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Blow-up Analysis for a Nonlocal Reaction-diffusion Equation with Robin Boundary Conditions

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Abstract. This work is concerned with the blow-up phenomena for a nonlocal reaction-diffusion equation with null Robin boundary conditions. We establish sufficient conditions to guarantee the solution exists globally or blows up at finite time under appropriate measure sense. Moreover, upper and lower bounds for the blow-up time are derived in higher dimensional spaces. Finally, some application examples are presented.

1. Introduction

Our main interest lies in the following reaction-diffusion equation with weighted nonlocal sources

$$u_t = \Delta u + a(x)f(u), \qquad (x,t) \in \Omega \times (0,t^*), \tag{1.1}$$

subject to null Robin boundary and initial conditions

$$\frac{\partial u}{\partial \nu} + \sigma u = 0, \qquad (x,t) \in \partial\Omega \times (0,t^*), \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \qquad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded region with smooth boundary $\partial\Omega$, ν is the unit outward normal vector on $\partial\Omega$, and t^* represents the blow-up time when blow-up occurs, otherwise $t^* = +\infty$. The nonlinearity $f(u)$ is assumed to be nonnegative continuous function satisfies appropriate nonlocal conditions, which include the form of $u^k \left(\int_\Omega u^{l+1}\,dx\right)^m$. Moreover, the weight function $a(x) \in C^0(\overline{\Omega})$ satisfies

(A1) $a(x) > 0$, $x \in \Omega$ and $a(x) = 0$, $x \in \partial\Omega$, or

(A2) $a(x) \geq c > 0$ for all $x \in \overline{\Omega}$,

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where c is a positive constant. Meanwhile, σ is a positive constant and the initial data $u_0(x)$ is a positive C^1-function which satisfies a compatibility condition. Therefore, by the classical parabolic theory, one can deduce that the solution of problem (1.1)–(1.3) is nonnegative and smooth, of maximal existence time $t^* \in (0, +\infty]$. Moreover, if $t^* < +\infty$, then u blows up in finite time in L^∞-norm. Our nonlocal model (1.1) can account for many natural phenomena, such as blasting model, compressible reactant gas model, population dynamics theories, some biological species with a human-controlled distribution model, and the model of phase separation in binary alloys (see [1, 3, 5, 8] and the references therein).

In the past decades, there have been many authors dealing with global existence and blow-up phenomena of the solutions to the local or nonlocal reaction-diffusion equations, and there have been many monographs as well as the survey paper (cf. [4,9,15]). Specially, Quittner and Souplet [15, Chapter 5] introduced the qualitative properties of the solution to nonlocal reaction-diffusion equation with Dirichlet boundary condition in detail. In some sense, the nonlocal models are more close to the actual model than the local models, but such nonlocal models do not seem to be so much investigated than local models, and now many local theories are no longer holding. Hence this problem is challenging and difficult. In this paper, we would like to investigate blow-up phenomena of the solution for a class of nonlocal reaction-diffusion equation with Robin boundary condition, and our main purpose is to derive the bounds of the blow-up time if the blow-up occurs in finite time. As far as we know, a variety of methods have been used to study upper bounds of the blow-up time to the parabolic equations (cf. [14]). However, due to the explosive nature of the solutions, it is very important in applications to determine lower bounds on the blow-up time. Presently, the research on the lower bound of the blow-up time for the nonlocal problems with Dirichlet or Neumann boundary condition had some new progress. We provide the reader to the literature [7,10,16,19] (constant coefficients case) and [2,12] (time-dependent coefficients case), and the references therein. Moreover, the study on the local parabolic equations with time-dependent coefficients and nonlinear boundary flux, one can refer to [6]. For some recent interesting research on the local reaction-diffusion equation with nonlocal boundary conditions see [13].

Specially, we are very concerned about the recent research works of Song and Lv [11,17]. They considered the semilinear parabolic equation with weighted local sources

$$u_t = \Delta u + a(x) f(u), \quad (x, t) \in \Omega \times (0, t^*),$$

where the weight function satisfied $a(x) \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with

(A1) $a(x) > 0$, $x \in \Omega$ and $a(x) = 0$, $x \in \partial\Omega$, or

(A2) $a(x) \geq c > 0$ for all $x \in \overline{\Omega}$, or

(A3) $a(x) \equiv 0$ for all $x \in \overline{\Omega}$, or

(A4) $0 < c_1 < a(x) < c_2$ for all $x \in \overline{\Omega}$.

When the initial boundary value problem with nonlinear Neumann boundary condition, and weight function $a(x)$ satisfied (A1) or (A3) or (A4), they obtained the bounds for the blow-up time of the solution in three-dimensional space (cf. [11]). In [17], the initial boundary value problem with homogeneous Dirichlet or Neumann boundary condition, and weight function $a(x)$ satisfied (A1) or (A2) were considered, where they derived the bounds for the blow-up rate and the blow-up time in any smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$. Notes that, their results involved the case that some nonlinearities $f(u)$ satisfied nonlocal condition.

By the above-mentioned works, the study on blow-up analysis for the reaction-diffusion equations with weighted nonlocal inner source terms and Robin boundary condition has not been proceeded yet in the higher dimensional spaces. At a glance, the main difficulty lies in finding the influence of weight function $a(x)$ and source terms to the blow-up phenomena. We pay our attention to establish sufficient conditions to guarantee the solution of problem (1.1)–(1.3) exists globally or blows up at finite time under appropriate measure sense. Moreover, upper and lower bounds for the blow-up time are derived in higher dimensional spaces.

The rest of the paper is organized as follows. In Section 2, we construct suitable super-solution of problem (1.1)–(1.3) to get the solution exists globally. In Section 3, we impose the sufficient conditions on weight function $a(x)$ and nonlocal source terms f to guarantee that the solution of problem (1.1)–(1.3) blows up at finite time, and obtain an upper bound for the blow-up time. In Section 4, we will be devoted to drive lower bounds for the blow-up time under two different measure in the higher dimensional spaces. Moreover, a few examples are given to illustrate applications of our main results in Section 5.

2. The global existence

In this section, we seek a global super-solution to derive the solution of problem (1.1)–(1.3) exists globally. More precisely, we obtain the following main results.

Theorem 2.1. *Suppose that the nonnegative function f satisfies*

$$f(s(x,t)) \leq (s(x,t))^k \left(\int_\Omega (s(x,t))^{l+1}\, dx \right)^m, \quad s(x,t) \geq 0, \tag{2.1}$$

where the function $s(x,t) \in C(\Omega \times (0,t^*))$, and the positive constants k, l, m such that $k + (l+1)m > 1$. Meanwhile, the weight function $a(x) \in C^0(\overline{\Omega})$ satisfies (A1) *or* (A2)*, the initial data $u_0(x) \leq A_1^{-\delta_1}\phi_1(x)$, where δ_1 is an arbitrary positive constant, $A_1 > 1$ is* *sufficiently large, and $\phi_1(x) > 0$ is the first eigenfunction which corresponding the first eigenvalue λ_1 of the following eigenvalue problem*

$$\Delta\phi_1 + \lambda_1\phi_1 = 0, \quad x \in \Omega,$$
$$\frac{\partial \phi_1}{\partial \nu} + \sigma\phi_1 = 0, \quad x \in \partial\Omega,$$

which is normalized by $\max_{x\in\overline{\Omega}} \phi(x) = 1$.

