Characterizations of Umbilic Points of Isometric Immersions in Riemannian and Lorentzian Manifolds

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Abstract. Several characterizations of umbilic points of submanifolds in arbitrary Riemannian and Lorentzian manifolds are given. As a consequence, we obtain new characterizations of spheres in the Euclidean space and of hyperbolic spaces in the Lorentz-Minkowski space. We also prove the Lorentzian version of a classical result by Cartan.

1. Introduction

Umbilic points and totally umbilic submanifolds are topics of interest in both Riemannian and Lorentzian settings.

Several authors have obtained different results on totally umbilic hypersurfaces of certain Riemannian manifolds. In this direction, M. Okumura in [18] got sufficient conditions for a complete constant mean curvature hypersurface of a Riemannian manifold of nonnegative constant curvature to be totally umbilic. Later, in [14] T. Hasanis gave a result of the same kind for Riemannian manifolds of dimension at least four and positive constant curvature.

In the Lorentzian setting, we can also find results providing sufficient conditions for a complete constant mean curvature spacelike hypersurface to be totally umbilic. In the de Sitter space, Goddard [13] conjectured that every complete spacelike hypersurface with constant mean curvature must be totally umbilic. Although the conjecture turned out to be false, it motivated the search for a positive answer under appropriate additional hypotheses. This is the case of the works by K. Akutagawa [3], S. Montiel [16], who solved the conjecture in the compact case, and more recently by C. P. Aquino and H. F. de Lima [5]. The last author, jointly with F. Camargo, A. Caminha and U. Parente, studied the same problem in the anti-de Sitter space, [8].

Without considering any hypothesis on the mean curvature, K. Nomizu and K. Yano in [17] characterized the connected totally umbilic submanifolds of a Euclidean space as the only submanifolds for which every circle of the submanifold is a circle of the space. More recently, in analogous terms, T. Adachi and S. Maeda in [1] characterized totally umbilic hypersurfaces in a Riemannian space form by a property of the extrinsic shape of circles on hypersurfaces. Finally, in [6,7], the authors showed dual characterizations of the sphere in the Euclidean space and the hyperbolic space in the Lorentz-Minkowski space in dimension 3 and in arbitrary dimension, making use of synthetic geometrical techniques.

In this work we deal with the notion of umbilic point of an immersed submanifold in an arbitrary Riemannian or Lorentzian manifold. We obtain a characterization of this concept by the study of the intersections of the submanifold with normal submanifolds which are totally geodesic at this point, which provides a natural extension of the notion of umbilic point of a regular surface in the three-dimensional Euclidean space. In this direction, when the submanifold has codimension one, we also get a result on the mean curvature of the hypersurface at a given point. As an application, we prove new characterizations of the sphere in the Euclidean space, as well as of the hyperbolic space in the Lorentz-Minkowski space. In particular, we get that the n-sphere with $n \geq 3$ is the only hypersurface in the Euclidean space such that its intersections at each point with 2 normal hyperplanes are $(n-1)$-spheres. Analogously, we prove that the n-hyperbolic space with $n \geq 3$ is the only spacelike hypersurface in the Lorentz-Minkowski space such that its intersections at each point with 2 normal hyperplanes are $(n-1)$-hyperbolic spaces. The first result is a wide generalization of Theorem 1 in [6].

We also prove the Lorentzian version of a classical result by Cartan. It asserts that for dimension bigger than 2 and fixed codimension r bigger than 1, the only Lorentzian manifolds for which for each point the existence of a totally geodesic r-submanifold with an arbitrary prescribed spacelike or timelike tangent space is assured, are those of constant sectional curvature.

The paper is organized as follows. In Section 2 we present the basic background and we get the Lorentzian version of Cartan's result. The main results are shown in Section 4, and the computations needed for the proofs can be found in Section 3. In the last section we apply the main results to the Euclidean space and the Lorentz-Minkowski space, in order to get characterizations of the sphere and the hyperbolic plane.

2. Preliminaries and a first result

Let $(\overline{M}^{n+m}, \overline{g})$ be a Riemannian manifold and consider an isometric immersion $x \colon (\Sigma^m, g) \to (\overline{M}, \overline{g})$. We denote by $\overline{\nabla}$ the Levi-Civita connection of $\overline{M}$ and by ∇^{Σ} the induced connection on Σ.

