresponding results in [1] and [4], respectively.

2. Preliminaries

In this section, we give some preliminaries whih will be used in the rest of this paper. From (1.2) we have

$$\|x-y\| \geq \lambda\|x-y-(Tx-Ty)\| \geq \lambda\|Tx-Ty\| - \lambda\|x-y\|,$$

so that

$$\|Tx-Ty\| \leq L\|x-y\|,\ \forall x, y \in K, \text{where}\, L = (1+\lambda)/\lambda.$$

Since $\|x-y\| \geq \lambda\|x-y-(Tx-Ty)\|$, we have

$$\begin{aligned} \langle x-Tx-(y-Ty), j_q(x-y)\rangle &= \|x-y\|^{q-2}\langle x-Tx-(y-Ty), j(x-y)\rangle \\ &\geq \lambda\|x-y\|^{q-2}\|x-Tx-(y-Ty)\|^2 \\ &\geq \lambda^{q-1}\|x-Tx-(y-Ty)\|^q. \end{aligned} \tag{2.1}$$

Recall that the modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq \tau\}.$$

E is uniformly smooth if and only if $\lim_{\tau\to 0^+}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. The space E is said to be $q-$uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c_q > 0$ such that $\rho_E(\tau) < c_q\tau^q$. Hilbert spaces, L_p, l_p spaces, $1 < p < \infty$, and the Sobolev spaces, $W_m^p, 1 < p < \infty$, are uniformly smooth. Hilbert spaces are 2-uniformly smooth while if $1 < p < 2$, then L_p, l_p and W_m^p is $p-$uniformly smooth; if $p \geq 2$, then L_p, l_p and W_m^p are 2-uniformly smooth.

Theorem 2.1. [6, p. 1130] *Let $q > 1$ and E be a real Banach space. Then the following are equivalent:*

(1) *E is $q-$uniformly smooth.*

(2) *There exists a constant $c_q > 0$ such that for all $x, y \in E$*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{2.2}$$

(3) *There exists a constant d_q such that for all $x, y \in E$ and $t \in [0,1]$*

$$\|(1-t)x+ty\|^q \geq (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x-y\|^q, \tag{2.3}$$

where $\omega_q(t) = t^q(1-t) + t(1-t)^q$.

Furthermore, Xu and Roach [7, Remark 5] proved that if E is $q-$uniformly smooth ($q > 1$), then for all $x, y \in E$, there exists a constant $L_* > 0$ such that

$$\|j_q(x) - j_q(y)\| \leq L_*\|x-y\|^{q-1}. \tag{2.4}$$

Lemma 2.1. [5, p. 303] *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers such that $\sum_{n=1}^\infty b_n < \infty$ and $a_{n+1} < a_n + b_n$, $\forall n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.*

3. Main Results

In this section, Let λ be the constant appearing in (1.1), $L = \frac{1+\lambda}{\lambda}$ be the Lipschitz constant of T, and c_q, d_q, $w_q(t)$, and L_* be the constants appearing in equations (2.2)-(2.4), respectively.

Lemma 3.1. *Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty convex subset of E, $T : K \to K$ be strictly pseudocontractive with $F(T) \neq \emptyset$. Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in K, $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ with $\alpha_n + \gamma_n \leq 1,\ \forall n \geq 1.$ Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the following Ishikawa iteration method with errors:*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n. \end{cases} \tag{3.1}$$

Then, for any x^* in $F(T)$,

$$\begin{aligned} \|x_{n+1}-x^*\|^q \ \leq\ & (1+2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q+a_n\beta_n^{q-1}qL_*(1+L)^{q+1} \\ & +a_n\beta_n q\lambda^{q-1}(1+L^2)^q)\|x_n-x^*\|^q \\ & -a_n(q\lambda^{q-1}-c_q a_n^{q-1})\|x_n-Ty_n\|^q \\ & +q\|e_n\|\|x_{n+1}-e_n-x^*\|^{q-1}+c_q\|e_n\|^q, \end{aligned} \tag{3.2}$$

where $a_n = \alpha_n + \gamma_n$, and $e_n = \gamma_n(u_n - Ty_n),\ \forall n \geq 1.$

