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Abelian Category of Cominimax and Weakly Cofinite Modules

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Abstract. Let R be a commutative Noetherian ring, I an ideal of R and M an arbitrary R-module. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. It is shown that the R-module $\operatorname{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $i \geq 0$, if and only if the R-module $\operatorname{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $0 \leq i \leq \operatorname{ara}(I)$. As an immediate consequence, we prove that if R is a Noetherian (resp. $(R, \mathfrak{m})$ is a Noetherian local) ring of dimension d, then the R-module $\operatorname{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $i \geq 0$ if and only if the R-module $\operatorname{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $0 \leq i \leq d+1$ (resp. for all $0 \leq i \leq d$). Also it is shown that if I is a principal ideal up to radical, then the category of I-cominimax (resp. I-weakly cofinite) modules is an Abelian full subcategory of the category of R-modules.

1. Introduction

Throughout this paper R is a commutative Noetherian ring with non-zero identity and I an ideal of R. Hartshorne in [8] defined a module M to be I-*cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$. He asked:

Question 1.1. Whether the category $\mathscr{M}(R, I)_{\mathrm{cof}}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules? That is, if $f \colon M \to N$ is an R-homomorphism of I-cofinite modules, are $\operatorname{Ker} f$ and $\operatorname{Coker} f$ I-cofinite?

With respect to this question, Hartshorne with an example showed that this is not true in general. However, he proved that if I is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is yes. In [5], Delfino and Marley extended this result to arbitrary complete local rings. Recently, Kawasaki [11], by using a spectral sequence argument, generalized the Delfino and Marley's result for an arbitrary ideal I of dimension one in a local ring R. Finally, in [15] it is shown that Hartshorne's question is true for all ideals of dimension one of any arbitrary Noetherian ring R. Also Melkersson in [14] (resp. Kawasaki in [10, Theorem 2.1]), proved that the Hartshorne's

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question is true for all Noetherian rings with dimension at most 2 (resp. for all principal ideals up to radical).

Recall that an R-module M is a *minimax* module if there exists a finitely generated submodule N of M such that the quotient module M/N is Artinian. Minimax modules have been studied in [17]. Recall too that an R-module M is called *weakly Laskerian* if $\mathrm{Ass}_R(M/N)$ is a finite set for each submodule N of M. The category of weakly Laskerian modules introduced in [6]. Note that these two class of R-modules are *Serre subcategory* of R-modules, in other words, they are closed under taking submodules, quotients and extensions. Let $\mathcal{S}$ be a Serre subcategory of R modules and I an ideal of R. As a generalization of I-cofinite modules in [1], the authors, introduced the concept of cofinite modules with respect to I and $\mathcal{S}$ or $(I, \mathcal{S})$-*cofinite* modules. An R-module M is $(I, \mathcal{S})$-cofinite module if $\mathrm{Supp}_R(M) \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$ for all $i \geq 0$. Note that when $\mathcal{S}$ is the category of minimax module (resp. weakly Laskerian) R-module, it is the same as I-cominimax (resp. I-weakly cofinite) modules, see also [2] and [7]. In this paper with a different method of proof from Kawasaki [10, Theorem 2.1] and using Koszul complex, when I is a principal ideal up to radical, we prove that for each $\mathcal{S}$ as a full Serre subcategory of R-modules, the category of $(I, \mathcal{S})$-cofinite modules is a full Abelian subcategory of R-modules. In particular the category of I-cominimax (resp. I-weakly cofinite) modules has the same property. More precisely we prove the following theorem.

Theorem 1.2. *Let R be a Noetherian ring, I be an ideal of R and M be a non-zero R-module. Let $\mathcal{S}$ be a Serre subcategory of R-modules. Then the following conditions are equivalent:*

(i) *The R-module $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $i \geq 0$.*

(ii) *The R-module $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $0 \leq i \leq \mathrm{ara}(I)$.*

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. We denote $\{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supseteq I\}$ by $V(I)$. The radical of I, denoted by $\mathrm{Rad}(I)$, is defined to be the set $\{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer the reader to [4] and [12].

2. Main results

We begin with a useful lemma.

Lemma 2.1. *Let R be a Noetherian ring, I be an ideal of R and $\mathcal{S}$ be a Serre subcategory of R-modules. Then for any R-modules T and any integer $k \geq 0$, the following conditions are equivalent:*

(i) $\operatorname{Ext}^n_R(R/I, T)$ *belongs to* $\mathcal{S}$ *for all* $0 \le n \le k$.