Then the nonnegative classical solution $u(x,t)$ of problem (1.1)–(1.3) *does not blow up; that is, $u(x,t)$ exists for all $t > 0$.*

Proof. Setting $\overline{w} = (A_1+t)^{-\delta_1}\phi_1(x)$, where $\delta_1 > 0$, $A_1 > 0$ are constants to be determined later. First, by virtue of $\max_{x\in\overline{\Omega}} \phi(x) = 1$, we can compute

$$\begin{aligned}
&\overline{w_t} - \Delta\overline{w} - a(x)\overline{w}^k \left(\int_\Omega \overline{w}^{l+1}\,dx\right)^m \\
&= -\delta_1(A_1+t)^{-\delta_1-1}\phi_1 + \lambda_1(A_1+t)^{-\delta_1}\phi_1 - a(x)(A_1+t)^{-\delta_1(k+(l+1)m)}\phi_1^k \left(\int_\Omega \phi_1^{l+1}\,dx\right)^m \\
&\geq \phi_1(A_1+t)^{-\delta_1}\left(-\delta_1(A_1+t)^{-1} + \lambda_1 - a(x)(A_1+t)^{-\delta_1(k+(l+1)m-1)}\right),
\end{aligned}$$

where the constant A_1 is sufficiently large such that

$$\overline{w_t} - \Delta\overline{w} - a(x)\overline{w}^k \left(\int_\Omega \overline{w}^{l+1}\,dx\right)^m \geq 0, \quad x \in \Omega,\ t > 0. \tag{2.2}$$

Next, we can derive the following equality on $\partial\Omega$:

$$\frac{\partial\overline{w}}{\partial\nu} + \sigma\overline{w} = (A_1+t)^{-\delta_1}\left(\frac{\partial\phi_1}{\partial\nu} + \sigma\phi_1\right) = 0, \quad x \in \partial\Omega,\ t > 0. \tag{2.3}$$

Finally, we require that the initial data satisfies

$$\overline{w}(x,0) = A_1^{-\delta_1}\phi_1(x) \geq u_0(x), \quad x \in \Omega. \tag{2.4}$$

Therefore, the relations (2.2)–(2.4) show that $\overline{w}(x,t)$ is a super-solution of the problem (1.1)–(1.3). It can be easily seen that $\overline{w}(x,t)$ exists globally. Thus, by the comparison principle and (2.2)–(2.4), $u(x,t)$ is global. The proof of Theorem 2.1 is completed. □

Remark 2.2. Indeed, if the nonnegative function f satisfies

$$f(s(x,t)) \geq (s(x,t))^k \left(\int_\Omega (s(x,t))^{l+1}\,dx\right)^m, \quad s(x,t) \geq 0, \tag{2.5}$$

with the function $s(x,t) \in C(\Omega \times (0,t^*))$, and the positive constants k, l, m such that $k + (l+1)m > 1$. Meanwhile, the weight function $a(x) \in C^0(\overline{\Omega})$ satisfies (A1) or (A2). Moreover, the initial data $u_0(x) \geq A_2 T^{-\delta_2}\phi_1(x)$, where $A_2 > 0$ is sufficiently large, and $T, \delta_2 > 0$. So it can be easily shown that $\underline{w}(x,t) = A_2(T-t)^{-\delta_2}\phi_1(x)$ is a sub-solution of the solution $u(x,t)$ for the problem of (1.1)–(1.3), which blows up at finite time $t = T$. Hence, the solution $u(x,t)$ of problem (1.1)–(1.3) blows up at some finite time $t^* \leq T$.

3. Blow-up and upper bound of t^*

In this section, we assume certain nonlocal condition on nonlinearity to guarantee that the solution of (1.1)–(1.3) blows up at finite time t^* and derive an upper bound for t^*. The result can be summarized as follows:

Theorem 3.1. *Suppose that $u(x,t)$ is a nonnegative solution of problem* (1.1)–(1.3), *and the nonnegative and integrable function f satisfies the condition*

$$\xi f(\xi) \geq 2(1+p)F(\xi), \quad \xi(x,t) \geq 0, \tag{3.1}$$

where the function $\xi = \xi(x,t) \in C(\Omega \times (0,t^*))$, and $F(\xi) = \int_0^\xi f(\eta)\,d\eta$, $p \geq 0$. Moreover, weight function $a(x) \in C^0(\overline{\Omega})$ satisfies (A1) *or* (A2). *Set*

$$\Theta(t) = -2(1+p)\left[\int_\Omega |\nabla u|^2\,dx + \sigma \int_{\partial\Omega} u^2\,ds\right] + 4(1+p)\int_\Omega a(x)F(u)\,dx$$

and let $\Theta(0) > 0$. Then the solution $u(x,t)$ of problem (1.1)–(1.3) *blows up in a finite time $t^* \leq T_0$ with*

$$T_0 = \frac{\Psi(0)}{p\Theta(0)}, \quad p > 0,$$

where $\Psi(0) = \int_\Omega u_0^2\,dx$. If $p = 0$, then $u(x,t)$ blows up at infinite time.

Remark 3.2. In fact, we can choose

$$f(u) = u^k \left(\int_\Omega u^{l+1}\,dx\right)^m, \quad F(u) = \int_0^u \eta^k \left(\int_\Omega \eta^{l+1}\,dx\right)^m d\eta,$$

$k > 0$, $l+1 > 0$, $m > 0$, $k + m(l+1) > 1$, which satisfies (3.1) in Theorem 3.1.