Hence, the Gauss formula for Σ reads as follows,

$$\overline{\nabla}_X Y = \nabla_X^\Sigma Y + \mathrm{II}_\Sigma(X, Y),$$

where II_Σ denotes the corresponding second fundamental form of Σ and X, Y are vector fields on Σ.

We define the mean curvature vector $\mathrm{H}(p)$ of the immersion at p as

$$\mathrm{H}(p) = \frac{1}{m} \operatorname{trace} \mathrm{II}_\Sigma(p).$$

If Σ is a hypersurface, we can chose (at least locally) a unitary normal vector field to the hypersurface. Then, the second fundamental form is proportional to it, and so does the mean curvature vector field, whose proportionality constant is called the mean curvature of the hypersurface.

The (immersed) submanifold is called totally geodesic at $p \in \Sigma$ if every geodesic $c_v(t)$ with initial vector $v \in T_p\Sigma$ is carried under x into a geodesic of $\overline{M}$, for all t in some neighborhood of 0, [15]. The immersion x is called totally geodesic if it is totally geodesic for all $p \in \Sigma$. Of course, if x is totally geodesic at p, then its second fundamental form at p vanishes.

Whereas totally geodesic immersions of dimension bigger than 1 and codimension bigger than 0 are the exception, there always exist (local) submanifolds which are totally geodesic at a single point $p \in \overline{M}$ with an arbitrary prescribed tangent space $T \subset T_p\overline{M}$. Indeed, choose $r > 0$ so small that the exponential map $\exp_{p|B(0,r)}$ is injective. Then, $\exp_p(B(0,r) \cap T)$ is such a submanifold. For dimension bigger than 2 and fixed codimension bigger than 1, the only manifolds for which for each point the existence of a totally geodesic submanifold with an arbitrary prescribed tangent space is assured, are those of constant sectional curvature. This result was proved by Cartan in [10].

A point p is called umbilic if its second fundamental form is equal to the metric tensor at p times a normal vector field at the point p, the mean curvature vector field at p. In the particular case of immersed surfaces of codimension 1, a point is umbilic if and only if the curvature of its normal sections coincide. The immersion x is called totally umbilic if it is umbilic for all $p \in \Sigma$.

The contents of this section work the same for spacelike submanifolds of a Lorentzian manifold, that is, isometric immersions of a Riemannian manifold into a Lorentzian one. The two paragraphs on totally geodesic submanifolds also stay true for timelike submanifolds (isometric immersions of a Lorentzian manifold into another Lorentzian manifold). There are only two things to be remarked. The first one is that it is possible to choose $r > 0$ small enough to assure that $\exp_p(B(0,r) \cap T)$ is a spacelike submanifold, if T is made up of spacelike vectors, or a timelike submanifold if T is a Lorentzian subspace. The second one is the Lorentzian version of Cartan's result, which constitutes our first result.

Before stating the result, we need some basic preliminaries on Lorentzian geometry. Let $(\overline{M}^{n+1}, \overline{g})$ be a Lorentzian manifold and $p \in \overline{M}$. $X \in T_p\overline{M} - \{0\}$ is called spacelike if $\overline{g}(X, X) > 0$, timelike if $\overline{g}(X, X) < 0$ and light-like if $\overline{g}(X, X) = 0$. An immersed submanifold of $\overline{M}$ is called non-degenerate if its induced metric is either Riemannian or Lorentzian.

As well as the Codazzi equation for a non-degenerate submanifold of $\overline{M}$, Σ,

$$(R(X, Y)Z)^{\perp} = (\nabla_X^{\perp} \mathrm{II}_{\Sigma})(Y, Z) - (\nabla_Y^{\perp} \mathrm{II}_{\Sigma})(X, Z),$$

where R is the curvature tensor of $\overline{M}$, X, Y, Z are vector fields on Σ, $\nabla^{\perp}$ is the normal connection of Σ and

$$(\nabla_X^{\perp} \mathrm{II}_{\Sigma})(Y, Z) = \nabla_X^{\perp} \mathrm{II}_{\Sigma}(Y, Z) - \mathrm{II}_{\Sigma}(\nabla_X Y, Z) - \mathrm{II}_{\Sigma}(Y, \nabla_X Z).$$

Theorem 2.1. *For dimension bigger than 2 and fixed codimension r bigger than 1, the only Lorentzian manifolds for which for each point the existence of a totally geodesic r-submanifold with an arbitrary prescribed spacelike or timelike tangent space is assured, are those of constant sectional curvature.*

Proof. The proof follows the ideas in [12]. Let $(\overline{M}^{n+1}, \overline{g})$ be a Lorentzian manifold with $2 \leq n$ and fixed $2 \leq r \leq n$ in such a way such that for each point the existence of a totally geodesic r-submanifold with an arbitrary prescribed spacelike or timelike tangent space is assured.

