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ADDITIVITY OF JORDAN MULTIPLICATIVE MAPS ON JORDAN OPERATOR ALGEBRAS

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Abstract. Let H be a Hilbert space and $\mathcal{N}$ a nest in H. Denote by $\mathcal{S}^a(H)$ the Jordan ring of all self-adjoint operators on H and Alg$\mathcal{N}$ the nest algebra associated to $\mathcal{N}$. We show that a bijective map $\Phi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ satisfying (1) $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for every pair of A, B, or (2) $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for every pair of A, B, or (3) $\Phi(\frac{1}{2}(AB+BA)) = \frac{1}{2}(\Phi(A)\Phi(B)+\Phi(B)\Phi(A))$ for every pair of A, B must be additive, that is, a Jordan ring isomorphism. We also show that if a bijective map Φ : Alg$\mathcal{N} \to$ Alg$\mathcal{N}$ satisfies the Jordan multiplicativity of the form (2) or (3), then Φ must be a Jordan isomorphism. Moreover, such Jordan multiplicative maps are characterized completely.

1. Introduction

It is a surprising result of Matindale [13] that every multiplicative bijective map from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. Thus the multiplicative structure determines the ring structure for some rings. Recently Matindale's result has attracted attention of many authors working on operator algebras. For example, it was utilized by Šemrl in [17] to characterize the semigroup isomorphisms of standard operator algebras on Banach spaces. Recall that a standard operator algebra on a Banach space X is a subalgebra of $\mathcal{B}(X)$ (the algebra of all bounded linear operators on X) which contains the identity and all finite rank operators. Some other results on the additivity of multiplicative maps between operator algebras can be found in [2, 3, 8, 9. 11, 14, 15]. Besides ring homomorphisms between rings, sometimes one has to consider Jordan ring homomorphisms. Note that, Jordan operator algebras have important

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applications in the mathematical foundations of quantum mechanics. So it is also interesting to ask when does the Jordan multiplicative structure determine the Jordan ring structure of operator Jordan rings (or algebras).

Let $\mathcal{R}$ and $\mathcal{R}'$ be Jordan rings and let $\Phi : \mathcal{R} \to \mathcal{R}'$ be a map. Recall that Φ is called a Jordan (ring) homomorphism if it is additive and Jordan multiplicative, i.e., $\Phi(A+B) = \Phi(A) + \Phi(B)$ and $\Phi(AB+BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for all $A, B \in \mathcal{R}$. In fact, there are three basic forms of Jordan multiplicative maps, namely, (1) $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for all A, B, (2) $\Phi(AB+BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for all A, B and (3) $\Phi(\frac{1}{2}(AB+BA)) = \frac{1}{2}(\Phi(A)\Phi(B) + \Phi(B)\Phi(A))$ for all A, B. It is clear that, if Φ is unital and additive, then these three forms of Jordan multiplicativity are equivalent. But in general, for a unital map, we do not know whether they are still equivalent without the additivity assumption.

The question of when a Jordan multiplicative map is additive was attacked by several authors. Let ϕ be a bijective map on a standard operator algebra. Molnár showed in [14] that if ϕ satisfies $\phi(ABA) = \phi(A)\phi(B)\phi(A)$, then ϕ is additive. Later, Molnár in [15] and then Lu in [11] considered the cases that ϕ preserves the operation $\frac{1}{2}(AB+BA)$ and the operation $AB+BA$, respectively, and proved that such a ϕ is also additive. The additivity of Jordan †-skew multiplicative maps was proved in [4]. Thus the Jordan multiplicative structure also determines the Jordan ring structure of the standard operator algebras. In this paper, we consider the same question and give affirmative answer for the cases of Jordan multiplicative maps on the real Jordan algebras of all self-adjoint operators and the nest algebras on Hilbert spaces.

Let us fix and recall some notations. Let H be a Hilbert space over $\mathbb{C}$, $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. Let $\mathcal{F}(H)$ denote the subspace of all finite rank operators in $\mathcal{B}(H)$ and I the identity operator on H. We denote by $\mathcal{S}_F^a(H)$ the real linear space of all finite rank self-adjoint operators in $\mathcal{F}(H)$ and $\mathcal{S}^a(H)$ the real linear space of all self-adjoint operators in $\mathcal{B}(H)$, which are obviously Jordan rings. A nest on H is a chain $\mathcal{N}$ of closed (under norm topology) subspaces of H contain $\{0\}$ and H, which is closed under the formation of arbitrary closed linear span (denoted by $\bigvee$) and intersection (denoted by $\bigwedge$). Alg$\mathcal{N}$ denotes the associated nest algebra, which is the set of all operators $T \in \mathcal{B}(H)$ such that $TN \subset N$ for every element $N \in \mathcal{N}$. When $\mathcal{N} \neq \{0, H\}$, we say that $\mathcal{N}$ is nontrivial. If $\mathcal{N}$ is trivial, then Alg$\mathcal{N} = \mathcal{B}(H)$. Let Alg$_F\mathcal{N}$ = Alg$\mathcal{N} \bigcap \mathcal{F}(H)$, the subalgebra of all finite rank operators in Alg$\mathcal{N}$. In [7], Erdos proved that Alg$_F\mathcal{N}$ is dense in Alg$\mathcal{N}$ in the strong operator topology. A subalgebra of Alg$\mathcal{N}$ is called a standard subalgebra if it contains Alg$_F\mathcal{N}$. In particular, Alg$_F\mathcal{N}$ is a standard subalgebra. Note that (nontrivial) nest algebras are important non-self-adjoint operator algebras that are neither prime nor semi-simple.

The present paper is organized as follows. Section 2 is devoted to proving the additivity of three kinds of Jordan multiplicative maps on the spaces of self-adjoint operators. Let $\Phi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ be a bijective map. We show precisely that Φ satisfies $\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$ for every $A, B \in \mathcal{S}^a(H)$ if and only if there exists a unitary or conjugate unitary operator U such that $\Phi(A) = \epsilon UAU^*$ for all $A \in \mathcal{S}^a(H)$, where $\epsilon = \pm 1$ (Theorem 2.1); Φ satisfies $\Phi(\frac{1}{2}AB + \frac{1}{2}BA) = \frac{1}{2}\Phi(A)\Phi(B) + \frac{1}{2}\Phi(B)\Phi(A)$ if and only if $\Phi(AB+BA) = \Phi(A)\Phi(B)+\Phi(B)\Phi(A)$ for every pair $A, B \in \mathcal{S}^a(H)$, and in turn, if and only if there exists a unitary or conjugate unitary operator U such that $\Phi(A) = UAU^*$ for all $A \in \mathcal{S}^a(H)$ (Theorem 2.2). In Section 3, we consider the additivity of Jordan multiplicative maps on standard subalgebras of nest algebras. Let $\mathcal{A}$ be a standard subalgebra of a nest algebra Alg$\mathcal{N}$ on a Hilbert space H, and $\Phi : \mathcal{A} \to \mathcal{A}$ be a bijective map satisfying $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for all $A, B \in \mathcal{A}$, or $\Phi(\frac{1}{2}AB + \frac{1}{2}BA) = \frac{1}{2}\Phi(A)\Phi(B) + \frac{1}{2}\Phi(B)\Phi(A)$ for all $A, B \in \mathcal{A}$, then Φ is additive (Theorem 3.1); and moreover, if $\mathcal{A} =$ Alg$\mathcal{N}$, then there is a bounded linear or conjugate linear invertible operator T with certain property concerning $\mathcal{N}$ such that $\Phi(A) = TAT^{-1}$ for every $A \in$ Alg$\mathcal{N}$ or $\Phi(A) = TA^*T^{-1}$ for every $A \in$ Alg$\mathcal{N}$ (Theorem 3.2).