Proof. In order to prove that the solution blows up in finite time under the assumption of Theorem 3.1 when $p > 0$, we first assume the solution $u(x,t)$ is global to get a contradiction. In this way, the auxiliary function $\Psi(t)$ is bounded for all $t \geq 0$. We compute the derivative of $\Psi(t)$ and utilize hypotheses in Theorem 3.1, Green's formula, we can get

$$\begin{aligned}\Psi'(t) &= 2\int_\Omega u u_t\,dx = 2\int_\Omega u(\Delta u + a(x)f(u))\,dx \\ &= -2\sigma\int_{\partial\Omega} u^2\,ds - 2\int_\Omega |\nabla u|^2\,dx + 2\int_\Omega a(x)uf(u)\,dx \\ &\geq -2(1+p)\left[\int_\Omega |\nabla u|^2\,dx + \sigma\int_{\partial\Omega} u^2\,ds\right] + 4(1+p)\int_\Omega a(x)F(u)\,dx \\ &= \Theta(t).\end{aligned} \tag{3.2}$$

Since the function $\Psi(t)$ is bounded for all $t \geq 0$, so is $\Theta(t)$. Now, differentiating $\Theta(t)$ and using Green's formula, we obtain

$$\begin{aligned}\Theta'(t) &= -4(1+p)\int_\Omega \nabla u \cdot \nabla u_t \, dx - 4\sigma(1+p)\int_{\partial\Omega} u u_t \, ds \\ &\quad + 4(1+p)\int_\Omega a(x) f(u) u_t \, dx \\ &= 4(1+p)\int_\Omega u_t(\Delta u + a(x) f(u)) \, dx \\ &= 4(1+p)\int_\Omega u_t^2 \, dx \geq 0,\end{aligned} \tag{3.3}$$

which implies $\Theta(t) > 0$ for all $t \geq 0$, since $\Theta(0) > 0$. Moreover, applying the Schwarz's inequality, we have

$$\Theta'(t)\Psi(t) = 4(1+p)\int_\Omega u_t^2 \, dx \int_\Omega u^2 \, dx \geq (1+p)(\Psi'(t))^2 \geq (1+p)\Psi'(t)\Theta(t),$$

which is equivalent to

$$(\Theta\Psi^{-(1+p)})' \geq 0. \tag{3.4}$$

Then integrating (3.4) from 0 to t, we can compute

$$\Theta(t)(\Psi(t))^{-(1+p)} \geq \Theta(0)(\Psi(0))^{-(1+p)}. \tag{3.5}$$

Substituting (3.5) into (3.2), we yield the differential inequality

$$-\frac{1}{p}(\Psi^{-p})' = \Psi'(t)(\Psi(t))^{-(1+p)} \geq \Theta(t)(\Psi(t))^{-(1+p)} \geq \Theta(0)(\Psi(0))^{-(1+p)}.$$

We denote $\Gamma = \Theta(0)(\Psi(0))^{-(1+p)}$, it turns to

$$-\frac{1}{p}(\Psi^{-p})' \geq \Gamma. \tag{3.6}$$

Now, integrating again, we have the following inequality

$$(\Psi(t))^{-p} \leq (\Psi(0))^{-p} - p\Gamma t. \tag{3.7}$$

Obviously, (3.7) cannot hold for all time, which is a contradiction. Hence the solution $u(x,t)$ blows up in finite time. Therefore, (3.7) leads to

$$t^* \leq T_0 = \frac{\Psi(0)}{p\Theta(0)}, \tag{3.8}$$

valid for $p > 0$.

In particular, if $p = 0$, by a direct calculation, we can compute

$$\Psi(t) \geq \Psi(0) e^{\Theta(0)(\Psi(0))^{-1} t},$$

which is valid for all $t > 0$, implying that the solution $u(x,t)$ blows up at infinite time. This completes the proof of Theorem 3.1. $\square$

4. Lower bounds for t^*

4.1. In the sense of L^{l+1}-norm

In this subsection, we assume nonlinearity f satisfies the nonlocal condition (2.1), and use the modified differential inequality to seek lower bounds for the blow-up time t^* in different cases.

Case 1: $0 \le k \le 1$.

Theorem 4.1. *Suppose that $u(x,t)$ is the nonnegative classical solution of problem* (1.1)–(1.3)*, $u(x,t)$ blows up at finite time t^* in the L^{l+1}-norm, and the nonnegative function f satisfies* (2.1) *with $0 \le k \le 1$, $l > 0$, $m > 0$, $k + (l+1)m > 1$. Meanwhile, the weight function $a(x) \in C^0(\overline{\Omega})$ satisfies* (A1) *or* (A2)*. Then the blow-up time t^* is bounded from below by*

$$t^* \ge T_1 = \frac{l+1}{I_1(m(l+1)+k-1)(\varphi(0))^{[m(l+1)+k-1]/(l+1)}},$$

in the measure of $\varphi(t) = \int_\Omega u^{l+1}\,dx$, where $\varphi(0) = \int_\Omega u_0^{l+1}\,dx$, and I_1 is a computable positive constant.

Proof. First, differentiating $\varphi(t)$ and using (1.1), (1.2), (2.1), and Green's formula, we have

$$\begin{aligned}\varphi'(t) &= (l+1)\int_\Omega u^l(\Delta u + a(x)f(u))\,dx \\ &= -\sigma(l+1)\int_{\partial\Omega} u^{l+1}\,ds - (l+1)l\int_\Omega u^{l-1}\,|\nabla u|^2\,dx + (l+1)\int_\Omega a(x)u^l f(u)\,dx \\ &\le -(l+1)l\int_\Omega u^{l-1}\,|\nabla u|^2\,dx + (l+1)\int_\Omega a(x)u^{l+k}\,dx\left(\int_\Omega u^{l+1}\,dx\right)^m.\end{aligned} \tag{4.1}$$

Now, since $0 \le k \le 1$, we can apply Hölder's inequality to last term on the right-hand side of (4.1), which yield

$$\begin{aligned}&(l+1)\int_\Omega a(x)u^{l+k}\,dx\left(\int_\Omega u^{l+1}\,dx\right)^m \\ &\le (l+1)\left(\int_\Omega (a(x))^{(l+1)/(1-k)}\,dx\right)^{(1-k)/(l+1)}\left(\int_\Omega u^{l+1}\,dx\right)^{m+(l+k)/(l+1)}.\end{aligned} \tag{4.2}$$

Next, inserting (4.2) into (4.1), we obtain

$$\varphi'(t) \le I_1(\varphi(t))^{m+(l+k)/(l+1)}, \tag{4.3}$$

where $I_1 = (l+1)\left(\int_\Omega (a(x))^{(l+1)/(1-k)}\,dx\right)^{(1-k)/(l+1)}$.

Since $\lim_{t\to t^*}\varphi(t) = \infty$, then integrating (4.3) from 0 to t^*, we can finally lead to

$$t^* \ge T_1 = \frac{l+1}{I_1(m(l+1)+k-1)(\varphi(0))^{[m(l+1)+k-1]/(l+1)}}.$$

Hence, the proof of Theorem 4.1 is completed. □

Case 2: $k > 1$.

We need $\Omega \subset R^N$ $(N \geq 3)$ is a convex bounded domain with smooth boundary, since we use the Sobolev type inequality in this case. More precisely, we state our results below.