From Schur's lemma, see [19], it is enough to show that at each point the sectional curvature is constant on the non-degenerate planes of the tangent space.

We fix $p \in \overline{M}$ and consider orthonormal $x, y, z \in T_p\overline{M}$. There exists a totally geodesic r-submanifold such that x, y belong to its tangent plane at p and z is orthogonal to it. Since the submanifold is totally geodesic, its second fundamental form vanishes, and the Codazzi equation assures us that $\overline{g}(R(x, y)z, x) = 0$.

Let $x, y, z \in T_p\overline{M}$ be spacelike orthonormal vectors. Then

$$K(x, y) - K(x, z) = 2\overline{g}(R(x, y')z', x) = 0,$$

where $K(x, y)$ is the sectional curvature of the plane spanned by x and y, $y' = \frac{1}{\sqrt{2}}(y + z)$ and $z' = \frac{1}{\sqrt{2}}(y - z)$.

Let $x, y, z \in T_p\overline{M}$ be orthonormal vectors, such that x, y are spacelike and z is timelike. Then

$$K(x, y) - K(x, z) = \frac{2}{\sqrt{3}}\overline{g}(R(x, y')z', x) = 0,$$

where $y' = \sqrt{\frac{1}{2}}y + \sqrt{\frac{3}{2}}z$ and $z' = \sqrt{\frac{3}{2}}y + \sqrt{\frac{1}{2}}z$.

The proof ends with the following two statements.

For each two non-degenerate planes, there exists a chain of planes from one to the other, such that in pairs they span a Lorentzian 3-dimensional space.

For each pair of non-degenerate planes in a Lorentzian space of dimension 3, there exists a chain of non-degenerate planes from one to the other, such that in pairs they intersect orthogonally and both planes of each pair are spacelike, or one is spacelike and the other timelike.

To prove the first statement, it is enough to consider the case of a Lorentzian plane, Π_1, and an spacelike plane, Π_2. Take $\{u_1, u_2\}$ an orthonormal basis of Π_1 such that u_1 is timelike, u_3 a unitary vector belonging to $\{u_2\}^\perp \cap \Pi_2$ and $\{u_4, u_5\}$ an orthonormal basis of Π_2 such that $\overline{g}(u_1, u_4) = 0$. It is clear that $\Pi_3 = \text{span}\{u_1, u_3\}$ is a timelike plane. It is easy to check that $\Pi_1 + \Pi_3$ and $\Pi_2 + \Pi_3$ are non-degenerate timelike 3-dimensional spaces.

To prove the second one, for each pair of non-degenerate planes in a 3-dimensional space, Π_1 and Π_2, we consider the plane generated by their orthogonal vectors, $\Pi_{1,2}$. We need to distinguish cases. If both planes are spacelike, $\Pi_{1,2}$ is timelike, and we have finished. If both planes are timelike, $\Pi_{1,2}$ can be spacelike or degenerate, in the last case, we can take Π_4 timelike such that $\Pi_{1,4}$ and $\Pi_{2,4}$ are spacelike. If one plane is timelike and the other is spacelike, $\Pi_{1,2}$ is timelike. Applying the previous case, we finish the proof. $\square$

3. Set up

Let $(\overline{M}^{n+m}, \overline{g})$ be an $n+m$-dimensional Riemannian manifold and $x\colon \Sigma^m \to \overline{M}$ an isometrically immersed submanifold. For all $q \in \Sigma$, we can consider the normal space to Σ at q by $T_q^\perp \Sigma$. We take $s \leq m - 1$ and we denote by M an $(s+n)$-submanifold in $\overline{M}$ such that it is totally geodesic at q and $T_q^\perp \Sigma \subset T_q M$.