(ii) $\operatorname{Ext}^n_R(N, T)$ *belongs to* $\mathcal{S}$ *for all* $0 \le n \le k$ *and for any finitely generated* R*-module* N *for which* $\operatorname{Supp}_R(N) \subseteq V(I)$.

Proof. It follows from the method of the proof of [9, Lemma 1]. □

The following lemma is a generalization of [15, Theorem 2.1] in the sense of Serre subcategory of the category of R-modules.

Lemma 2.2. *Let* R *be a Noetherian ring and* $I = (x_1, \ldots, x_n)$ *be an ideal of* R *and let* M *be an* R*-module. Let* $\mathcal{S}$ *be a Serre subcategory of the category of* R*-modules. Then the following statements are equivalent:*

(i) *The* R*-module* $\operatorname{Ext}^i_R(R/I, M)$ *belongs to* $\mathcal{S}$*, for all integers* $i \ge 0$,

(ii) *The* R*-module* $\operatorname{Tor}^R_i(R/I, M)$ *belongs to* $\mathcal{S}$*, for all integers* $i \ge 0$,

(iii) *The Koszul cohomology module* $\mathrm{H}^i(x_1, \ldots, x_n; M)$ *belongs to* $\mathcal{S}$*, for all integers* $i = 0, \ldots, n$.

Proof. This lemma follows from the method of the proof of [16, Theorem 2]. □

The next remark is needed in the proof of the next lemma.

Remark 2.3. Let I be an ideal of R and $\mathcal{S}$ a full Serre subcategory of R-modules. Let M be a finitely generated R-module and N belong to $\mathcal{S}$. As R is Noetherian and M is finitely generated, it follows that M possesses a free resolution

$$\mathbb{F}_\bullet\colon \cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

whose free modules have finite ranks. Thus $\operatorname{Ext}^i_R(M, N) = \mathrm{H}^i(\operatorname{Hom}_R(\mathbb{F}_\bullet, N))$ is a subquotient of a direct sum of finitely many copies of N. Therefore, since $\mathcal{S}$ is full Serre subcategory of R-modules, it follows that $\operatorname{Ext}^i_R(M, N)$ belongs to $\mathcal{S}$ for all $i \ge 0$.

The next results are of assistance in the proof of the main theorems in this paper.

Lemma 2.4. *Let* R *be a Noetherian ring,* $I := Rx_1 + \cdots + Rx_n$ $(n \ge 1)$ *be an ideal of* R *and* M *be a non-zero* R*-module. Then for each* $\mathcal{S}$ *as a full Serre subcategory of* R*-modules, the following conditions are equivalent:*

(i) *The* R*-module* $\operatorname{Ext}^i_R(R/I, M)$ *belongs to* $\mathcal{S}$*, for all* $i \ge 0$.

(ii) *The* R*-module* $\operatorname{Ext}^i_R(R/I, M)$ *belongs to* $\mathcal{S}$*, for all* $0 \le i \le n$.

Proof. (i) $\Rightarrow$ (ii): It's clear.

(ii) $\Rightarrow$ (i): Let

$$K^{\bullet}(\underline{x}, M)\colon 0 \longrightarrow M \xrightarrow{f_0} \bigoplus_{k=1}^{C_n^1} M \xrightarrow{f_1} \bigoplus_{k=1}^{C_n^2} M \longrightarrow \cdots \longrightarrow \bigoplus_{k=1}^{C_n^{n-1}} M \xrightarrow{f_{n-1}} M \longrightarrow 0$$

be the Koszul complex of M with respect to $\underline{x} = x_1, \ldots, x_n$. Then by the definition we have

$$\mathrm{H}^0(\underline{x}; M) = \mathrm{Ker}(f_0) = 0 :_M I \cong \mathrm{Hom}_R(R/I, M)$$

and so it follows from the hypothesis that the R-module $\mathrm{H}^0(\underline{x}; M)$ belongs to $\mathcal{S}$. Consider the following exact sequence

$$0 \longrightarrow \mathrm{Ker}(f_0) \longrightarrow M \longrightarrow \mathrm{Im}(f_0) \longrightarrow 0.$$