2. Jordan Multiplicative Maps on $\mathcal{S}^a(H)$

In this section, we characterize the Jordan multiplicative maps on self-adjoint operator space $\mathcal{S}^a(H)$ by checking their additivity.

The following is one of our main results.

Theorem 2.1. *Let H be a complex Hilbert space with $\dim H > 1$ and $\Phi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ be a bijective map. Then Φ satisfies*

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A) \tag{2.1}$$

for every $A, B \in \mathcal{S}^a(H)$ if and only if there exists a unitary or conjugate unitary operator U such that $\Phi(A) = \epsilon UAU^*$ for all $A \in \mathcal{S}^a(H)$, where $\epsilon = \pm 1$.

Proof. The "if" part is obvious. We divide the proof of "only if" part into a few steps.

Step 1. $\Phi(0) = 0$.

There exists an operator $A \in \mathcal{S}^a(H)$ such that $\Phi(A) = 0$ by the bijectivity of Φ. Thus $\Phi(0) = \Phi(A0A) = \Phi(A)\Phi(0)\Phi(A) = 0$.

Step 2. $\Phi(I) = \pm I$ and $\Phi(P)^2 = \Phi(P)$ for all projections P.

Take $A = I$ in the equation (2.1), we get $\Phi(B) = \Phi(I)\Phi(B)\Phi(I)$ which implies that $\Phi(I)$ is invertible. Let $A = B = I$ in (2.1), we see that $\Phi(I) = \Phi(I)^3$.

Multiplying this formula by $\Phi(I)^{-1}$, we have $\Phi(I)^2 = I$. Therefore $\Phi(I) = \pm I$ according to the fact that $\Phi(I)$ is self-adjoint.

If $\Phi(I) = -I$, let $\Psi = -\Phi$. Then Ψ meets the equation (2.1) and $\Psi(I) = I$. So, with no loss of generality, we assume that $\Phi(I) = I$.

Now, letting $A = P, B = I$ in (2.1) yields that $\Phi(P)^2 = \Phi(P)$ for all projections P.

Step 3. Φ preserves the order and the orthogonality of the projections in both directions.

If projections P, Q satisfy $P \leq Q$, that is, $PQ = QP = P$, then $\Phi(P)\Phi(Q)\Phi(P) = \Phi(PQP) = \Phi(P) = \Phi(QPQ) = \Phi(Q)\Phi(P)\Phi(Q)$. Multiplying this formula by $\Phi(Q)$ from both sides, we obtain that $\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = \Phi(P)$, that is, Φ preserves the order of projections. If $PQ = QP = 0$, then $PQP = QPQ = 0$. By (2.1) we have

$$\Phi(P)\Phi(Q)\Phi(P) = \Phi(P)\Phi(Q)\Phi(Q)\Phi(P) = \Phi(P)\Phi(Q)(\Phi(P)\Phi(Q))^* = 0$$

and

$$\Phi(Q)\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P)\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P)(\Phi(Q)\Phi(P))^* = 0.$$

Therefore $\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$ and Φ preserves the orthogonality of projections. Considering Φ^{-1} we see that Φ preserves the order and the orthogonality of the projections in both directions.

Step 4. Φ preserves the rank of projections in both directions.

We show that for every $P \in \mathcal{S}^a(H)$, rank$(P) =$ rank$(\Phi(P))$. Let $P \in \mathcal{S}^a(H)$ be a rank-1 projection. Then rank$(\Phi(P))$ is at least 1. If the rank of $\Phi(P)$ is greater than one, then there exist two mutually orthogonal rank-1 projections Q_1, Q_2 such that $Q_1, Q_2 < \Phi(P)$. Let $P_1 = \Phi^{-1}(Q_1)$ and $P_2 = \Phi^{-1}(Q_2)$. Then, by Step 3, $P_1 < P$, $P_2 < P$ and $P_1P_2 = P_2P_1 = 0$, which is contrary to the fact that rank$(P) = 1$. Suppose now that $\Phi(P)$ is a rank-k projection if and only if $P \in \mathcal{S}^a$ is a rank-k projection, $k = 1, 2, \ldots n-1$. Let P be a rank-n projection, then rank$(\Phi(P))$ is at least n. If the rank of $\Phi(P)$ is greater than n, then there exist two projections $Q_1 \neq Q_2$ with rank$(Q_1) = n$, rank$(Q_2) = n$ such that $Q_1, Q_2 < \Phi(P)$. Let $P_1 = \Phi^{-1}(Q_1)$, $P_2 = \Phi^{-1}(Q_2)$, then rank $P_i \geq n$ and by Step 3, we have $P_1 < P$, $P_2 < P$, $P_1 \neq P_2$, which is contrary to the fact that rank$(P) = n$.

Step 5. Φ is orthogonally additive on the finite rank projections.

If $P, Q \in \mathcal{S}^a(H)$ are orthogonal finite rank projections, we know that $\Phi(P)$, $\Phi(Q)$ are orthogonal finite rank projections. As Φ preserves the order, we have

$\Phi(P)$, $\Phi(Q) \leq \Phi(P+Q)$, which implies that $\Phi(P) + \Phi(Q) \leq \Phi(P+Q)$. Since Φ preserves also the rank of projections, it follows that

$$\Phi(P) + \Phi(Q) = \Phi(P+Q). \tag{2.2}$$

This means that Φ is orthogonally additive on the set of all finite rank projections in $\mathcal{S}^a(H)$.

Step 6. Φ maps finite rank self-adjoint operators into finite rank self-adjoint operators.

Every finite rank self-adjoint operator can be written as a real linear combination of mutually orthogonal rank-one projections. Therefore, if A is a finite rank self-adjoint operator, we can find a projection P with rank(P) =rank(A) such that $PAP = A$. Since $\Phi(A) = \Phi(P)\Phi(A)\Phi(P)$, by (2.1) and Step 3, we see that $\Phi(A)$ has finite rank, as desired.

Step 7. $\Phi(\lambda A) = \lambda\Phi(A)$ for every $\lambda \in \mathbb{R}$ and $A \in \mathcal{S}^a(H)$.

If P is a rank-1 projection and $\lambda \in \mathbb{R}$ is a scalar, then we have

$$\Phi(\lambda P) = \Phi(P(\lambda P)P) = \Phi(P)\Phi(\lambda P)\Phi(P) = h_P(\lambda)\Phi(P)$$

for some scalar $h_P(\lambda) \in \mathbb{R}$. It follows from the fact that $\Phi(P)$ has rank 1, we have that

$$h_P(\lambda^2\mu)\Phi(P) = \Phi(\lambda^2\mu P) = \Phi(\lambda P)\Phi(\mu P)\Phi(\lambda P) = h_P(\lambda)^2 h_P(\mu)\Phi(P),$$

which gives that

$$h_P(\lambda^2\mu) = h_P(\lambda)^2 h_P(\mu) \tag{2.3}$$

for every $\lambda, \mu \in \mathbb{R}$. Choosing $\mu = 1$ entails $h_P(\lambda^2) = h_P(\lambda)^2$. Thus from (2.3) we obtain that h_P is a multiplicative function.