Consider $S_M = M \cap \Sigma$ and denote by ∇^M and ∇^{S_M} the induced connections on M and S_M, respectively.

Let X, Y be two vector fields on S_M. Taking into account that M is totally geodesic at q, we have

$$(3.1) \qquad \overline{\nabla}_X Y(q) = \nabla_X^M Y(q).$$

Moreover, the respective orthogonal projections satisfy

$$(3.2) \qquad \begin{aligned} \nabla_X^\Sigma Y(q) &= \text{Tan}_{|\Sigma}(\overline{\nabla}_X Y(q)) = \text{Tan}_{|\Sigma}(\nabla_X^M Y(q)) \\ &= \text{Tan}_{|S_M}(\nabla_X^M Y(q)) = \text{Tan}_{|S_M}(\overline{\nabla}_X Y(q)) = \nabla_X^{S_M} Y(q). \end{aligned}$$

Hence, by comparing the Gauss formula for Σ and for S_M, we deduce the following equality

$$(3.3) \qquad \text{II}_\Sigma(X, Y)(q) = \text{II}_{S_M}(X, Y)(q).$$

4. Main results

When studying regular surfaces in $\mathbb{R}^3$, we see that the usual definition of umbilic point states that a point is umbilic if the principal curvatures of the surface at the point coincide, [9], equivalently, if the curvature of all the normal sections to the surface at the point coincide. Motivated by this definition, we give the following result.

Theorem 4.1. *Let Σ be an m-dimensional submanifold immersed in a Riemannian manifold $\overline{M}^{m+n}$ with $m \geq 2$ and take $q \in \Sigma$. For a fixed $0 < s \leq m-1$, we assume that for each $(s+n)$-submanifold M such that it is totally geodesic at q and $T_q^\perp \Sigma \subset T_q M$, we get the same mean curvature vector at q as that of the isometrically immersed s-submanifold $S_M := M \cap \Sigma$. Then, the point $q \in \Sigma$ is umbilic.*

Proof. For any two arbitrary unitary vectors $u, u' \in T_q\Sigma$, take $u_2, \ldots, u_s \in T_q\Sigma$ such that $\{u, u_2, \ldots, u_s\}$ and $\{u', u_2, \ldots, u_s\}$ are orthonormal sets, and consider the totally umbilic submanifolds at q spanned by them and $T_q^\perp \Sigma$, M and M' respectively.

Our hypothesis guaranty that

$$\mathrm{II}_{S_M}(u, u) + \sum \mathrm{II}_{S_M}(u_i, u_i) = \mathrm{II}_{S_{M'}}(u', u') + \sum \mathrm{II}_{S_{M'}}(u_i, u_i),$$

and as a direct consequence of (3.3) we obtain $\mathrm{II}_\Sigma(u, u) = \mathrm{II}_\Sigma(u', u')$. $\square$

Note that for $m = 2$ and codimension 1, Theorem 4.1 holds trivially. In this case, the curves S_M must be considered as 1-dimensional submanifolds immersed in Σ.

When we focus on the case of codimension 1, thanks to (3.1) and (3.2), by using the Gauss formula for S_M in M, we can deduce that

$$(4.1) \qquad \overline{g}(D_M(X), Y)(q) = \overline{g}(AX, Y)(q),$$

for any two tangent vector fields of S_M, X and Y, where A is the Weingarten operator of Σ in $\overline{M}$ and D_M is the Weingarten operator of S_M in M.

Now fix $0 < s \leq m-1$ and take the orthonormal basis $\{e_1, \ldots, e_m\}$ of $T_q\Sigma$ consisting of the eigenvectors of the Weingarten operator of Σ. We define the submanifolds $\{M_\alpha\}_\alpha$ as the totally geodesic at q submanifolds spanned by the unitary normal vector field to the hypersurface Σ at q, $N(q)$, and any subset of $\{e_1, \ldots, e_m\}$ with cardinal s. In this setting, Theorem 4.1 reads as follows.