Using the hypothesis and Remark 2.3, it follows from this exact sequence that the R-module $\mathrm{Ext}^i_R(R/I, \mathrm{Im}(f_0))$ belongs to $\mathcal{S}$, for each $0 \leq i \leq n$. Now the following exact sequence

$$0 \longrightarrow \mathrm{Im}(f_0) \longrightarrow \mathrm{Ker}(f_1) \longrightarrow \mathrm{H}^1(\underline{x}; M) \longrightarrow 0 \tag{2.1}$$

induces the exact sequence

$$\mathrm{Hom}_R(R/I, \mathrm{Ker}(f_1)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}^1(\underline{x}; M)) \longrightarrow \mathrm{Ext}^1_R(R/I, \mathrm{Im}(f_0)). \tag{2.2}$$

Now as by hypothesis the R-module $\mathrm{Hom}_R(R/I, M)$ belongs to $\mathcal{S}$, it is easy to see that the R-module $\mathrm{Hom}_R(R/I, \mathrm{Ker}(f_1))$ also belongs to $\mathcal{S}$. Therefore, the exact sequence (2.2) implies that the R-module $\mathrm{Hom}_R(R/I, \mathrm{H}^1(\underline{x}; M))$ also belongs to $\mathcal{S}$, (note that $n \geq 1$). By the definition of Koszul complex $I\,\mathrm{H}^1(\underline{x}; M) = 0$. Consequently, the R-module $\mathrm{H}^1(\underline{x}; M) = 0 :_{\mathrm{H}^1(\underline{x}; M)} I$ belongs to $\mathcal{S}$. Now it follows from exact sequence (2.1) that the R-module $\mathrm{Ext}^i_R(R/I, \mathrm{Ker}(f_1))$ also belongs to $\mathcal{S}$ for all $0 \leq i \leq n$. Now the following exact sequence

$$0 \longrightarrow \mathrm{Ker}(f_1) \longrightarrow \bigoplus_{k=1}^{C_n^1} M \longrightarrow \mathrm{Im}(f_1) \longrightarrow 0$$

implies that the R-module $\mathrm{Ext}^i_R(R/I, \mathrm{Im}(f_1))$ belongs to $\mathcal{S}$ for each $0 \leq i \leq n-1$. So proceeding in the same way we can see the Koszul cohomoloy modules $\mathrm{H}^i(\underline{x}; M)$ belong to $\mathcal{S}$ for all $0 \leq i \leq n$. Now the assertion follows from Lemma 2.2. □

Now we are ready to state and prove the first main result of this paper.

Before bringing this main result, recall that, for any proper ideal I of R, the *arithmetic rank* of I, denoted by $\mathrm{ara}(I)$, is the least number of elements of I required to generate an ideal which has the same radical as I, i.e.,

$$\mathrm{ara}(I) := \min\left\{n \geq 0 : \exists\, x_1, \ldots, x_n \in I \text{ with } \mathrm{Rad}((x_1, \ldots, x_n)) = \mathrm{Rad}(I)\right\}.$$

Theorem 2.5. *Let R be a Noetherian ring, I be an ideal of R and M be a non-zero R-module. Let $\mathcal{S}$ be a Serre subcategory of R-modules. Then the following conditions are equivalent:*

(i) *The R-module $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $i \geq 0$.*

(ii) *The R-module $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $0 \leq i \leq \mathrm{ara}(I)$.*

Proof. The assertion follows immediately from Lemmas 2.4 and 2.1. □

Corollary 2.6. *Let R be a Noetherian (resp. $(R, \mathfrak{m})$ be a Noetherian local) ring of dimension d, I be an ideal of R and M be a non-zero R-module. Then for each $\mathcal{S}$ as a full Serre subcategory of R-modules, the following statements are equivalent:*

(i) *The R-module $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $i \geq 0$.*

(ii) *The R-module $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $0 \leq i \leq d+1$ (resp. for all $0 \leq i \leq d$).*

Proof. The assertion follows immediately from Theorem 2.5 and [13, Corollaries 2.7 and 2.8]. □

Corollary 2.7. *Let R be a Noetherian ring, $I := Rx_1 + \cdots + Rx_n$ $(n \geq 1)$ be an ideal of R and M be a non-zero R-module with support in $V(I)$. Then for each $\mathcal{S}$ as a full Serre subcategory of R-modules, the following conditions are equivalent:*