We now assert that h_P does not depend on P. Let $Q \in \mathcal{S}^a(H)$ be a rank one projection with the property that $PQP \neq 0$. Then

$$\Phi(\lambda P\mu^2 Q\lambda P) = \Phi(\mu P)\Phi(\lambda^2 P)\Phi(\mu P) = h_P(\mu)^2 h_Q(\lambda^2)\Phi(P)\Phi(Q)\Phi(P).$$

This yields that

$$h_P(\lambda)^2 h_Q(\mu^2) = h_P(\mu)^2 h_Q(\lambda^2).$$

Hence we have that $h_P = h_Q$. If $PQP = 0$ then we can choose a rank one projection R such that $PRP \neq 0$ and $QRQ \neq 0$, which implies that $h_P = h_Q = h_R$. Thus h_P does not depend on P. So there exists a multiplicative function

$h : \mathbb{R} \to \mathbb{R}$ such that $\Phi(\lambda P) = h(\lambda)\Phi(P)$ holds for every rank one projection $P \in \mathcal{S}^a(H)$ and every $\lambda \in \mathbb{R}$. Note that

$$h(-\lambda) = -h(\lambda) \tag{2.4}$$

for every real number λ.

For any $A \in \mathcal{S}^a(H)$ we have $\Phi(P\lambda^2 AP) = \Phi(P)\Phi(\lambda^2 A)\Phi(P) = \Phi(\lambda P)\Phi(A)\Phi(\lambda P) = h(\lambda)^2\Phi(P)\Phi(A)\Phi(P)$. Since this holds for every rank one projection P and $\Phi(P)$ runs through the whole set of rank one projections, we obtain that $\Phi(\lambda^2 A) = h(\lambda)^2\Phi(A)$ for every $\lambda \in \mathbb{R}$. This yields that

$$\Phi(\lambda A) = h(\lambda)\Phi(A)$$

for every $A \in \mathcal{S}^a(H)$ and every $\lambda \in \mathbb{R}$.

We further prove that h is additive. Let $x, y \in H$ be vectors with the property that $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = 0$. Let $A = (\lambda x + \mu y) \otimes (\lambda x + \mu y)$, $P = x \otimes x$ and $Q = y \otimes y$, here $\lambda, \mu \in \mathbb{R}$. Then we have $h(\lambda^2 + \mu^2)\Phi(A) = \Phi((\lambda^2 + \mu^2)A) = \Phi(A(P + Q)A) = \Phi(A)[\Phi(P) + \Phi(Q)]\Phi(A) = \Phi(A)\Phi(P)\Phi(A) + \Phi(A)\Phi(Q)\Phi(A) = \Phi(APA) + \Phi(AQA) = \Phi(\lambda^2 A) + \Phi(\mu^2 A) = (h(\lambda^2) + h(\mu^2))\Phi(A)$. So h is additive on $\mathbb{R}^+$. If $\lambda > 0$, $\mu < 0$, then $h(\lambda + (-\mu)) = h(\lambda) + h(-\mu) = h(\lambda) - h(\mu)$. Thus h is additive and then is an automorphism of $\mathbb{R}$. Therefore h is the identity on $\mathbb{R}$.

Step 8. Φ is additive.

Let $A, B \in \mathcal{S}_F^a(H)$ be arbitrary. For any rank one projection $P = x \otimes x$ on H, by Step 7, we have $\Phi(P)\Phi(A+B)\Phi(P) = \Phi(P(A+B)P) = \langle (A+B)x, x\rangle\Phi(P) = \langle Ax, x\rangle\Phi(P) + \langle Bx, x\rangle\Phi(P) = \Phi(PAP) + \Phi(PBP) = \Phi(P)\Phi(A)\Phi(P) + \Phi(P)\Phi(B)\Phi(P) = \Phi(P)(\Phi(A) + \Phi(B))\Phi(P)$, which forces that $\Phi(A + B) = \Phi(A) + \Phi(B)$, i.e., Φ is additive on $\mathcal{S}_F^a(H)$.

Step 9. There exists an unitary or conjugate unitary operator such that $\Phi(A) = UAU^*$ for all $A \in \mathcal{S}^a(H)$.

By Step 8, Φ is a Jordan ring isomorphism of $\mathcal{S}^a(H)$. Thus, the conclusion follows from Corollary 4 in [6] directly, completing the proof. ∎

Next let us consider other two forms of Jordan multiplicative maps.

Theorem 2.2. *Let H be a complex Hilbert space with $\dim H > 1$ and let $\Phi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ be a bijective map. Then the following statements are equivalent.*

(i) $\Phi(\frac{1}{2}AB + \frac{1}{2}BA) = \frac{1}{2}\Phi(A)\Phi(B) + \frac{1}{2}\Phi(B)\Phi(A)$ *for every pair* $A, B \in \mathcal{S}^a(H)$.

(*ii*) $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ *for every pair* $A, B \in \mathcal{S}^a(H)$.

(*iii*) *There exists a unitary or conjugate unitary operator* U *such that* $\Phi(A) = UAU^*$ *for all* $A \in \mathcal{S}^a(H)$.

Proof. (iii)⇒(ii) and (iii)⇒(i) are obviously. (i)⇒(ii) or (ii)⇒(i) is not easily checked directly. So we will show that (i)⇒(iii) and (ii)⇒(iii).

Let us show that (i)⇒(iii) by several steps. Assume that (i) holds true.

Step 1. $\Phi(0) = 0$.

Indeed, there exists some $A \in \mathcal{S}^a(H)$ such that $\Phi(A) = 0$. Thus we have

$$\Phi(0) = \Phi(\frac{1}{2}(A0 + 0A)) = \frac{1}{2}(\Phi(A)\Phi(0) + \Phi(0)\Phi(A)) = 0.$$

Step 2. Φ preserves projections in both directions and $\Phi(I) = I$.

Since $\Phi(P) = \Phi(\frac{1}{2}(P^2 + P^2)) = \frac{1}{2}(\Phi(P)^2 + \Phi(P)^2) = \Phi(P)^2$, we see that Φ preserves projections. Since Φ^{-1} has the same properties as Φ, it follows that Φ preserves projections in both directions. For every $A \in \mathcal{S}^a(H)$ we have $\Phi(A) = \Phi(\frac{1}{2}(AI + IA)) = \frac{1}{2}(\Phi(A)\Phi(I) + \Phi(I)\Phi(A))$. Multiplying this equality by $\Phi(I)$ from the right and the left sides respectively, we have $\Phi(A)\Phi(I) = \Phi(I)\Phi(A)$. Hence $\Phi(I) = \lambda I$ for some $\lambda \in \mathbb{R}$. Since $\Phi(I)^2 = \Phi(I)$ and $\Phi(I) \neq 0$, we have $\Phi(I) = I$.

Step 3. Φ preserves the order and the orthogonality of projections in both directions.

If projections $P, Q \in \mathcal{S}^a(H)$ are orthogonal to each other, then we have

$$0 = \Phi(0) = \Phi(\frac{1}{2}(PQ + QP)) = \frac{1}{2}(\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P)).$$

Multiplying this equality by $\Phi(Q)$ from the left and from the right respectively, we have $\Phi(Q)\Phi(P)\Phi(Q) = -\Phi(P)\Phi(Q)$ and $\Phi(Q)\Phi(P)\Phi(Q) = -\Phi(Q)\Phi(P)$. Hence

$$\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$$

and Φ preserves the orthogonality of projections in both directions.

We assert that Φ preserves the partial order relation $\leq$ between projections. If $P, Q \in \mathcal{S}^a(H)$ are projections and $P \leq Q$, then we obtain

$$\Phi(P) = \Phi(\frac{1}{2}(PQ + QP)) = \frac{1}{2}(\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P)).$$

Multiplying this equality by $\Phi(Q)$ from both sides, we get that $\Phi(Q)\Phi(P)\Phi(Q) = \Phi(P)\Phi(Q)$ and $\Phi(Q)\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P)$, that is, $\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = \Phi(P)$. Thus Φ preserves the partial order.

Step 4. Φ preserves the rank of projections in both directions, Φ is orthogonally additive on the finite rank projections.

The argument is similar to that of Step 4 and Step 5 in the proof of Theorem 2.1.