Theorem 4.2. *Suppose that $u(x,t)$ is the nonnegative classical solution of problem* (1.1)–(1.3)*, $u(x,t)$ blows up at finite time t^* in the L^{l+1}-norm, and the nonnegative function f satisfies* (2.1) *with*

$$k > 1, \quad l + 1 > \max\left\{1, \frac{2(k-1)(N-2)}{2N-3-2(N-2)(m+1)}\right\}, \quad 0 < m < \frac{1}{2N-4}.$$

Meanwhile, weight function $a(x) \in C^0(\overline{\Omega})$ satisfies (A1) *or* (A2)*. Then the blow-up time t^* is bounded from below by*

$$t^* \geq T_2 = \int_{\varphi(0)}^{\infty} \frac{d\eta}{J_1 + J_2 \eta^{\frac{2N-3}{2(N-2)}} + J_3 \eta^{\frac{3(N-2)}{3N-8}}},$$

in the measure of $\varphi(t)$, where J_1, J_2 and J_3 are some computable positive constants.

Remark 4.3. Because of $k > 1$, $l + 1 > \max\left\{1, \frac{2(k-1)(N-2)}{2N-3-2(N-2)(m+1)}\right\}$, and $0 < m < \frac{1}{2N-4}$, it can be easily seen that $k + (l+1)m > 1$.

Proof. First, by using similar arguments as used in Theorem 4.1, we have

$$\varphi'(t) \leq -\frac{4l}{l+1} \int_\Omega \left|\nabla u^{(l+1)/2}\right|^2 dx + (l+1) \int_\Omega a(x) u^{l+k}\, dx \left(\int_\Omega u^{l+1}\, dx\right)^m. \tag{4.4}$$

Since $k > 1$, using Hölder's inequality twice to the last term on the right-hand side of (4.4), we can derive the following inequalities, respectively:

$$\left(\int_\Omega u^{l+1}\, dx\right)^m \leq \left(\int_\Omega (a(x))^{-\frac{l+1}{k-1}}\, dx\right)^{\frac{(k-1)m}{l+k}} \left(\int_\Omega a(x) u^{l+k}\, dx\right)^{\frac{(l+1)m}{l+k}}, \tag{4.5}$$

$$\int_\Omega a(x) u^{l+k}\, dx \leq \left(\int_\Omega (a(x))^{\frac{(l+1)m+l+k}{(l+1)m}}\, dx\right)^{\frac{(l+1)m}{(l+1)m+l+k}} \left(\int_\Omega u^{(l+1)m+l+k}\, dx\right)^{\frac{l+k}{(l+1)m+l+k}}. \tag{4.6}$$

Now, substituting (4.5), (4.6) into the last term in (4.4), we obtain

$$\begin{aligned} &(l+1)\int_\Omega a(x) u^{l+k}\, dx \left(\int_\Omega u^{l+1}\, dx\right)^m \\ &\leq (l+1)\left(\int_\Omega (a(x))^{-\frac{l+1}{k-1}}\, dx\right)^{\frac{(k-1)m}{l+k}} \left(\int_\Omega (a(x))^{\frac{(l+1)m+l+k}{(l+1)m}}\, dx\right)^{\frac{(l+1)m}{l+k}} \int_\Omega u^{(l+1)m+l+k}\, dx. \end{aligned} \tag{4.7}$$

Afterwards, applying Hölder's and Young's inequalities to (4.7), we can compute

$$\begin{aligned} \int_\Omega u^{(l+1)m+l+k}\, dx &\leq \left(\int_\Omega u^{\frac{(l+1)(2N-3)}{2(N-2)}}\, dx\right)^{q_1} |\Omega|^{1-q_1} \\ &\leq q_1 \int_\Omega u^{\frac{(l+1)(2N-3)}{2(N-2)}}\, dx + (1-q_1)\,|\Omega|, \end{aligned} \tag{4.8}$$

where $q_1 = \frac{2(N-2)((l+1)m+l+k)}{(l+1)(2N-3)}$. Note that $q_1 \in (0,1)$ in view of $l+1 > \frac{2(k-1)(N-2)}{2N-3-2(N-2)(m+1)}$, and $0 < m < \frac{1}{2N-4}$.

Next, by virtue of Hölder's inequality again to the first term on the right-hand side of (4.8), we get

$$\int_\Omega u^{\frac{(l+1)(2N-3)}{2(N-2)}}\,dx \le \left(\int_\Omega \left(u^{\frac{l+1}{2}}\right)^{\frac{2N}{N-2}}\,dx\right)^{1/4} (\varphi(t))^{3/4}. \tag{4.9}$$

Then using Sobolev inequality with $N \ge 3$ in [18], which show that

$$\left\| u^{(l+1)/2} \right\|_{L^{2N/(N-2)}(\Omega)} \le C_s \left\| u^{(l+1)/2} \right\|_{W^{1,2}(\Omega)},$$

where C_s is the Sobolev optimal constant. Moreover, we apply Jensen's inequality to derive

$$\left(\int_\Omega \left(u^{\frac{l+1}{2}}\right)^{\frac{2N}{N-2}}\,dx\right)^{1/4} \le C_b \left[(\varphi(t))^{\frac{N}{4(N-2)}} + \left(\int_\Omega \left|\nabla u^{(l+1)/2}\right|^2 dx\right)^{\frac{N}{4(N-2)}} \right], \tag{4.10}$$

where

$$C_b = \begin{cases} 2^{1/2}(C_s)^{3/2} & \text{for } N = 3, \\ (C_s)^{N/[2(N-2)]} & \text{for } N > 3. \end{cases} \tag{4.11}$$

Now, inserting (4.8)–(4.10) into (4.7), meanwhile, using Young's inequality, we obtain

$$\begin{aligned}
&(l+1)\int_\Omega a(x)u^{l+k}\,dx \left(\int_\Omega u^{l+1}\,dx\right)^m \\
&\le (l+1)\left(\int_\Omega (a(x))^{-\frac{l+1}{k-1}}\,dx\right)^{\frac{(k-1)m}{l+k}} \left(\int_\Omega (a(x))^{\frac{(l+1)m+l+k}{(l+1)m}}\,dx\right)^{\frac{(l+1)m}{l+k}} \\
&\quad \times \Bigg[(1-q_1)\,|\Omega| + q_1 C_b (\varphi(t))^{\frac{2N-3}{2(N-2)}} + \frac{q_1 N \varsigma_1}{4(N-2)} \int_\Omega \left|\nabla u^{\frac{l+1}{2}}\right|^2 dx \\
&\quad + \frac{q_1(3N-8) C_b^{\frac{4(N-2)}{3N-8}}}{4(N-2)\varsigma_1^{\frac{N}{3N-8}}} (\varphi(t))^{\frac{3(N-2)}{3N-8}} \Bigg],
\end{aligned} \tag{4.12}$$

for arbitrary $\varsigma_1 > 0$ to be determined.