(i) *M is $(I, \mathcal{S})$-cofinite.*

(ii) *The R-module $\mathrm{Ext}^i_R(R/I, M)$ belongs to $\mathcal{S}$, for all $0 \leq i \leq n$.*

Proof. The assertion follows from Lemma 2.4. □

Theorem 2.8. *Let I be an ideal of a Noetherian ring R such that $\mathrm{ara}(I) = 1$. Let $\mathscr{M}(R, I, \mathcal{S})_{\mathrm{cof}}$ denote the category of $(I, \mathcal{S})$-cofinite R-modules. Then $\mathscr{M}(R, I, \mathcal{S})_{\mathrm{cof}}$ is an Abelian category.*

Proof. Let $M, N \in \mathscr{M}(R, I, \mathcal{S})_{\mathrm{cof}}$ and let $f \colon M \to N$ be an R-homomorphism. It is enough to show that the R-modules $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are $(I, \mathcal{S})$-cofinite.

To this end, the exact sequence

$$0 \longrightarrow \mathrm{Ker}\, f \longrightarrow M \longrightarrow \mathrm{Im}\, f \longrightarrow 0$$

induces an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{Ker}\, f) \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{Im}\, f) \\ \longrightarrow \mathrm{Ext}^1_R(R/I, \mathrm{Ker}\, f) \longrightarrow \mathrm{Ext}^1_R(R/I, M) \end{aligned}$$

that implies the R-modules $\mathrm{Hom}_R(R/I, \mathrm{Ker}\, f)$ and $\mathrm{Ext}^1_R(R/I, \mathrm{Ker}\, f)$ are finitely generated. Therefore it follows from Theorem 2.5 that $\mathrm{Ker}\, f$ is $(I, \mathcal{S})$-cofinite. Now, the assertion follows from the following exact sequences

$$0 \longrightarrow \mathrm{Ker}\, f \longrightarrow M \longrightarrow \mathrm{Im}\, f \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Im}\, f \longrightarrow N \longrightarrow \mathrm{Coker}\, f \longrightarrow 0. \qquad \square$$

Kawasaki in [10, Theorem 2.1] proved the following corollary by using Noetherian property but our method of proof is quite different and use Koszul cohomology.

Corollary 2.9. *Let I be an ideal of a Noetherian ring R such that $\mathrm{ara}(I) = 1$. Let $\mathscr{M}(R, I)_{\mathrm{cof}}$ denote the category of I-cofinite R-modules. Then $\mathscr{M}(R, I)_{\mathrm{cof}}$ is an Abelian category.*

The following corollary is our last main result in this paper.

Corollary 2.10. *Let I be an ideal of a Noetherian ring R such that $\mathrm{ara}(I) = 1$. Let $\mathscr{M}(R, I)_{\mathrm{comin}}$ (resp. $\mathscr{M}(R, I)_{\mathrm{wcof}}$) denote the category of I-cominimax (resp. the category of I-weakly cofinite) R-modules. Then $\mathscr{M}(R, I)_{\mathrm{comin}}$ (resp. $\mathscr{M}(R, I)_{\mathrm{wcof}}$) is an Abelian category.*

Corollary 2.11. *Let R be a Noetherian ring and I a proper ideal of R. Let M be a non-zero I-cominimax (resp. I-weakly cofinite) R-module. Then, the R-modules $\mathrm{Ext}^i_R(N, M)$ and $\mathrm{Tor}^R_i(N, M)$ are I-cominimax (resp. I-weakly cofinite) R-modules, for all finitely generated R-modules N and all integers $i \geq 0$.*

Proof. Since N is finitely generated, it follows that N has a free resolution of finitely generated free modules. Now the assertion follows using Theorem 2.8 and computing the modules $\mathrm{Ext}^i_R(N, M)$ and $\mathrm{Tor}^R_i(N, M)$, by this free resolution. $\square$

If $\mathrm{ara}(I) = 1$ then $\mathrm{cd}\, I = 1$ but the converse is not true in general. We close this paper by offering a question and problem for further research. The following question is at present far from being solved.

Question 2.12. Let R be a commutative Noetherian ring with non-zero identity and I an ideal of R with $\mathrm{cd}\, I = 1$. Is $\mathscr{M}(R, I)_{\mathrm{cof}}$ an Abelian full subcategory of R-modules?

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