Step 5. Φ maps finite rank self-adjoint operators into finite rank self-adjoint operators.

If $A \in \mathcal{S}_F^a(H)$, then there exists a finite rank projection P such that $PA = AP = A$. A simple computation gives

$$\Phi(A) = \Phi(\frac{1}{2}(AP + PA)) = \frac{1}{2}(\Phi(A)\Phi(P) + \Phi(P)\Phi(A)),$$

which implies that $\Phi(A)$ is also finite rank.

Step 6. $\Phi(-P) = -\Phi(P)$ for every finite rank projection.

Let $P_1, P_2, \cdots, P_n \in \mathcal{S}^a(H)$ be pairwise orthogonal finite rank projections and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$. Using the orthoadditivity of Φ we have

$$\begin{aligned} \Phi(\textstyle\sum_k \lambda_k P_k) &= \Phi(\tfrac{1}{2}((\textstyle\sum_k \lambda_k P_k)(\sum_l P_l) + (\sum_l P_l)(\sum_k \lambda_k P_k))) \\ &= \frac{1}{2}(\Phi(\sum_k \lambda_k P_k)\Phi(\sum_l P_l) + \Phi(\sum_l P_l)\Phi(\sum_k \lambda_k P_k)) \\ &= \frac{1}{2}(\Phi(\sum_k \lambda_k P_k)\sum_l \Phi(P_l) + \sum_l \Phi(P_l)\Phi(\sum_k \lambda_k P_k)) \\ &= \textstyle\sum_l \tfrac{1}{2}(\Phi(\sum_k \lambda_k P_k)\Phi(P_l) + \Phi(P_l)\Phi(\sum_k \lambda_k P_k)) \\ &= \textstyle\sum_l \Phi(\tfrac{1}{2}((\sum_k \lambda_k P_k)P_l + P_l \sum_k \lambda_k P_k))) \\ &= \textstyle\sum_l \Phi(\lambda_l P_l). \end{aligned} \tag{2.5}$$

Let P be of rank 1, we have

$$\begin{aligned} \Phi(\lambda P) &= \Phi(\frac{1}{2}((\lambda P)P + P(\lambda P))) \\ &= \frac{1}{2}(\Phi(\lambda P)\Phi(P) + \Phi(P)\Phi(\lambda P). \end{aligned}$$

Multiplying this equality by $\Phi(P)$ from both sides, we have that $\Phi(P)\Phi(\lambda P)\Phi(P) = \Phi(P)\Phi(\lambda P) = \Phi(\lambda P)\Phi(P)$. It follows that

$$\Phi(\lambda P) = \Phi(P)\Phi(\lambda P)\Phi(P).$$

Since $\Phi(P)$ is of rank 1, the equality above ensures that

$$\Phi(\lambda P) = \mu\Phi(P) \tag{2.6}$$

for some $\mu \in \mathbb{R}$. So, we obtain that $\Phi(-P) = c\Phi(P)$ for some scalar $c \in \mathbb{R}$. Since

$$c^2\Phi(P) = (c\Phi(P))^2 = \Phi(-P)^2 = \Phi(P),$$

we have $c = \pm 1$. By the injectivity of Φ we get $\Phi(-P) = -\Phi(P)$. Thus, (2.5) implies that

$$\Phi(-P) = -\Phi(P) \tag{2.7}$$

for every finite rank projection $P \in \mathcal{S}^a(H)$.

Step 7. $\Phi(PTP) = \Phi(P)\Phi(T)\Phi(P)$ for every $T \in \mathcal{S}^a_F(H)$ and every projection $P \in \mathcal{S}^a_F(H)$.

For any $A, B \in \mathcal{S}^a(H)$ we write

$$A \circ B = \frac{1}{2}(AB + BA).$$

With this notation the condition (i) can be restated as

$$\Phi(A \circ B) = \Phi(A) \circ \Phi(B)$$

for every pair $A, B \in \mathcal{S}^a(H)$

Let $T \in \mathcal{S}^a_F(H)$ be arbitrary and let $P \in \mathcal{S}^a_F(H)$ be a projection. Choose a finite rank projection $Q \in \mathcal{S}^a_F(H)$ such that $QP = PQ = P$ and $TQ = QT = T$. It is trivial to compute that

$$(2P - Q) \circ (T \circ P) = PTP.$$

Thus we get

$$\Phi(2P - Q) \circ (\Phi(T) \circ \Phi(P)) = \Phi(PTP).$$

We prove that $\Phi(2P - Q) = 2\Phi(P) - \Phi(Q)$. Indeed, since $Q - P$ is a projection which is orthogonal to P, by (2.5) and (2.7) we see that

$$\begin{aligned}
\Phi(2P - Q) &= \Phi(P - (Q - P)) = \Phi(P) + \Phi(-(Q - P)) \\
&= \Phi(P) - \Phi(Q - P) = \Phi(P) - \Phi(Q - P) \\
&= \Phi(P) - (\Phi(Q) - \Phi(P)) = 2\Phi(P) - \Phi(Q).
\end{aligned}$$

So, we have

$$(2\Phi(P) - \Phi(Q)) \circ (\Phi(T) \circ \Phi(P)) = \Phi(PTP).$$

We assert that $\Phi(Q)\Phi(T)\Phi(Q) = \Phi(T)$ and $\Phi(Q)\Phi(P)\Phi(Q) = \Phi(P)$. In fact, these follow from the equalities

$$\Phi(T) = \frac{1}{2}(\Phi(T)\Phi(Q) + \Phi(Q)\Phi(T))$$

and

$$\Phi(P) = \frac{1}{2}(\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P))$$

after multiplying them by $\Phi(Q)$ from both sides. Similarly, one can now easily check that

$$(2\Phi(P) - \Phi(Q)) \circ (\Phi(P) \circ \Phi(Q)) = \Phi(P)\Phi(T)\Phi(P).$$

Therefore, we have $\Phi(PTP) = \Phi(P)\Phi(T)\Phi(P)$.

Step 8. $\Phi(\lambda A) = \lambda\Phi(A)$ for every $\lambda \in \mathbb{R}$ and $A \in \mathcal{S}_F^a(H)$.

Fix a rank 1 projection $P \in \mathcal{S}_F^a(H)$, by Step 6, there is a bijective function $h_P : \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(\lambda P) = h_P(\lambda)\Phi(P) \qquad (\lambda \in \mathbb{R}). \tag{2.8}$$

Now we claim that h_P is multiplicative. To see this, let $P \in \mathcal{S}_F^a(H)$ be a rank 1 projection. We have

$$\begin{aligned} h_P(\lambda\mu)\Phi(P) &= \Phi(\frac{1}{2}((\lambda P)(\mu P) + (\mu P)(\lambda P))) \\ &= \frac{1}{2}(h_P(\lambda)\Phi(P)h_P(\mu)\Phi(P) + h_P(\mu)\Phi(P)h_P(\lambda)\Phi(P)) \\ &= h_P(\lambda)h_P(\mu)\Phi(P) \end{aligned}$$

and this shows that h_P is multiplicative. Therefore, $h_P(-\lambda) = -h_P(\lambda)$.