Proof. For each $n \geq 1$, set $a_n = \alpha_n + \gamma_n$ and $e_n = \gamma_n(u_n - Ty_n)$. Then it follows from (3.1) that for each $n \geq 1$,

$$x_{n+1} = (1-a_n)x_n + a_n T y_n + e_n.$$

Let x^* be an arbitrary fixed point of T. Then from (2.2) we obtain

$$\begin{aligned} \|x_{n+1}-x^*\|^q &= \|(1-a_n)x_n + a_nTy_n + e_n - x^*\|^q \\ &\leq \|(1-a_n)x_n + a_nTy_n - x^*\|^q \\ &\quad +q\langle e_n, j_q(x_{n+1}-e_n-x^*)\rangle + c_q\|e_n\|^q \\ &\leq \|(1-a_n)x_n + a_nTy_n - x^*\|^q \\ &\quad +q\|e_n\|\|x_{n+1}-e_n-x^*\|^{q-1} + c_q\|e_n\|^q. \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} \|(1-a_n)x_n+a_nTy_n-x^*\|^q &= \|x_n-x^*-a_n(x_n-Ty_n)\|^q \\ &\leq \|x_n-x^*\|^q - qa_n\langle x_n-Ty_n, j_q(x_n-x^*)\rangle \\ &\quad +a_n^q c_q\|x_n-Ty_n\|^q, \end{aligned} \tag{3.4}$$

$$(3.5)\quad \begin{aligned} \langle x_n - Ty_n, j_q(x_n - x^*)\rangle &= \langle x_n - y_n, j_q(x_n - x^*)\rangle + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle \\ &= \beta_n \langle x_n - Tx_n - (x^* - Tx^*), j_q(x_n - x^*)\rangle \\ &\quad + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle \\ &\geq \beta_n \lambda^{q-1} \|x_n - Tx_n - (x^* - Tx^*)\|^q \\ &\quad + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle \\ &= \beta_n \lambda^{q-1} \|x_n - Tx_n\|^q + \langle y_n - Ty_n, j_q(x_n - x^*)\rangle, \end{aligned}$$

and

$$(3.6)\quad \begin{aligned} \langle y_n - Ty_n, j_q(x_n - x^*)\rangle &= \langle y_n - Ty_n - (x^* - Tx^*), j_q(x_n - x^*) - j_q(y_n - x^*)\rangle \\ &\quad + \langle y_n - Ty_n - (x^* - Tx^*), j_q(y_n - x^*)\rangle \\ &\geq \lambda^{q-1} \|y_n - Ty_n - (x^* - Tx^*)\|^q \\ &\quad + \langle y_n - Ty_n - (x^* - Tx^*), j_q(x_n - x^*) - j_q(y_n - x^*)\rangle. \end{aligned}$$

Furthermore, using (2.3), we have

$$(3.7)\quad \begin{aligned} \|y_n - Ty_n\|^q &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)\|^q \\ &\geq (1-\beta_n)\|x_n - Ty_n\|^q + \beta_n \|Tx_n - Ty_n\|^q \\ &\quad - \omega_q(\beta_n) d_q \|x_n - Tx_n\|^q. \end{aligned}$$

Thus, from (3.4)-(3.7) we get

$$\begin{aligned} \|x_{n+1} - x^*\|^q &\leq \|x_n - x^*\| - q a_n \{\beta_n \lambda^{q-1} \|x_n - Tx_n\|^q + \lambda^{q-1}(1-\beta_n)\|x_n - Ty_n\|^q \\ &\quad + \lambda^{q-1} \beta_n \|Tx_n - Ty_n\|^q - \lambda^{q-1} \omega_q(\beta_n) d_q \|x_n - Tx_n\|^q \\ &\quad + \langle y_n - Ty_n, j_q(x_n - x^*) - j_q(y_n - y^*)\rangle\} \\ &\quad + a_n^q c_q \|x_n - Ty_n\|^q + q\|e_n\| \|x_{n+1} - e_n - x^*\|^{q-1} + c_q \|e_n\|^q \\ &\leq \|x_n - x^*\|^q - a_n(q\lambda^{q-1}(1-\beta_n) - a_n^{q-1} c_q)\|x_n - Ty_n\|^q \\ &\quad + q d_q \lambda^{q-1} a_n \omega_q(\beta_n) \|x_n - Tx_n\|^q \\ &\quad + q a_n \|y_n - Ty_n\| \|j_q(x_n - x^*) - j_q(y_n - x^*)\| \\ &\quad + q\|e_n\| \|x_{n+1} - e_n - x^*\|^{q-1} + c_q \|e_n\|^q. \end{aligned}$$