Finally, substituting (4.12) into (4.4), we can deduce

$$\varphi'(t) \le J_1 + J_2 (\varphi(t))^{\frac{2N-3}{2(N-2)}} + J_3 (\varphi(t))^{\frac{3(N-2)}{3N-8}} + J_4 \int_\Omega \left|\nabla u^{\frac{l+1}{2}}\right|^2 dx,$$

where

$$
\begin{aligned}
J_1 &= (1-q_1)J_5\,|\Omega|\,,\\
J_2 &= q_1 C_b J_5,\\
J_3 &= \frac{q_1(3N-8)C_b^{\frac{4(N-2)}{3N-8}}}{4(N-2)\varsigma_1^{\frac{N}{3N-8}}}J_5,\\
J_4 &= \frac{q_1 N J_5 \varsigma_1}{4(N-2)} - \frac{4l}{l+1},\\
J_5 &= (l+1)\left(\int_\Omega (a(x))^{-\frac{l+1}{k-1}}\,dx\right)^{\frac{(k-1)m}{l+k}}\left(\int_\Omega (a(x))^{\frac{(l+1)m+l+k}{(l+1)m}}\,dx\right)^{\frac{(l+1)m}{l+k}}.
\end{aligned}
$$

Thus, we can choose $\varsigma_1 = \frac{16l(N-2)}{q_1 N J_5(l+1)} > 0$ such that $J_4 = 0$. Therefore, it turns to

$$
\varphi'(t) \le J_1 + J_2(\varphi(t))^{\frac{2N-3}{2(N-2)}} + J_3(\varphi(t))^{\frac{3(N-2)}{3N-8}}. \tag{4.13}
$$

Since $\lim_{t\to t^*}\varphi(t)=\infty$, then we integrate (4.13) from 0 to t^* which can compute

$$
t^* \ge T_2 = \int_{\varphi(0)}^{\infty} \frac{d\eta}{J_1 + J_2\eta^{\frac{2N-3}{2(N-2)}} + J_3\eta^{\frac{3(N-2)}{3N-8}}}.
$$

The proof of Theorem 4.2 is completed. □

4.2. In the sense of weighted L^{l+1}-norm

In this subsection, we investigate the nonnegative classical solution of problem (1.1)–(1.3) that blows up in weighted L^{l+1}-norm. Here, we assume nonlinearity f satisfies the following nonlocal condition

$$
a(x)f(s(x,t)) \le (s(x,t))^k\left(\int_\Omega b(x)(s(x,t))^{l+1}\,dx\right)^m, \quad s(x,t)\ge 0, \tag{4.14}
$$

where the function $s(x,t)\in C(\Omega\times(0,t^*))$, and the weight function $b(x)\in C^1(\Omega)\cap C^0(\overline{\Omega})$ satisfies

$$
b(x) > 0,\ x\in\Omega \quad \text{and} \quad b(x)=0,\ x\in\partial\Omega, \tag{4.15}
$$

or

$$
b(x) \ge c_0 > 0 \quad \text{for all } x\in\overline{\Omega}, \tag{4.16}
$$

with c_0 is a positive constant, moreover,

$$
-b(x)B \le \nabla b(x) \le b(x)B \iff \left|\frac{\partial b(x)}{\partial x_i}\right| \le B_i b(x) \quad \text{for all } x\in\Omega, \tag{4.17}
$$

where each $B=(B_1,B_2,\ldots,B_N)$ is a positive constant vector.

Case 1: $0\le k\le 1$.

Theorem 4.4. *Suppose that $u(x,t)$ is the nonnegative classical solution of problem* (1.1)–(1.3), *$u(x,t)$ blows up at finite time t^* in weighted L^{l+1}-norm, and the nonnegative function f satisfies* (4.14) *with $0 \le k \le 1$, $l > 0$, $m > 0$, $k+(l+1)m > 1$. Meanwhile, the weight function $b(x) \in C^1(\Omega) \cap C^0(\overline{\Omega})$ satisfies* (4.15) *or* (4.16) *and* (4.17). *Then the blow-up time t^* is bounded from below by*

$$t^* \ge T_3 = \int_{\Phi(0)}^{\infty} \frac{d\eta}{K_1\eta + K_2\eta^{m+\frac{l+k}{l+1}}},$$

in the measure of $\Phi(t) = \int_\Omega b(x)u^{l+1}\,dx$, where $\Phi(0) = \int_\Omega b(x)u_0^{l+1}\,dx$, and K_1, K_2 are computable positive constants.

Proof. First, differentiating $\Phi(t)$ and utilizing (1.1), (1.2), (4.14), (4.17) and Green's formula, we have

$$\begin{aligned}
\Phi'(t) &= (l+1)\int_\Omega b(x)u^l(\Delta u + a(x)f(u))\,dx,\\
&= -\sigma(l+1)\int_{\partial\Omega} b(x)u^{l+1}\,ds - (l+1)\int_\Omega \nabla(b(x)u^l)\cdot\nabla u\,dx\\
&\quad + (l+1)\int_\Omega b(x)u^l a(x)f(u)\,dx\\
&\le (l+1)\,|B|\int_\Omega b(x)u^l\,|\nabla u|\,dx - (l+1)l\int_\Omega b(x)u^{l-1}\,|\nabla u|^2\,dx\\
&\quad + (l+1)\int_\Omega b(x)u^{l+k}\,dx\left(\int_\Omega b(x)u^{l+1}\,dx\right)^m.
\end{aligned} \tag{4.18}$$

We now apply Schwarz's and Young's inequalities to the first term on the right-hand side of (4.18) to yield

$$\begin{aligned}
&(l+1)\,|B|\int_\Omega b(x)u^l\,|\nabla u|\,dx\\
&\le (l+1)\,|B|\left(\int_\Omega b(x)u^{l-1}\,|\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega b(x)u^{l+1}\,dx\right)^{1/2}\\
&\le \frac{(l+1)^2\,|B|^2\,\gamma_1}{2}\int_\Omega b(x)u^{l-1}\,|\nabla u|^2\,dx + \frac{1}{2\gamma_1}\int_\Omega b(x)u^{l+1}\,dx,
\end{aligned} \tag{4.19}$$

where γ_1 is a positive constant to be chosen. Next, since $0 \le k \le 1$, we can use Hölder's inequality to the last term on the right-hand side of (4.18) to obtain

$$\begin{aligned}
&(l+1)\int_\Omega b(x)u^{l+k}\,dx\left(\int_\Omega b(x)u^{l+1}\,dx\right)^m\\
&\le (l+1)\left(\int_\Omega b(x)\,dx\right)^{\frac{1-k}{l+1}}\left(\int_\Omega b(x)u^{l+1}\,dx\right)^{m+\frac{l+k}{l+1}}.
\end{aligned} \tag{4.20}$$

Afterwards, inserting (4.19), (4.20) into (4.18), we can lead to

$$\begin{aligned}\Phi'(t) &\le \frac{1}{2\gamma_1}\Phi(t) + \left[\frac{(l+1)^2\,|B|^2\,\gamma_1}{2} - (l+1)l\right]\int_\Omega b(x)u^{l-1}\,|\nabla u|^2\,dx \\ &\quad + (l+1)\left(\int_\Omega b(x)\,dx\right)^{\frac{1-k}{l+1}}(\Phi(t))^{m+\frac{l+k}{l+1}}.\end{aligned} \tag{4.21}$$

We can select $\gamma_1 = \frac{2l}{(l+1)|B|^2}$, then (4.21) turns into

$$\Phi'(t) \le K_1\Phi(t) + K_2(\Phi(t))^{m+\frac{l+k}{l+1}}, \tag{4.22}$$

where $K_1 = \frac{(l+1)|B|^2}{4l}$, $K_2 = (l+1)\left(\int_\Omega b(x)\,dx\right)^{\frac{1-k}{l+1}}$.