We show that h_P does not depend on P. If $Q \in \mathcal{S}_F^a(H)$ is another rank-1 projection not orthogonal to P, then we have

$$\begin{aligned} \Phi(\tfrac{1}{2}((\lambda P)Q + Q(\lambda P))) &= \frac{1}{2}(h_P(\lambda)\Phi(P)\Phi(Q) + h_P(\lambda)\Phi(Q)\Phi(P)) \\ &= \frac{1}{2}h_P(\lambda)(\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P)). \end{aligned}$$

Similarly,

$$\Phi(\frac{1}{2}(P(\lambda Q) + (\lambda Q)P)) = \frac{1}{2}h_Q(\lambda)(\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P)).$$

Since $PQ \neq 0$, we must have $PQ+QP \neq 0$ and hence $\Phi(P)\Phi(Q)+\Phi(Q)\Phi(P) \neq 0$. It follows that $h_P = h_Q$. If Q is orthogonal to P, then we can choose a rank-1 projection $R \in \mathcal{S}_F^a(H)$ such that R is neither orthogonal to P nor orthogonal to Q. By what we just proved, we get $h_P = h_Q = h_R$. Therefore, there is a bijective function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(\lambda P) = h(\lambda)\Phi(P)$$

for every $\lambda \in \mathbb{R}$ and every rank-1 projection $P \in \mathcal{S}_F^a(H)$.

Next we show that

$$\Phi(\lambda A) = h(\lambda)\Phi(A) \tag{2.9}$$

for every $A \in \mathcal{S}_F^a(H)$ and $\lambda \in \mathbb{R}$. By (2.5), we see that (2.9) holds if A is any finite rank projection. Let $A \in \mathcal{S}_F^a(H)$ be arbitrary, then there exists a finite rank projection P such that $PA = AP = A$. A simple computation gives

$$\begin{aligned}\Phi(\lambda A) &= \Phi((\frac{1}{2}(A(\lambda P) + (\lambda P)A)) \\ &= \frac{1}{2}(\Phi(A)h(\lambda)\Phi(P) + h(\lambda)\Phi(P)\Phi(A)) = h(\lambda)\Phi(A).\end{aligned}$$

We claim that h is the identity map on $\mathbb{R}$. Let $x, y \in H$ be orthogonal unital vectors and $\lambda, \mu \in \mathbb{R}$ be such that $\lambda^2 + \mu^2 = 1$. Put $P = x \otimes x$, $Q = y \otimes y$ and $R = (\lambda x + \mu y) \otimes (\lambda x + \mu y)$. It is obvious that R, P, Q are rank-1 projections and P is orthogonal to Q. By Step 7. we have

$$\begin{aligned}h(\lambda^2 + \mu^2)\Phi(R) &= \Phi((\lambda^2 + \mu^2)R) = \Phi(R(P+Q)R) = \Phi(R)\Phi(P+Q)\Phi(R) \\ &= \Phi(R)\Phi(P)\Phi(R) + \Phi(R)\Phi(Q)\Phi(R) = \Phi(RPR) + \Phi(RQR) \\ &= \Phi(\lambda^2 R) + \Phi(\mu^2 R) = (h(\lambda^2) + h(\mu^2))\Phi(R).\end{aligned}$$

By the multiplicativity of h, we get $h(\lambda)^2 + h(\mu)^2 = 1$ whenever $\lambda^2 + \mu^2 = 1$. This equality implies that, for any $0 \le a \le 1$, we have $0 \le h(a) \le 1$, $h(a) + h(1-a) = 1$ and h preserves the order. In particular, $h(1) = 1$ and $h(\frac{1}{2}) = \frac{1}{2}$. Next we show that the restriction of h to the rational numbers is the identity. By the multiplicativity of h we have $h(2) = 2$. Assume that $h(n) = n$. It follows form $1 - h(\frac{1}{n+1}) = h(1 - \frac{1}{n+1}) = h(\frac{n}{n+1}) = nh(\frac{1}{n+1})$ that $h(n+1) = n+1$. By induction, this implies that $h(n) = n$ for all natural numbers n. Hence, $h(r) = r$ for every rational r. Because h is also order preserving, we see that $h(\lambda) = \lambda$ holds for all $\lambda \in \mathbb{R}$, as desired.

Step 9. $\Phi(\lambda A) = \lambda\Phi(A)$ for every $\lambda \in \mathbb{R}$ and $A \in \mathcal{S}^a(H)$.

Let $A \in \mathcal{S}^a(H)$ be arbitrary and $\lambda \in \mathbb{R}$, a computation gives that

$$\begin{aligned}\frac{1}{2}(\Phi(\lambda A)\Phi(x \otimes x) + \Phi(x \otimes x)\Phi(\lambda A)) &= \Phi(\frac{1}{2}\lambda(A(x \otimes x) + (x \otimes x)A)) \\ &= \frac{1}{2}\Phi(A)\Phi(\lambda x \otimes x) + \frac{1}{2}\Phi(\lambda x \otimes x)\Phi(A) \\ &= \frac{1}{2}\lambda(\Phi(A)\Phi(x \otimes x) + \Phi(x \otimes x)\Phi(A)),\end{aligned}$$

which implies that $(\lambda\Phi(A)-\Phi(\lambda A))\Phi(x\otimes x)=\Phi(x\otimes x)(\lambda\Phi(A)-\Phi(\lambda A))$ for every rank 1 operator. Hence $\lambda\Phi(A)=\Phi(\lambda A)$.

Step 10. Φ is additive on $\mathcal{S}_F^a(H)$, and thus a Jordan ring isomorphism.
The proof is the same as that of Step 8 in Theorem 2.1.

Step 11. There exists an unitary or conjugate unitary operator such that $\Phi(A)=UAU^*$ for all $A\in\mathcal{S}^a(H)$.

This follows from [6, Corollary 4]. Here we give a direct proof. By Step 5 and the additivity of Φ, it is easily seen that Φ preserves adjacency in both directions on $\mathcal{S}_F^a(H)$. By [6] there exist a unitary or conjugate unitary operator $U:H\to H$ and a real number c such that $\Phi(x\otimes x)=cUx\otimes xU^*$ for all $x\in H$. By Step 9 we see that $c=1$. For every $A\in\mathcal{S}_F^a(H)$, applying the claims in Step 7 and 9, one gets $\langle Ax,x\rangle\Phi(x\otimes x)=\Phi((x\otimes x)A(x\otimes x))=\Phi(x\otimes x)\Phi(A)\Phi(x\otimes x)=Ux\otimes xU^*\Phi(A)Ux\otimes xU^*=\langle\Phi(A)Ux,Ux\rangle Ux\otimes xU^*=\langle U^*\Phi(A)Ux,x\rangle\Phi(x\otimes x)$. So, $\langle U^*\Phi(A)Ux,x\rangle=\langle Ax,x\rangle$ holds for all $x\in H$ and consequently, $\Phi(A)=UAU^*$ as H is a complex Hilbert space. If $A\in\mathcal{S}^a(H)$ is not of finite rank, the following equality

$$\begin{aligned}\frac{1}{2}U(A(x\otimes x)+(x\otimes x)A)U^* &= \Phi(\frac{1}{2}(A(x\otimes x)+(x\otimes x)A))\\ &= \frac{1}{2}(\Phi(A)U(x\otimes x)U^*+U(x\otimes x)U^*\Phi(A))\end{aligned}$$

implies that $(UAx-\Phi(A)Ux)\otimes Ux=Ux\otimes(\Phi(A)Ux-Ax)$. Hence, for every $x\in H$, there exists some $\lambda_x\in\mathbb{R}$ such that $UAx-\Phi(A)Ux=\lambda_xUx$. This entails that $UA-\Phi(A)U=\lambda U$ for some $\lambda\in\mathbb{R}$. So there exists a functional $\varphi:\mathcal{S}^a(H)\to\mathbb{R}$ such that

$$\Phi(A)=UAU^*+\varphi(A)I$$

for every $A\in\mathcal{S}^a(H)$. Thus we have

$$\begin{aligned}&\frac{1}{2}U(A(x\otimes x)+(x\otimes x)A)U^*\\ =&\frac{1}{2}(\Phi(A)U(x\otimes x)U^*+U(x\otimes x)U^*\Phi(A))\\ =&\frac{1}{2}((UAU^*+\varphi(A)I)Ux\otimes Ux+(Ux\otimes Ux)(UAU^*+\varphi(A)I)).\end{aligned}$$

It follows that $2\varphi(A)Ux\otimes Ux=0$, and hence $\varphi(A)=0$ for every A, as desired.