On the other hand, observe that

$$\omega_q(\beta_n) = \beta_n(1-\beta_n)^q + \beta_n^q(1-\beta_n) \leq 2\beta_n,$$

$$\|x_n - Tx_n\| \leq (1+L)\|x_n - x^*\|,$$

$$\begin{aligned}\|j_q(x_n - x^*) - j_q(y_n - x^*)\| &\leq L_*\beta_n^{q-1}\|x_n - Tx_n\|^{q-1} \text{ (using (2.4))} \\ &\leq L_*(1+L)^{q-1}\beta_n^{q-1}\|x_n - x^*\|^{q-1},\end{aligned}$$

and

$$\begin{aligned}\|y_n - Ty_n\| &\leq (1+L)\|y_n - x^*\| \\ &\leq (1+L)[(1-\beta_n)\|x_n - x^*\| + \beta_n L\|x_n - x^*\|] \\ &\leq (1+L)^2\|x_n - x^*\|.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_{n+1} - x^*\|^q \leq\ & (1 + 2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q \\ & +a_n\beta_n^{q-1}qL_*(1+L)^{q+1})\|x_n - x^*\|^q \\ & -a_n(q\lambda^{q-1}(1-\beta_n) - a_n^{q-1}c_q)\|x_n - Ty_n\|^q \\ & +q\|e_n\|\|x_{n+1} - e_n - x^*\|^{q-1} + c_q\|e_n\|^q.\end{aligned} \tag{3.8}$$

Note that

$$\|Ty_n - x^*\| \leq L\|y_n - x^*\| \leq L((1-\beta_n)\|x_n - x^*\| + \beta_n\|Tx_n - x^*\|) \leq L^2\|x_n - x^*\|,$$

and

$$\|x_n - Ty_n\| \leq \|x_n - x^*\| + \|Ty_n - x^*\| \leq (1+L^2)\|x_n - x^*\|.$$

Therefore, from (3.8) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^q \leq\ & (1 + 2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q + a_n\beta_n^{q-1}qL_*(1+L)^{q+1} \\ & +a_n\beta_n q\lambda^{q-1}(1+L^2)^q)\|x_n - x^*\|^q \\ & -a_n(q\lambda^{q-1} - a_n^{q-1}c_q)\|x_n - Ty_n\|^q \\ & +q\|e_n\|\|x_{n+1} - e_n - x^*\|^{q-1} + c_q\|e_n\|^q.\end{aligned}$$

Lemma 3.2. *Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty convex subset of E, and $T : K \to K$ be strictly pseudocontractive with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1, \ \forall n \geq 1;$

(ii) $\overline{\lim}_{n\to\infty}\alpha_n < \lambda(q/c_q)^{1/(q-1)}$, $\overline{\lim}_{n\to\infty}\beta_n < 1/L$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n \beta_n^{\tau} < \infty$, *where* $\tau = \min\{1, (q-1)\}$.

Let $\{x_n\}$ *be the bounded sequence generated from an arbitrary* $x_1 \in K$ *by the Ishikawa iteration method (3.1) with errors. Then,*

(a) *for each* $x^* \in F(T), \lim_{n\to\infty}\|x_n - x^*\|$ *exists;*

(b) *there exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *such that* $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$.

Proof. From Lemma 3.1, we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^q \ \leq \ & (1+\delta_n)\|x_n - x^*\|^q - a_n(q\lambda^{q-1} - c_q a_n^{q-1})\|x_n - Ty_n\|^q \\ & + q\|e_n\|\|x_{n+1} - e_n - x^*\|^{q-1} + c_q\|e_n\|^q, \end{aligned} \tag{3.9}$$

where $a_n = \alpha_n + \gamma_n, e_n = \gamma_n(u_n - Ty_n)$, and

$$\delta_n = 2a_n\beta_n\lambda^{q-1}qd_q(1+L)^q + a_n\beta_n^{q-1}qL_*(1+L)^{q+1} + a_n\beta_n q\lambda^{q-1}(1+L^2)^q, \ \forall n \geq 1.$$