Finally, since $\lim_{t\to t^*}\Phi(t) = \infty$, then we integrate (4.22) from 0 to t^*, which can derive

$$t^* \ge T_3 = \int_{\Phi(0)}^{\infty}\frac{d\eta}{K_1\eta + K_2\eta^{m+\frac{l+k}{l+1}}}.$$

The proof of Theorem 4.4 is completed. □

Case 2: $k > 1$.

Similarly to the Case 2 in Subsection 4.1, here we assume $\Omega \subset R^N$ $(N \ge 3)$ is a convex bounded domain with smooth boundary. Our result can be summarized as follows:

Theorem 4.5. *Suppose that $u(x,t)$ is the nonnegative classical solution of problem* (1.1)–(1.3)*, $u(x,t)$ blows up at finite time t^* in weighted L^{l+1}-norm, and the nonnegative function f satisfies* (4.14) *with*

$$k > 1, \quad l+1 > \max\left\{1, \frac{2(k-1)(N-2)}{2N-3-2(N-2)(m+1)}\right\}, \quad 0 < m < \frac{1}{2N-4}.$$

Meanwhile, the weighted function $b(x) \in C^1(\Omega)\cap C^0(\overline{\Omega})$ satisfies (4.15) *or* (4.16) *and* (4.17)*. Then the blow-up time t^* is bounded from below by*

$$t^* \ge T_4 = \int_{\Phi(0)}^{\infty}\frac{d\eta}{L_1 + L_2\eta + L_3\eta^{\frac{2N-3}{2(N-2)}} + L_4\eta^{\frac{3(N-2)}{3N-8}}}$$

in the measure of weighted L^{l+1}-norm $\Phi(t)$, which is defined in Theorem 4.4*. Here L_1, L_2, L_3 and L_4 are some computable positive constants.*

Proof. By virtue of the similar arguments as used in Theorem 4.4, we have

$$\begin{aligned}\Phi'(t) &\le \frac{\mu_1}{2}\Phi(t) + \left[2\,|B|^2\,\mu_1 - \frac{4l}{l+1}\right]\int_\Omega b(x)\left|\nabla u^{(l+1)/2}\right|^2 dx \\ &\quad + (l+1)\int_\Omega b(x)u^{l+k}\,dx\left(\int_\Omega b(x)u^{l+1}\,dx\right)^m\end{aligned} \tag{4.23}$$

for an arbitrary $\mu_1 > 0$ to be specified later. Because of $k > 1$, we can use Hölder's inequality twice to derive

$$\left(\int_\Omega b(x)u^{l+1}\,dx\right)^m \le \left(\int_\Omega b(x)\,dx\right)^{\frac{(k-1)m}{l+k}} \left(\int_\Omega b(x)u^{l+k}\,dx\right)^{\frac{(l+1)m}{l+k}} \tag{4.24}$$

and

$$\int_\Omega b(x)u^{l+k}\,dx \le \left(\int_\Omega b(x)\,dx\right)^{\frac{(l+1)m}{(l+1)m+l+k}} \left(\int_\Omega b(x)u^{(l+1)m+l+k}\,dx\right)^{\frac{l+k}{(l+1)m+l+k}}. \tag{4.25}$$

Now, substituting (4.24), (4.25) into the last term of (4.23), we compute

$$\begin{aligned} &(l+1)\int_\Omega b(x)u^{l+k}\,dx\left(\int_\Omega b(x)u^{l+1}\,dx\right)^m \\ &\le (l+1)\left(\int_\Omega b(x)\,dx\right)^m \int_\Omega b(x)u^{(l+1)m+l+k}\,dx. \end{aligned} \tag{4.26}$$

Next, applying Hölder's and Young's inequalities to (4.26), we can obtain

$$\begin{aligned} &\int_\Omega b(x)u^{(l+1)m+l+k}\,dx \\ &\le \left(\int_\Omega (b(x))^{\frac{2N-3}{2(N-2)}} u^{\frac{(l+1)(2N-3)}{2(N-2)}}\,dx\right)^{q_1} \left(\int_\Omega (b(x))^{\frac{(l+1)(1-m)-l-k}{(l+1)(1-q_1)}}\,dx\right)^{1-q_1} \\ &\le q_1\int_\Omega (b(x))^{\frac{2N-3}{2(N-2)}} u^{\frac{(l+1)(2N-3)}{2(N-2)}}\,dx + (1-q_1)\int_\Omega (b(x))^{\frac{(l+1)(1-m)-l-k}{(l+1)(1-q_1)}}\,dx, \end{aligned} \tag{4.27}$$

where q_1 is given in Theorem 4.2. Then using Hölder's inequality again to the first term on the right-hand side of (4.27), we get

$$\int_\Omega (b(x))^{\frac{2N-3}{2(N-2)}} u^{\frac{(l+1)(2N-3)}{2(N-2)}}\,dx \le \left(\int_\Omega \left((b(x))^{\frac{1}{2}} u^{\frac{l+1}{2}}\right)^{\frac{2N}{N-2}}\,dx\right)^{1/4} (\Phi(t))^{3/4}. \tag{4.28}$$

We now introduce the weighted Sobolev inequality for $N \ge 3$,

$$\left\|(b(x))^{\frac{1}{2}} u^{\frac{l+1}{2}}\right\|_{L^{\frac{2N}{N-2}}(\Omega)} \le C_s \left\|(b(x))^{\frac{1}{2}} u^{\frac{l+1}{2}}\right\|_{W^{1,2}(\Omega)},$$

where C_s is the Sobolev optimal constant. Meanwhile, using (4.17) and Jensen's inequality, we can obtain

$$\left(\int_\Omega \left((b(x))^{\frac{1}{2}} u^{\frac{l+1}{2}}\right)^{\frac{2N}{N-2}}\,dx\right)^{\frac{1}{4}} \le C_B\left[\Phi^{\frac{N}{4(N-2)}} + \left(\int_\Omega b(x)\left|\nabla u^{\frac{l+1}{2}}\right|^2\,dx\right)^{\frac{N}{4(N-2)}}\right], \tag{4.29}$$

where $C_B = \max\left\{(1+(\frac{1}{2}|B|^2)^{\frac{N}{4(N-2)}})C_b, 2^{\frac{N}{4(N-2)}}C_b\right\}$, and C_b is the constant given in (4.11).