(ii)$\Rightarrow$(iii). To do this, we need a claim which is also interesting of itself.

Claim. *Let H be a complex Hilbert space. If $A \in \mathcal{S}^a(H)$ is such that $BA + AB \geq 0$ holds for every $0 \leq B \in \mathcal{S}^a(H)$, then A is a nonnegative scalar multiple of the identity.*

We first observe that A is positive. Indeed, for $I \in \mathcal{S}^a(H)$, we have $AI + IA \geq 0$, that is $A \geq 0$. If $0 \neq x \in H$ is arbitrary, then we have $x \otimes Ax + Ax \otimes x \geq 0$. It follows from this inequality that for each $y \in H$, we have $\langle (x \otimes Ax + Ax \otimes x)y, y\rangle = \langle y, Ax\rangle\langle x, y\rangle + \langle y, x\rangle\langle Ax, y\rangle \geq 0$, which implies that

$$\text{Re}(\langle y, x\rangle\langle Ax, y\rangle) \geq 0. \tag{2.10}$$

We can write $Ax = \lambda x + u$, where $\lambda \in \mathbb{C}$ and $u \in x^\perp \in H$ is a vector orthogonal to x. Let $y = \mu x + v$ with $\mu \in \mathbb{C}$ and $v \in H$ a vector orthogonal to x. It follows from (2.10) that $\text{Re}((\lambda\overline{\mu}\|x\|^2 + \langle u, v\rangle)\mu) \geq 0$. This implies that

$$|\mu|^2\text{Re}(\lambda\|x\|^2) + \text{Re}(\mu\langle u, v\rangle) \geq 0$$

holds for all $\mu \in \mathbb{C}$ and $v \in x^\perp$. It is easy to see that we necessarily have $\|u\|^2 = 0$, that is, $u = 0$.

The above observation shows that, for every $x \in H$, the vectors Ax and x are linearly dependent. Therefore, A is a scalar multiple of the identity. This completes the proof of the claim.

Now let us turn to the proof of (ii)⇒(iii).

Note that, if $A \in \mathcal{S}^a(H)$ is positive, then there is a positive element $B \in \mathcal{S}^a(H)$ such that $2B^2 = A$. Thus $\Phi(A) = \Phi(2B^2) = 2\Phi(B)^2 \geq 0$. Since Φ^{-1} has the same properties as Φ, we see that Φ preserves the positivity in both directions.

Let $A \in \mathcal{S}^a(H)$ be positive. We have $\Phi(A)\Phi(I) + \Phi(I)\Phi(A) = \Phi(2A) \geq 0$. Since $\Phi(A)$ runs through all positive elements of $\mathcal{S}^a(H)$, by the above Claim, we infer that $\Phi(I) = \lambda I$ for some scalar $\lambda > 0$. Considering the transformation $\Psi : \mathcal{S}^a(H) \to \mathcal{S}^a(H)$ defined by $\Psi(A) = \frac{1}{\lambda}\Phi(A)$. Since $\Phi(A) = \Phi(I\frac{A}{2} + \frac{A}{2}I) = 2\lambda\Phi(\frac{A}{2})$, one can easily check that $\Psi(\frac{1}{2}AB + \frac{1}{2}BA) = \frac{1}{2}\Psi(A)\Psi(B) + \frac{1}{2}\Psi(B)\Psi(A)$. Now by (i)⇒(iii), there exists a unitary or conjugate unitary operator U such that $\Phi(A) = \lambda UAU^*$ for every $A \in \mathcal{S}^a(H)$. Since $\lambda U(AB + BA)U^* = \Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A) = \lambda^2 U(AB + BA)U^*$, we must have $\lambda = 1$. So, (iii) holds true, completing the proof. ■

3. Jordan Multiplicative Maps on Nest Algebras

Let H be a (real or complex) Hilbert space and $\mathcal{N}$ be a nest on H. It is well known that $x \otimes f$ is in $\text{Alg}_F\mathcal{N}$ if and only if there exists an element $N \in \mathcal{N}$ such that $x \in N$ and $f \in (I - N_-)H$. It follows that $\text{Alg}_F\mathcal{N}$ contains no idempotents if $N_- = N$ for every $N \in \mathcal{N}$. For more information on nest algebras, we refer to [5].

The following is our main result in this section which states that every bijective Jordan multiplicative map on nest algebras is in fact a Jordan isomorphism.

Theorem 3.1. *Let* $\mathcal{A}$ *be a standard subalgebra of a nest algebra* $\mathrm{Alg}(\mathcal{N})$ *on a Hilbert space* H *with* $\dim H > 1$*, and* $\Phi : \mathcal{A} \to \mathcal{A}$ *be a bijective map satisfying*

$$\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A), \tag{3.1}$$

then Φ *is additive.*

Proof. As $\mathrm{Alg}\mathcal{N} = \mathcal{B}(H)$ is a standard operator algebra if $\mathcal{N} = \{0, I\}$, we always assume in the sequel that the nest $\mathcal{N}$ is not trivial.

Our idea is similar to that in [11] for the case that Φ acts on a standard operator algebra. The main technique we will use is the following argument which will be termed a standard argument. Suppose $A, B, S \in \mathcal{A}$ are such that $\Phi(S) = \Phi(A) + \Phi(B)$. Multiplying this equality by $\Phi(T)$ ($T \in \mathcal{A}$) from the right and the left, respectively, we get $\Phi(T)\Phi(S) = \Phi(T)\Phi(A) + \Phi(T)\Phi(B)$ and $\Phi(S)\Phi(T) = \Phi(A)\Phi(T) + \Phi(B)\Phi(T)$. Summing them, we get

$$\Phi(S)\Phi(T) + \Phi(T)\Phi(S) = \Phi(A)\Phi(T) + \Phi(T)\Phi(A) + \Phi(T)\Phi(B) + \Phi(B)\Phi(T).$$

It follows from (3.1) that

$$\Phi(ST + TS) = \Phi(AT + TA) + \Phi(BT + TB).$$

Moreover, if

$$\Phi(AT + TA) + \Phi(BT + TB) = \Phi(AT + TA + BT + TB),$$

then by the injectivity of Φ, we will reach that

$$ST + TS = AT + TA + BT + TB.$$

We give the proof by several Steps.

Step 1. $\Phi(0) = 0$

Obvious.

Fix an element E in $\mathcal{N}$ with $0 < E < I$. For the sake of simplicity, we write $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{11} = E\mathcal{A}E$, $\mathcal{A}_{12} = E\mathcal{A}(I - E)$ and $\mathcal{A}_{22} = (I - E)\mathcal{A}(I - E)$.

Step 2. The following statements are true.

(i) $T_{22} \in \mathcal{A}_{22}$, $\mathcal{A}_{12}T_{22} = 0$ implies $T_{22} = 0$. Dually, $T_{11} \in \mathcal{A}_{11}$, $T_{11}\mathcal{A}_{12} = 0$ implies $T_{11} = 0$.

(ii) Let $T_{12} \in \mathcal{A}_{12}$. Then $\mathcal{A}_{11}T_{12} = 0$ implies $T_{12} = 0$ and $T_{12}\mathcal{A}_{22} = 0$ implies $T_{12} = 0$.