Since $\|x_n - Ty_n\| \leq (1 + L^2)\|x_n - x^*\|$, it follows from the boundedness of $\{x_n\}$ that $\{Ty_n\}$ is bounded. Hence, we know that $\{u_n - Ty_n\}$ is bounded. Note that $\sum_{n=1}^{\infty} \gamma_n < \infty$. Thus, we infer that

$$\sum_{n=1}^{\infty} \|e_n\| = \sum_{n=1}^{\infty} \|\gamma_n(u_n - Ty_n)\| < \infty,$$

which hence implies that

$$\sum_{n=1}^{\infty} \|e_n\|^q < \infty.$$

Since $\{e_n\}$ and $\{x_n\}$ are both bounded, there exists a number $M > 0$ such that

$$\|x_n - x^*\| \leq M \text{ and } \|x_{n+1} - e_n - x^*\| \leq M, \ \forall n \geq 1$$

Hence, from (3.9) we get

$$\begin{aligned} \|x_{n+1} - x^*\|^q \ \leq \ & \|x_n - x^*\|^q - a_n(q\lambda^{q-1} - a_n^{q-1}c_q)\|x_n - Ty_n\|^q \\ & + \delta_n M^q + q\|e_n\|M^{q-1} + c_q\|e_n\|^q. \end{aligned} \tag{3.10}$$

Since $\overline{\lim}_{n\to\infty}\alpha_n < \lambda(q/c_q)^{1/(q-1)}$, we have $\overline{\lim}_{n\to\infty}a_n < \lambda(q/c_q)^{1/(q-1)}$. So, for any given $\varepsilon > 0$, there exists an integer $N_0 \geq 1$ such that $\sup_{n\geq N_0} a_n <$

$\lambda(q/c_q)^{1/(q-1)}$. Let $b = \sup_{n\geq N_0} a_n$. Then for all $n \geq N_0$, we have $a_n \leq b < \lambda(q/c_q)^{1/(q-1)}$. Obviously, it is easy to see that

$$q\lambda^{q-1} - a_n^{q-1}c_q \geq q\lambda^{q-1} - b^{q-1}c_q = c_q(\lambda^{q-1}(q/c_q) - b^{q-1}) > 0.$$

Consequently, (3.10) reduces to

$$\begin{aligned} \|x_{n+1} - x^*\|^q \;&\leq\; \|x_n - x^*\|^q - a_n(q\lambda^{q-1} - b^{q-1}c_q)\|x_n - Ty_n\|^q \\ &\quad +\delta_n M^q + \|e_n\|qM^{q-1} + c_q\|e_n\|^q,\ \forall n \geq N_0, \end{aligned} \tag{3.11}$$

which hence implies that

$$\|x_{n+1} - x^*\|^q \leq \|x_n - x^*\|^q + \delta_n M^q + \|e_n\|qM^{q-1} + c_q\|e_n\|^q.$$

Since

$$\sum_{n=1}^{\infty}\|e_n\| < \infty, \sum_{n=1}^{\infty}\|e_n\|^q < \infty, \sum_{n=1}^{\infty}\gamma_n < \infty \text{ and } \sum_{n=1}^{\infty}\alpha_n\beta_n^{\tau} < \infty,$$

we conclude that

$$\sum_{n=1}^{\infty}(\delta_n M^q + \|e_n\|qM^{q-1} + c_q\|e_n\|^q) < \infty.$$

Hence, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

On the other hand, from (3.11) we deduce that for all $n \geq N_0$

$$\begin{aligned} (q\lambda^{q-1} - b^{q-1}c_q)a_n\|x_n - Ty_n\|^q \;&\leq\; \|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q + \delta_n M^q \\ &\quad +\|e_n\|qM^{q-1} + c_q\|e_n\|^q \end{aligned}$$

from which it follows

$$\begin{aligned} & (q\lambda^{q-1} - b^{q-1}c_q)\textstyle\sum_{j=N_0}^{n} a_j\|x_j - Ty_j\|^q \\ \leq\; & \|x_{N_0} - x^*\|^q - \|x_{n+1} - x^*\|^q \\ & +\textstyle\sum_{j=N_0}^{n}(\delta_j M^q + \|e_j\|qM^{q-1} + c_q\|e_j\|^q) \\ \leq\; & \|x_{N_0} - x^*\|^q + \textstyle\sum_{j=1}^{\infty}(\delta_j M^q + \|e_j\|qM^{q-1} + c_q\|e_j\|^q) \\ <\; & \infty. \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} a_n\|x_n - Ty_n\|^q < \infty$.