Corollary 4.2. *Let Σ be a hypersurface of a Riemannian manifold $\overline{M}$ with dimension at least 3, $q \in \Sigma$ and suppose that the mean curvature of S_{M_α} at q in M_α does not depend on α. Then, the point $q \in \Sigma$ is umbilic and its mean curvature is that of each S_{M_α}.*

Note that this last criterion requires its checking on a finite number of submanifolds, contrary to Theorem 4.1, which requires a fine behavior for all hypersurface M satisfying the conditions imposed.

Remark 4.3. The previous corollary does not hold if we change the basis of eigenvectors by any orthonormal basis. Indeed, take a surface of $\mathbb{R}^3$ admitting a zero mean curvature non umbilic point, for instance the hyperbolic paraboloid. The asymptotic directions at this point give us an orthonormal basis for which the normal sections have zero curvature at the point. On the contrary, the thesis concerning the mean curvature of the point remains true.

Thanks to (4.1), a generalization of the last statement of the previous remark can be given, which constitutes a natural characterization of the mean curvature of a hypersurface at a point.

Corollary 4.4. *Let Σ be a hypersurface of a Riemannian manifold $\overline{M}$ with dimension at least 3, take $q \in \Sigma$ and an arbitrary orthonormal basis $\{v_1, \ldots, v_m\}$ of $T_q\Sigma$. For a fixed $0 < s \leq m - 1$, consider the submanifolds $\{M_\alpha\}_\alpha$ which are totally geodesic at q and are spanned by $N(q)$ and any subset of $\{v_1, \ldots, v_m\}$ with cardinal s. Then, the mean curvature of Σ at q is the mean of the mean curvature of the hypersurfaces $S_{M_\alpha} = M_\alpha \cap \Sigma$ of M_α at q.*

On the other hand, the converse of Theorem 4.1 holds.

Theorem 4.5. *Let Σ be an m-dimensional submanifold of the Riemannian manifold $\overline{M}^{m+n}$ with $m \geq 2$ and let $q \in \Sigma$ be a umbilic point. Fix $0 < s \leq m - 1$. Then, the mean curvature vector field at q of the submanifold $\Sigma \cap M$ coincides with the mean curvature vector field of Σ at q, for each $(s+n)$-submanifold M such that it is totally geodesic at q and $T_q^\perp \Sigma \subset T_q M$.*

Proof. The proof follows from (3.3) and the definition of the mean curvature vector. $\square$

As well as the converse of Corollary 4.2.

Corollary 4.6. *Let Σ be a hypersurface of the Riemannian manifold $\overline{M}$ with dimension at least 3 and let $q \in \Sigma$ be a umbilic point. Then, the mean curvature of S_{M_α} at q in M_α coincides with the mean curvature of Σ at q, for all M_α.*

And finally, as a consequence of (3.3), we get two more results. They are the first one for submanifolds and the second one for hypersurfaces.

Theorem 4.7. *Let Σ be an m-dimensional submanifold of a Riemannian manifold $\overline{M}^{m+n}$ with $m \geq 3$ and take $q \in \Sigma$. Fix $2 \leq s \leq m - 1$ and assume that for each $(s+n)$-submanifold M such that it is totally geodesic at q and $T_q^\perp \Sigma \subset T_q M$, the point q is umbilic*

in $S_M := M \cap \Sigma$ as a submanifold of $\overline{M}$ (equivalently as submanifold of M). Then, the point $q \in \Sigma$ is umbilic.

Proof. Consider equation (3.3) for a submanifold S_M in $\overline{M}$. Then

$$\mathrm{II}_\Sigma(X, Y)(q) = \mathrm{II}_{S_M}(X, Y)(q),$$

where $X, Y \in \mathfrak{X}(S_M)$. If we suppose that the point $q \in S_M$ is umbilic, then

$$\mathrm{II}_{S_M}(X, Y)(q) = \overline{g}(X, Y)(q)\,\mathrm{H}(q),$$

where $\mathrm{H}(q)$ denotes the mean curvature vector field of S_M at q.

Thanks to (3.3), $\mathrm{H}(q)$ does not depend on M, and so, q is also umbilic in Σ. $\qquad\square$

In fact, the hypothesis of the previous theorem can be weakened.

Remark 4.8. Consider an arbitrary basis $\{w_1, \ldots, w_m\}$ for $T_q\Sigma$ and take the submanifolds $\{M_\alpha\}_\alpha$ which are totally geodesic at q and are spanned by $T_q^\perp\Sigma$ and any subset of $\{w_1, \ldots, w_m\}$ with cardinal $s \geq 2$. If we suppose that the point q is umbilic in S_{M_α} for all α, then q is also umbilic in Σ.