(iii) $T_{ii}S_{ii} + S_{ii}T_{ii} = 0$ for every $S_{ii} \in \mathcal{A}_{ii}$ $(i = 1, 2)$ implies $T_{ii} = 0$.

Obvious.

Step 3. $\Phi(A_{11} + A_{12}) = \Phi(A_{11}) + \Phi(A_{12})$.

Since Φ is surjective, we may find an element $S = S_{11} + S_{12} + S_{22} \in \mathcal{A}$ such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(A_{12}). \tag{3.2}$$

For $T_{22} \in \mathcal{A}_{22}$, applying a standard argument to (3.2), we get that $\Phi(ST_{22}+T_{22}S) = \Phi(A_{11}T_{22} + T_{22}A_{11}) + \Phi(A_{12}T_{22} + T_{22}A_{12}) = \Phi(A_{12}T_{22})$. Therefore,

$$ST_{22} + T_{22}S = S_{12}T_{22} + S_{22}T_{22} + T_{22}S_{22} = A_{12}T_{22}. \tag{3.3}$$

Multiplying Eq.(3.3) by E from the left side, we get that $S_{12}T_{22} = A_{12}T_{22}$. Thus $S_{12} = A_{12}$ by Step 2.(ii). It follows that $S_{22} = 0$ by Step 2.(iii). For $T_{12} \in \mathcal{A}_{12}$, applying a standard argument to (3.1) again, we have $\Phi(ST_{12}+T_{12}S) = \Phi(A_{11}T_{12}+T_{12}A_{11}) + \Phi(A_{12}T_{12} + T_{12}A_{12})$. Hence, $ST_{12} + T_{12}S = S_{11}T_{12} + T_{12}S_{22} = S_{11}T_{12} = A_{11}T_{12}$, this implies that $S_{11} = A_{11}$ by Step 2.(i). Consequently, $S = A_{11} + A_{12}$.

Similarly, one can check the claim of Step 4.

Step 4. $\Phi(A_{12} + A_{22}) = \Phi(A_{12}) + \Phi(A_{22})$.

Step 5. $\Phi(A_{12} + B_{12}A_{22}) = \Phi(A_{12}) + \Phi(B_{12}A_{22})$.

A simple computation gives $A_{12} + B_{12}A_{22} = (E + B_{12})(A_{12} + A_{22}) + (A_{12} + A_{22})(E + B_{12})$. Then, by Step 3 and 4, we have that

$$\begin{aligned}
\Phi(A_{12} + B_{12}A_{22}) &= \Phi(A_{12} + A_{22})\Phi(E + B_{12}) + \Phi(E + B_{21})\Phi(A_{12} + A_{22}) \\
&= (\Phi(A_{12}) + \Phi(A_{22}))(\Phi(E) + \Phi(B_{12})) \\
&\quad +((\Phi(E) + \Phi(B_{12}))(\Phi(A_{12}) + \Phi(A_{22})) \\
&= \Phi(A_{12})\Phi(E) + \Phi(A_{12})\Phi(B_{12}) + \Phi(A_{22})\Phi(E) \\
&\quad +\Phi(A_{22})\Phi(B_{12}) + \Phi(E)\Phi(A_{12}) \\
&\quad +\Phi(B_{12})\Phi(A_{12}) + \Phi(E)\Phi(A_{22}) + \Phi(B_{12})\Phi(A_{22}) \\
&= \Phi(A_{12}E + EA_{12}) + \Phi(A_{22}B_{12} + B_{12}A_{22}) \\
&\quad +\Phi(A_{12}B_{12} + B_{12}A_{12}) + \Phi(A_{22}E + EA_{22}) \\
&= \Phi(A_{12}) + \Phi(B_{12}A_{22}).
\end{aligned}$$

Step 6. Φ is additive on $\mathcal{A}_{12}$.

Let $A_{12}, B_{12} \in \mathcal{A}_{12}$ and choose $S = S_{11} + S_{12} + S_{22} \in \mathcal{A}$ such that

$$\Phi(S) = \Phi(A_{12}) + \Phi(B_{12}). \tag{3.4}$$

For $T_{22} \in \mathcal{A}_{22}$, applying a standard argument to (3.4), we get that

$$\begin{aligned}\Phi(ST_{22} + T_{22}S) &= \Phi(A_{12}T_{22} + T_{22}A_{12}) + \Phi(B_{12}T_{22} + T_{22}B_{12}) \\ &= \Phi(A_{12}T_{22}) + \Phi(B_{12}T_{22}) = \Phi((A_{12} + B_{12})T_{22}).\end{aligned}$$

Hence,

$$T_{22}S + ST_{22} = T_{22}S_{22} + S_{12}T_{22} + S_{22}T_{22} = (A_{12} + B_{12})T_{22}.$$

Multiplying the above equality by $I - E$ from the left side, we see that $T_{22}S_{22} + S_{22}T_{22} = 0$ and $S_{22} = 0$. Furthermore the equality $S_{12}T_{22} = (A_{12} + B_{12})T_{22}$ implies $S_{12} = A_{12} + B_{12}$ by Step 2.(ii).

Now there remains to prove that $S_{11} = 0$. For $T_{12} \in \mathcal{A}_{12}$, applying a standard argument to (3.4) again, we get that $T_{12}S + ST_{12} = T_{12}S_{22} + S_{11}T_{12} = 0$. Since we have proved that $S_{22} = 0$, we have that $S_{11}T_{12} = 0$ for every $T_{12} \in \mathcal{A}_{12}$. Hence, by Step 2.(i), $S_{11} = 0$.

Step 7. Φ is additive on $\mathcal{A}_{11}$.

Let $A_{11}, B_{11} \in \mathcal{A}_{11}$, and choose $S = S_{11} + S_{12} + S_{22} \in \mathcal{A}$ such that

$$\Phi(S) = \Phi(A_{11}) + \Phi(B_{11}) \tag{3.5}$$

For $T_{22} \in \mathcal{A}_{22}$, applying a standard argument to (3.5), we get that $\Phi(ST_{22} + T_{22}S) = \Phi(A_{11}T_{22} + T_{22}A_{11}) + \Phi(B_{11}T_{22} + T_{22}B_{11}) = 0$. Hence $ST_{22} + T_{22}S = S_{12}T_{22} + S_{22}T_{22} + T_{22}S_{22} = 0$. By a simple computation, we see that $S_{12} = 0$ and $S_{22} = 0$. For $T_{12} \in \mathcal{A}_{12}$, applying a standard argument to (3.5) again, we get $\Phi(ST_{12} + T_{12}S) = \Phi(A_{11}T_{12} + T_{12}A_{11}) + \Phi(B_{11}T_{12} + T_{12}B_{11}) = \Phi(A_{11}T_{12}) + \Phi(B_{11}T_{12}) = \Phi((A_{11}+B_{11})T_{12})$ by Step 6. Hence $ST_{12}+T_{12}S = (A_{11}+B_{11})T_{12}$. Since we have proved that $S_{22} = 0$, by Step 2.(i), we get $S_{11} = A_{11} + B_{11}$.

Similarly, one can prove

Step 8. Φ is additive on $\mathcal{A}_{22}$.

Step 9. Φ is additive on $E\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12}$.