Next, we claim that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0.$$

Indeed, since $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} a_n = \infty$ and we have $\underline{\lim}_{n\to\infty}\|x_n - Ty_n\| = 0$. If it is false, then $\underline{\lim}_{n\to\infty}\|x_n - Ty_n\| = \delta > 0$. Hence, there exists an integer $N_1 > 1$ such that $\inf_{n\geq N_1}\|x_n - Ty_n\| > \delta/2$. This implies that

$$\infty = (\frac{\delta}{2})^q \sum_{n=N_1}^{\infty} a_n \leq \sum_{n=1}^{\infty} a_n\|x_n - Ty_n\|^q < \infty,$$

which leads to a contradiction. Thus, $\underline{\lim}_{n\to\infty}\|x_n - Ty_n\| = 0$. Since

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - Ty_n\| + \|Ty_n - Tx_n\| \\ &\leq \|x_n - Ty_n\| + L\|y_n - x_n\| \\ &\leq \|x_n - Ty_n\| + L\beta_n\|x_n - Tx_n\|,\end{aligned}$$

we have

$$(1 - L\beta_n)\|x_n - Tx_n\| \leq \|x_n - Ty_n\|.$$

So, we derive

$$L(\frac{1}{L} - \overline{\lim}_{n\to\infty}\beta_n) \cdot \underline{\lim}_{n\to\infty}\|x_n - Tx_n\| \leq \underline{\lim}_{n\to\infty}\|x_n - Ty_n\| = 0.$$

Note that $\overline{\lim}_{n\to\infty}\beta_n < 1/L$. Hence we have $\underline{\lim}_{n\to\infty}\|x_n - Tx_n\| = 0$. This shows that there exists a subsequences $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0.$$

Now we can state and prove our main results in this paper.

Theorem 3.1. *Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E, and $T : K \to K$ be compact and strictly pseudocontactive with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1,\ \forall n \geq 1;$

(ii) $\overline{\lim}_{n\to\infty}\alpha_n < \lambda(q/c_q)^{1/(q-1)}, \overline{\lim}_{n\to\infty}\beta_n < 1/L$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty;$

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n\beta_n^{\tau} < \infty$, *where* $\tau = \min\{1, (q - 1)\}$.

Let $\{x_n\}$ be the bounded sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration method (3.1) with errors. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From Lemma 3.2, it follows that for each $x^* \in F(T), \lim_{n\to\infty}\|x_n - x^*\|$ exists, and that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$. Since $\{x_{n_i}\}$ is bounded and T is compact, so, $\{Tx_{n_i}\}$ has a strongly convergent subsequence. Without loss of generality, we may assume that $\{Tx_{n_i}\}$ converges strongly to some $p \in K$. Observe that

$$\|x_{n_i} - p\| \leq \|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - p\| \to 0 \ (i \to \infty).$$

Hence, we know that $\{x_{n_i}\}$ converges strongly to $p \in K$. Obviously, according to the Lipschitz continuity of T, it is easy to see that

$$p = \lim_{i\to\infty} x_{n_i} = \lim_{n\to} Tx_{n_i} = Tp,$$

that is, $p \in F(T)$. Therefore, we have

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{i\to\infty}\|x_{n_i} - p\| = 0,$$

which hence implies that $\{x_n\}$ converges strongly to $p \in F(T)$.