In the particular case of hypersurfaces cut by hypersurfaces, only two cuts are needed.

Theorem 4.9. *Let Σ be a hypersurface of a Riemannian manifold $\overline{M}^{m+1}$ with $m \geq 3$ and take $q \in \Sigma$. If there exist two different normal hypersurfaces, M_1 and M_2, totally geodesic at q and such that q is umbilic in $S_{M_i} := M_i \cap \Sigma$ as a submanifold of $\overline{M}$ (equivalently as submanifold of M_i) for $i = 1, 2$, then the point $q \in \Sigma$ is umbilic.*

Observe that the converse of Theorem 4.7 and that of Theorem 4.9 hold trivially.

4.1. Spacelike submanifolds in Lorentzian manifolds

Suppose now that $(\overline{M}^{n+m}, \overline{g})$ is a Lorentzian manifold and Σ^m is a spacelike submanifold of $\overline{M}$. Let $q \in \Sigma$ be an arbitrary point and let $T \subset T_q\Sigma$ be a subspace of $T_q\Sigma$. Making use of the exponential map, we can find a Lorentzian submanifold M in $\overline{M}$ through q, with $T_q\Sigma = \mathrm{span}\left(T \cup T_q^\perp\Sigma\right)$ such that M is totally geodesic at q. Thus, the submanifold $S_M = M \cap \Sigma$ is spacelike.

In this setting, the set up and all the results of this section remain true.

5. Several consequences

5.1. For submanifolds of the Euclidean space

In the sequel, we say that an affine subspace of the Euclidean space through a point is normal to a submanifold at that point, if it contains the normal subspace to the submanifold at the point.

A first consequence is obtained from Theorem 4.7 and Remark 4.8.

Corollary 5.1. *Let Σ be an m-dimensional submanifold of $\mathbb{R}^{m+n}$ ($m \geq 3$), $q \in \Sigma$ and $2 \leq s \leq m-1$ fixed. Consider an arbitrary basis of $T_q\Sigma$ and the family consisting of all the $(s+n)$-dimensional normal affine subspaces through q spanned by the vectors of the basis, $\{\Pi_i\}_{i \in I}$. If $q \in \Sigma \cap \Pi_i$ is umbilic in Π_i for all $i \in I$, then q is a umbilic point of Σ.*

In particular, we can state the following characterization.

Corollary 5.2. *Let Σ be an m-dimensional submanifold of $\mathbb{R}^{m+n}$ ($m \geq 3$), $q \in \Sigma$ and $2 \leq s \leq m-1$ fixed. Consider an arbitrary basis of $T_q\Sigma$ and the family consisting of all the $(s+n)$-dimensional normal affine subspaces through q spanned by the vectors of the basis, $\{\Pi_i\}_{i \in I}$. If for each $i \in I$ the intersection $\Sigma \cap \Pi_i$ is a piece of an s-sphere, then q is a umbilic point of Σ.*

We deal now with totally umbilic submanifolds. An m-dimensional non-totally geodesic, totally umbilic submanifold of a Euclidean $(m+n)$-space is contained in a sphere of an affine $(m+1)$-subspace, [11]. Hence, from now on we can focus on hypersurfaces.

As a consequence of Remark 4.8 we can also enunciate a characterization of the sphere in the Euclidean space of dimension at least four.

Corollary 5.3. *The sphere $\mathbb{S}^m$ is the only connected and complete hypersurface of $\mathbb{R}^{m+1}$ ($m \geq 3$) such that, for some $2 \leq s \leq m$ fixed, its intersection at each point q with the $(s+1)$-dimensional normal affine subspaces through q spanned by an arbitrary basis of the tangent space at q, are pieces of spheres.*

In the previous result, the completeness can be omitted if we ask the intersections to be spheres. In the particular case of hypersurfaces, Theorem 4.9 gives us the next result, in which we have not included the completeness as an hypothesis.