Inserting (4.27)–(4.29) into (4.26) and using Young's inequality, we compute

$$
\begin{aligned}
&(l+1)\int_\Omega b(x)u^{l+k}\,dx\left(\int_\Omega b(x)u^{l+1}\,dx\right)^m\\
&\le (l+1)\left(\int_\Omega b(x)\,dx\right)^m\left[q_1C_B\Phi^{\frac{2N-3}{2(N-2)}}+\frac{q_1N\mu_2}{4(N-2)}\int_\Omega b(x)\left|\nabla u^{\frac{l+1}{2}}\right|^2dx\right.\\
&\qquad\left.+\frac{q_1(3N-8)C_B^{\frac{4(N-2)}{3N-8}}}{4(N-2)\mu_2^{\frac{N}{3N-8}}}\Phi^{\frac{3(N-2)}{3N-8}}+(1-q_1)\int_\Omega(b(x))^{\frac{(l+1)(1-m)-l-k}{(l+1)(1-q_1)}}\,dx\right],
\end{aligned}
\tag{4.30}
$$

where $\mu_2>0$ is a constant to be determined.

Afterwards, inserting (4.30) into (4.23), we obtain

$$
\Phi'(t)\le L_1+L_2\Phi(t)+L_3(\Phi(t))^{\frac{2N-3}{2(N-2)}}+L_4(\Phi(t))^{\frac{3(N-2)}{3N-8}}+L_5\int_\Omega b(x)\left|\nabla u^{\frac{l+1}{2}}\right|^2dx,
$$

where

$$
\begin{aligned}
L_1&=(l+1)(1-q_1)\left(\int_\Omega b(x)\,dx\right)^m\int_\Omega(b(x))^{\frac{(l+1)(1-m)-l-k}{(l+1)(1-q_1)}}\,dx,\\
L_2&=\frac{\mu_1}{2},\\
L_3&=(l+1)q_1C_B\left(\int_\Omega b(x)\,dx\right)^m,\\
L_4&=(l+1)\frac{q_1(3N-8)C_B^{\frac{4(N-2)}{3N-8}}}{4(N-2)\mu_2^{\frac{N}{3N-8}}}\left(\int_\Omega b(x)\,dx\right)^m,\\
L_5&=2|B|^2\mu_1+(l+1)\frac{q_1N\mu_2}{4(N-2)}\left(\int_\Omega b(x)\,dx\right)^m-\frac{4l}{l+1}.
\end{aligned}
$$

For $\mu_1>0$ small enough, we select $\mu_2>0$ such that $L_5=0$.

Finally, it turns into

$$
\Phi'(t)\le L_1+L_2\Phi(t)+L_3(\Phi(t))^{\frac{2N-3}{2(N-2)}}+L_4(\Phi(t))^{\frac{3(N-2)}{3N-8}}.
\tag{4.31}
$$

Since $\lim_{t\to t^*}\Phi(t)=\infty$, then we integrate (4.31) from 0 to t^*, which can lead to

$$
t^*\ge T_4=\int_{\Phi(0)}^{\infty}\frac{d\eta}{L_1+L_2\eta+L_3\eta^{\frac{2N-3}{2(N-2)}}+L_4\eta^{\frac{3(N-2)}{3N-8}}}.
$$

The proof of Theorem 4.5 is completed. □

5. Applications

In this section, we present five illustrations to demonstrate the applications of Theorems 3.1, 4.1, 4.2, 4.4 and 4.5.

Example 5.1. Let $u(x,t)$ be a nonnegative solution of the following problem:

$$\begin{aligned}
&u_t = \Delta u + (1-|x|^2)u^2 \int_\Omega u\,dx, && (x,t) \in \Omega \times (0,t^*),\\
&\frac{\partial u}{\partial \nu} + u = 0, && (x,t) \in \partial\Omega \times (0,t^*),\\
&u(x,0) = e^{-|x|} > 0, && x \in \Omega,
\end{aligned}$$

where $\Omega = \left\{ x = (x_1,x_2,x_3) \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 1 \right\}$ is the unit ball in $\mathbb{R}^3$, we then have

$$a(x) = 1 - x^2, \quad f(u) = u^2 \int_\Omega u\,dx, \quad \sigma = 1, \quad u_0 = e^{-|x|}.$$

Now we set $p = 1/10$, and then, it is easy to verify that, (A1) and (3.1) hold. By the definition of $\Theta(t)$ in Theorem 3.1, we obtain

$$\begin{aligned}
\Theta(0) = &-2\left(1 + \frac{1}{10}\right)\left[\int_\Omega |\nabla u_0|^2\,dx + \int_{\partial\Omega} u_0^2\,ds\right]\\
&+ 4\left(1 + \frac{1}{10}\right)\int_\Omega u_0\,dx \int_\Omega (1-x^2)\int_0^{u_0} \eta^2\,d\eta dx\\
= &\ 59.76 > 0.
\end{aligned}$$

Thus, it follows from Theorem 3.1 that $u(x,t)$ must blow up in finite time t^*, and we have an upper bound for blow-up time that

$$t^* \le T_0 = \frac{10\Psi(0)}{\Theta(0)} = 1.43,$$

where $\Psi(0) = \int_\Omega u_0^2\,dx = 8.53$. If $p = 0$, then $t^* = \infty$. This shows that the solution blows up at infinite time.

Example 5.2. Let $u(x,t)$ be a nonnegative classical solution of the following problem:

$$\begin{aligned}
&u_t = \Delta u + \left(\frac{1}{\sqrt{10}} - |x|\right) u^{1/3} \int_\Omega u^2\,dx, && (x,t) \in \Omega \times (0,t^*),\\
&\frac{\partial u}{\partial \nu} + \frac{1}{\sqrt{10}} u = 0, && (x,t) \in \partial\Omega \times (0,t^*),\\
&u(x,0) = \frac{1}{\sqrt{10}} e^{-|x|} > 0, && x \in \Omega,
\end{aligned}$$

where $\Omega = \left\{ x = (x_1,x_2,x_3) \mid |x|^2 = \sum_{i=1}^3 x_i^2 < 1/10 \right\}$ is the ball with radius equal to $1/\sqrt{10}$ in $\mathbb{R}^3$. We then have

$$a(x) = \frac{1}{\sqrt{10}} - |x|, \quad f(u) = u^{1/3} \int_\Omega u^2\,dx, \quad \sigma = \frac{1}{\sqrt{10}}, \quad u_0(x) = \frac{1}{\sqrt{10}} e^{-|x|}.$$

Setting $k = 1/3$, $l = 1$, $m = 1$, it can be easily seen that (A1), (2.1), and the constraints on parameters in Theorem 4.1 are satisfied. Then substituting k, l, m into I_1, we can compute $I_1 = 0.73$.