Remark 3.1. If K is a compact subset of E, then it follows immediately from Theorem 3.1 that $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 3.2. *Let $q > 1$ and E be a real $q-$uniformly smooth Banach space. Let K be a nonempty closed convex subset of E, and $T : K \to K$ be demicompact and strictly pseudocontractive with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K, and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be as in Theorem 3.1. Let $\{x_n\}$ be the bounded sequence generated from an arbitrary $x_1 \in K$ by the Ishikawa iteration method (3.1) with errors. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 3.2, it follows that for each $x^* \in F(T)$, $\lim_{n\to\infty}\|x_n - x^*\|$ exists, and that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$. Since $\{x_{n_i}\}$ is bounded and $\{x_{n_i} - Tx_{n_i}\}$ is strongly convergent, it follows from the demicompactness of T that there exists a subsequence of $\{x_{n_i}\}$ which converges strongly to some $p \in K$. Without loss of generality, we may assume that $\{x_{n_i}\}$ converges strongly to $p \in K$. Hence, taking into account that $\lim_{i\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$ and the Lipschitz continuity of T, we derive $p \in F(T)$. Observe that

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{i\to\infty}\|x_{n_i} - p\| = 0.$$

Therefore, $\{x_n\}$ converges strongly to $p \in F(T)$.

Remark 3.2. If we take $\beta_n = 0, \forall n \geq 1$ in Lemmas 3.1, 3.2 and Theorems 3.1, 3.2, respectively, then we can obtain the results corresponding to Mann iteration method with errors

$$x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T x_n + \gamma_n u_n,\ \forall n \geq 1.$$

In addition, if we take $\gamma_n = 0,\ \forall n \geq 1,$ in (3.1), then under the lack of the assumption that $\{x_n\}$ is bounded, Lemmas 3.1, 3.2 and Theorems 3.1, 3.2 are still valid. Indeed, if $\gamma_n = 0,\ \forall n \geq 1$, then it follows from (3.9) that for all $n \geq N_0$,

$$\begin{aligned} \|x_{n+1} - x^*\|^q &\leq (1 + \delta_n)\|x_n - x^*\|^q \\ &\leq (1 + \delta_n)(1 + \delta_{n-1}) \ldots (1 + \delta_{N_0})\|x_{N_0} - x^*\|^q \\ &\leq e^{\sum_{j=1}^{\infty} \delta_j} \|x_{N_0} - x^*\|^q \\ &< \infty. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, Theorems 3.1 and 3.2 generalize Theorems 1.1 and 1.2, respectively.

Remark 3.3. It is well known that in the sense of Xu [2], the Ishikawa iteration method with errors is defined as the following: for an arbitrary $x_1 \in K$, the sequence $\{x_n\}$ is generated by the iterative scheme

$$(3.12) \qquad \begin{cases} y_n = (1 - \beta_n - \theta_n)x_n + \beta_n T x_n + \theta_n v_n, \\ x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n,\ n \geq 1, \end{cases}$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in K, and $\{\alpha_n\}, \{\beta_n\}, \{\theta_n\}, \{\gamma_n\}$ are real sequences in [0,1] satisfying the restrictions: $\alpha_n + \gamma_n \leq 1,\ \beta_n + \theta_n \leq 1,\ \forall n \leq 1.$ Naturally, we put forth the following open question.

Open Question. Can the Ishikawa iteration method (3.12) with errors in the sense of Xu [2] be extended to Theorems 3.1 and 3.2, respectively?

References

1. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, **20** (1967) 197-228.
2. Y. G. Xu, , Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations, *J. Math. Anal. Appl.*, **224** (1998), 91-101.
3. M. O. Osilike and A. Udomene, Demiclosedness principle and convergence theorems for strictly pseudocontractive mappings of Browder-Petryshyn type, *J. Math. Anal. Appl.*, **256** (2001), 431-445.

4. B. E. Rhoades, Fixed point iteratios using matrices, *Trans .Amer. Math. Soc.*, **196** (1974), 161-176.

5. K. K. Tan and H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.*, **178** (1993), 301-308.

6. H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal. TMA.*, bf 16 (1991), 1127-1138.

7. Z. B. Xu, and G. F. Roach, Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces, *J. Math. Anal. Appl.*, **157** (1991), 189-210.

L. C. Zeng
Department of Mathematics,
Shanghai Normal University,
Shanghai 200234, China.

G. M. Lee
Department of Applied Mathematics,
Pukyong National University,
Pusan 608-737, Korea.

N. C. Wong
Department of Applied Mathematics,
National Sun Yat-Sen University,
Kaohsiung, Taiwan 804, R.O.C.