Corollary 5.4. *The only connected hypersurface Σ in $\mathbb{R}^{m+1}$ ($m \geq 3$) such that at each point $q \in \Sigma$ its intersection with 2 normal hyperplanes are spheres, is the sphere $\mathbb{S}^m$.*

The connectedness can be omitted as an hypothesis. Assume Σ is not connected. Then we can apply the previous result to each connected component. We get that Σ is made up of spheres. Then, there are points of Σ such that for any normal hypersurface at the point, the intersection with Σ is not an sphere, but many. This is a contradiction.

We get a result that constitutes a wide generalization of Theorem 1 of [6].

Corollary 5.5. *The only hypersurface Σ in $\mathbb{R}^{m+1}$ ($m \geq 3$), such that at each point $q \in \Sigma$ its intersection with 2 normal hyperplanes are spheres, is the sphere $\mathbb{S}^m$.*

Making use of Alexandrov's theorem [4] and Corollary 4.4, we can include the case $m = 2$ in the following result.

Corollary 5.6. *The only compact and connected hypersurface Σ in $\mathbb{R}^{m+1}$ such that its intersection at each point $q \in \Sigma$ with 2 orthonormal normal hyperplanes (i.e., its normal vectors at q are orthonormal) are pieces of spheres with radius r, is a sphere with radius r.*

5.2. For submanifolds of the Lorentz-Minkowski space

The second part of the section is devoted to the Lorentz-Minkowski space. Taking into account Section 5.1, we obtain results for spacelike submanifolds analogous to those in the Euclidean space.

Corollary 5.7. *Let Σ be an m-dimensional spacelike submanifold of $\mathbb{L}^{m+n}$ $(m \geq 3)$, $q \in \Sigma$ and $2 \leq s \leq m - 1$ fixed. Consider an arbitrary basis of $T_q\Sigma$ and the family consisting of all the $(s+n)$-dimensional normal affine subspaces through q spanned by the vectors of the basis, $\{\Pi_i\}_{i \in I}$. If $q \in \Sigma \cap \Pi_i$ is umbilic in Π_i for all $i \in I$, then q is a umbilic point of Σ.*

In particular, we can state the following characterization.

Corollary 5.8. *Let Σ be an m-dimensional spacelike submanifold of $\mathbb{L}^{m+n}$ $(m \geq 3)$, $q \in \Sigma$ and $2 \leq s \leq m - 1$ fixed. Consider an arbitrary basis of $T_q\Sigma$ and the family consisting of all the $(s+n)$-dimensional normal affine subspaces through q spanned by the vectors of the basis, $\{\Pi_i\}_{i \in I}$. If for each $i \in I$ the intersection $\Sigma \cap \Pi_i$ is a piece of an s-hyperbolic plane, then q is a umbilic point of Σ.*

When studying totally umbilic submanifolds, it is possible to focus on the case of hypersurfaces. The reason is that an m-dimensional spacelike non-totally geodesic, totally umbilic submanifold of the Lorentz-Minkowski $(m + n)$-space is either contained in a hyperbolic space of an affine timelike $(m+1)$-subspace or in a sphere of an affine spacelike $(m + 1)$-subspace, [2].

Corollary 5.9. *The hyperbolic space $\mathbb{H}^m$ is the only connected and complete spacelike hypersurface of $\mathbb{L}^{m+1}$ $(m \geq 3)$ such that, for some $2 \leq s \leq m$ fixed, its intersection at each point q with the $(s+1)$-dimensional normal affine subspaces through q spanned by an arbitrary basis of the tangent space at q, are pieces of hyperbolic spaces.*

Reasoning as in the Euclidean case, we get the following result.

Corollary 5.10. *The only spacelike hypersurface Σ in $\mathbb{L}^{m+1}$ $(m \geq 3)$ such that its intersection at each point $q \in \Sigma$ with two normal hyperplanes are $(m-1)$-hyperbolic spaces, is the hyperbolic m-space.*

Observe that if we consider the standard realization of the hyperbolic space $\mathbb{H}^n(1)$ given by the upper component of the hyperboloid of two sheets, as a direct consequence of our result it is clear that the intersection of any vectorial timelike hyperplane with $\mathbb{H}^n(1)$, must be a $(n-1)$-hyperbolic space, with mean and sectional curvature equal to 1, i.e., a copy of $\mathbb{H}^{n-1}(1)$ embedded in $\mathbb{L}^{n+1}$ and totally umbilic.

Acknowledgments

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