Let $A_{11}, B_{11} \in \mathcal{A}_{11}$, $A_{12}, B_{12} \in \mathcal{A}_{12}$. It follows from Step 3, 6 and 7 that

$$\begin{aligned}\Phi((A_{11} + A_{12}) + (B_{11} + B_{12})) &= \Phi((A_{11} + B_{11}) + (A_{12} + B_{12})) \\ &= \Phi(A_{11} + B_{11}) + \Phi(A_{12} + B_{12}) \\ &= \Phi(A_{11}) + \Phi(B_{11}) + \Phi(A_{12}) + \Phi(B_{12}) \\ &= \Phi(A_{11} + A_{12}) + \Phi(B_{11} + B_{12}).\end{aligned}$$

Step 10. $\Phi(A_{11}+A_{22})=\Phi(A_{11})+\Phi(A_{22})$.
Let $S=S_{11}+S_{12}+S_{22}\in\mathcal{A}$ such that

$$\Phi(S)=\Phi(A_{11})+\Phi(A_{22}). \tag{3.6}$$

For E, applying a standard argument to (3.6), we have that $\Phi(SE+ES)=\Phi(A_{11}E+EA_{11})+\Phi(A_{22}E+EA_{22)})=\Phi(2A_{11})$ and hence, $SE+ES=2S_{11}+S_{12}=2A_{11}$. Multiplying this equality by $I-E$ from the right side, we get that $S_{12}=0$ and $S_{11}=A_{11}$. Similarly, for $I-E$ applying a standard argument to (3.6), we can show that $S_{22}=A_{22}$.

Step 11. $\Phi(A_{11}+A_{12}+A_{22})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{22})$.
Let $S=S_{11}+S_{12}+S_{22}\in\mathcal{A}$ such that

$$\Phi(S)=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{22}). \tag{3.7}$$

For E, applying a standard argument to (3.7) we have that $\Phi(SE+ES)=\Phi(2A_{11})+\Phi(A_{12})=\Phi(2A_{11}+A_{12})$. Hence $SE+ES=2A_{11}+A_{12}$, that is, $2S_{11}+S_{12}=2A_{11}+A_{12}$. By a simple computation, we get that $S_{11}=A_{11}$ and $S_{12}=A_{12}$. Similarly, for $I-E$ applying a standard argument to (3.7), one gets $S_{22}=A_{22}$.

Now let us complete the proof of Theorem 3.1. For any $A=A_{11}+A_{12}+A_{22}$ and $B=B_{11}+B_{12}+B_{22}$ in $\mathcal{A}$, all steps are used in seeing the following equalities,

$$\begin{aligned}\Phi(A+B)&=\Phi((A_{11}+B_{11})+(A_{12}+B_{12})+(A_{22}+B_{22}))\\&=\Phi(A_{11}+B_{11})+\Phi(A_{12}+B_{12})+\Phi(A_{22}+B_{22})\\&=\Phi(A_{11})+\Phi(B_{11})+\Phi(A_{12})+\Phi(B_{12})+\Phi(A_{22})+\Phi(B_{22})\\&=\Phi(A_{11}+A_{12}+A_{22})+\Phi(B_{11}+B_{12}+B_{22})\\&=\Phi(A)+\Phi(B).\end{aligned}$$

Hence Φ is additive and hence a Jordan isomorphism. ∎

Theorem 3.1 still holds if Φ is a Jordan multiplicative map of form (3) on a nest algebra. The proof is similar to that of Theorem 3.1, and we omit it here. We guess that every Jordan multiplicative map of form (1) on a nest algebra is also additive, however we are not able to prove this in the present paper.

Theorem 3.1′. *Let $\mathcal{A}$ be a standard subalgebra of a nest algebra* $\mathrm{Alg}(\mathcal{N})$ *on a Hilbert space H with* $\dim H>1$, *and $\Phi:\mathcal{A}\to\mathcal{A}$ be a bijective map satisfying*

$$\Phi(\frac{1}{2}(AB+BA))=\frac{1}{2}\Phi(A)\Phi(B)+\frac{1}{2}\Phi(B)\Phi(A),$$

then Φ *is additive.*

The following results generalize the main results in [12] by omitting the additivity assumption.

Theorem 3.2. *Let* H *be an infinite dimensional real or complex Hilbert space, and* Alg$\mathcal{N}$ *be a nest algebra on* H*. Let* $\Phi : \text{Alg}\mathcal{N} \to \text{Alg}\mathcal{N}$ *be a bijective map. Then the following are equivalent.*

(1) Φ *satisfies* $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ $\forall A, B \in \text{Alg}\mathcal{N}$.

(2) Φ *satisfies* $\Phi(\frac{1}{2}(AB + BA)) = \frac{1}{2}\Phi(A)\Phi(B) + \frac{1}{2}\Phi(B)\Phi(A)$ $\forall A, B \in \text{Alg}\mathcal{N}$.

(3) *Either there exists a dimension preserving order isomorphism* $\theta : \mathcal{N} \to \mathcal{M}$ *and an invertible bounded linear or conjugate-linear operator* $T : X \to Y$ *satisfying* $T(N) = \theta(N)$ *for every* $N \in \mathcal{N}$ *such that*

$$\Phi(A) = TAT^{-1} \quad \textit{for all } A \in \text{Alg}\mathcal{N};$$

or there exists a dimension preserving order isomorphism $\theta : \mathcal{N}^{\perp} \to \mathcal{M}$ *and an invertible bounded linear or conjugate-linear operator* $T : X^* \to Y$ *satisfying* $T(N^{\perp}) = \theta(N^{\perp})$ *for every* $N \in \mathcal{N}$ *such that*

$$\Phi(A) = TA^*T^{-1} \quad \textit{for all } A \in \text{ Alg}\mathcal{A}.$$

Moreover, in the latter case, X *and* Y *are reflexive.*

By $M_n(\mathbb{F})$ we denote the algebra of $n \times n$ matrices over $\mathbb{F}$ (the real field or the complex field). For every finite sequence of positive integers $n_1, n_2, \ldots n_k$, satisfying $n_1 + n_2 + \ldots + n_k = n$, we associate an algebra $\mathcal{T}(n_1, \ldots, n_k)$ consisting of all $n \times n$ matrices of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1k} \\ 0 & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{kk} \end{pmatrix}$$

where A_{ij} is an $n_i \times n_j$ matrix. We will call such an algebra $\mathcal{T}(n_1, \ldots, n_k)$ a block upper triangular algebra in $M_n(\mathbb{F})$. Let $\mathcal{V}(n_1, \ldots, n_k) = \{T = (T_{ij})_{k\times k} \mid T_{ij} = 0$ whenever $i + j > k + 1\}$.

Theorem 3.3. *Let* $\mathcal{T}(n_1, \ldots, n_k)$ *be a block upper triangular algebra in* $M_n(\mathbb{F})$ *and* $\Phi : \mathcal{T}(n_1, \ldots, n_k) \to \mathcal{T}(n_1, \ldots, n_k)$ *be a bijective map. Then the following are equivalent.*

(1) Φ *satisfies* $\Phi(AB+BA) = \Phi(A)\Phi(B)+\Phi(B)\Phi(A)$ $\forall A, B \in \mathcal{T}(n_1, \ldots, n_k)$.

(2) Φ *satisfies* $\Phi(\frac{1}{2}(AB + BA)) = \frac{1}{2}\Phi(A)\Phi(B) + \frac{1}{2}\ \Phi(B)\Phi(A)\ \forall A, B \in \mathcal{T}(n_1, \ldots, n_k)$.

(3) *There is a ring automorphism* $h : \mathbb{F} \to \mathbb{F}$ *and either there is an invertible matrix* $T \in \mathcal{T}(n_1, \ldots, n_k)$ *such that* Φ *is of the form* $\Phi(A) = TA_hT^{-1}$ *for all* $A \in \mathcal{T}(n_1, \ldots, n_k)$*; or there is an invertible matrix* $T \in \mathcal{V}(n_1, \ldots, n_k)$ *such that* $\Phi(A) = TA_h^{tr}T^{-1}$ *for all* $A \in \mathcal{A}$*, where* $[a_{ij}]_h = [h(a_{ij})]$ *and* A^{tr} *stands for the transpose of* A.

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