Therefore, by Theorem 4.1, we obtain a lower bound for the blow-up time t^* as follows:

$$t^* \geq T_1 = \frac{3}{2I_1(\varphi(0))^{2/3}} = 3.48,$$

where $\varphi(0) = \int_\Omega u_0^2 \, dx = 0.45$.

Example 5.3. Let $u(x,t)$ be a nonnegative classical solution of the following problem:

$$\begin{aligned} & u_t = \Delta u + \left(e^{|x|^2} + 1\right) u^{3/2} \left(\int_\Omega u^3 \, dx\right)^{1/4}, && (x,t) \in \Omega \times (0,t^*), \\ & \frac{\partial u}{\partial \nu} + \frac{2}{9} u = 0, && (x,t) \in \partial\Omega \times (0,t^*), \\ & u(x,0) = \frac{1}{10} - |x|^2 \times 10^{-2} > 0, && x \in \Omega, \end{aligned}$$

where $\Omega = \left\{ x = (x_1, x_2, x_3) \mid |x|^2 = \sum_{i=1}^3 x_i^2 < (1/10)^2 \right\}$ is the ball with radius equal to $1/10$ in $\mathbb{R}^3$. We then have

$$a(x) = e^{|x|^2} + 1, \quad f(u) = u^{3/2} \left(\int_\Omega u^3 \, dx\right)^{1/4}, \quad \sigma = \frac{2}{9}, \quad u_0(x) = \frac{1}{10} - |x|^2 .$$

Setting $k = 3/2$, $l = 2$, $m = 1/4$, it can be easy to know that (A2), (2.1), and the constraints on parameters in Theorem 4.2 are satisfied. Meanwhile, the Sobolev optimal constant $C_s = 3^{-1/2} 4^{1/3} \pi^{-2/3}$ in three-dimensional space, so $C_b = 0.40$ by (4.11).

Now, substituting k, l, m, into q_1, we can compute $q_1 = 17/18$. Next, we choose $\varsigma_1 = 0.53$ such that $J_4 = 0$. Inserting the above parameters into J_1, J_2, J_3 and J_5 we derive

$$J_1 = 1.66 \times 10^{-3}, \quad J_2 = 2.69, \quad J_3 = 0.14, \quad J_5 = 7.12.$$

Therefore, by Theorem 4.2, we obtain a lower bound for the blow-up time t^* as follows:

$$t^* \geq T_2 = \int_{\varphi(0)}^{\infty} \frac{d\eta}{J_1 + J_2 \eta^{\frac{3}{2}} + J_3 \eta^3} = 9.10,$$

where $\varphi(0) = \int_\Omega u_0^3 \, dx = 1.79 \times 10^{-3}$.

Example 5.4. Let $u(x,t)$ be a nonnegative classical solution of the following problem:

$$\begin{aligned} & u_t = \Delta u + u^{1/2} \int_\Omega e^{|x|} u^3 \, dx, && (x,t) \in \Omega \times (0,t^*), \\ & \frac{\partial u}{\partial \nu} + 2u = 0, && (x,t) \in \partial\Omega \times (0,t^*), \\ & u(x,0) = \frac{3}{20} - |x| > 0, && x \in \Omega, \end{aligned}$$

where $\Omega = \left\{ x = (x_1, x_2, x_3) \mid |x|^2 = \sum_{i=1}^3 x_i^2 < (1/10)^2 \right\}$ is the ball with radius equal to $1/10$ in $\mathbb{R}^3$. We then have

$$b(x) = e^{|x|}, \quad f(u) = u^{1/2} \int_\Omega u^3 \, dx, \quad \sigma = 2, \quad u_0(x) = \frac{3}{20} - |x| \,.$$

Setting $k = 1/2$, $l = 2$, $m = 1$, and choosing $B = (1, 1, 1)$, it is easy to verify that (4.16), (4.17), (4.14), and the constraints on the parameters in Theorem 4.4 are satisfied.

Then inserting the above parameters into K_1 and K_2, we derive

$$K_1 = \frac{9}{8}, \quad K_2 = 3.39.$$

Therefore, by Theorem 4.4, we obtain

$$t^* \geq T_3 = \int_{\Phi(0)}^{\infty} \frac{d\eta}{K_1 + K_2 \eta^{\frac{11}{6}}} = 4.15,$$

where $\Phi(0) = \int_\Omega e^{|x|} u_0^3 \, dx = 2.56 \times 10^{-3}$.

Example 5.5. Let $u(x,t)$ be a nonnegative classical solution of the following problem:

$$\begin{cases} u_t = \Delta u + u^{3/2} \left(\displaystyle\int_\Omega (1 + |x|) u^3 \, dx \right)^{1/4}, & (x,t) \in \Omega \times (0, t^*), \\ \dfrac{\partial u}{\partial \nu} + \dfrac{1}{10} u = 0, & (x,t) \in \partial\Omega \times (0, t^*), \\ u(x, 0) = \dfrac{1}{10} e^{-|x|} > 0, & x \in \Omega, \end{cases}$$

where $\Omega = \left\{ x = (x_1, x_2, x_3) \mid |x|^2 = \sum_{i=1}^3 x_i^2 < (1/10)^2 \right\}$ is the ball with radius equal to $1/10$ in $\mathbb{R}^3$. We then have

$$b(x) = 1 + |x| \,, \quad f(u) = u^{3/2} \left(\int_\Omega u^3 \, dx \right)^{1/4}, \quad \sigma = \frac{1}{10}, \quad u_0(x) = \frac{1}{10} e^{-|x|}.$$

Setting $k = 3/2$, $l = 2$, $m = 1/4$, $B = (1, 1, 1)$, it is easy to verify that (4.16), (4.17), (4.14), and the constraints on the parameters in Theorem 4.5 are satisfied. Meanwhile, q_1 is similar to Example 5.3, and $C_B = 0.92$.

Now, we select $\mu_1 = 1/6$ and $\mu_2 = 1.96$ so that $L_5 = 0$. Inserting the above parameters into L_1, L_2, L_3 and L_4, we compute

$$L_1 = 0.13, \quad L_2 = \frac{1}{12}, \quad L_3 = 3.13, \quad L_4 = 0.08.$$

Therefore, by Theorem 4.5, we obtain

$$t^* \geq T_4 = \int_{\Phi(0)}^{\infty} \frac{d\eta}{L_1 + L_2 \eta + L_3 \eta^{\frac{3}{2}} + L_4 \eta^3} = 1.94,$$

where $\Phi(0) = \int_\Omega (1 + |x|) u_0^3 \, dx = 1.78 \times 10^{-3}$.

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