TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 5, pp. 1139–1147, October 2016
DOI: 10.11650/tjm.20.2016.7154
This paper is available online at http://journal.tms.org.tw

Approximate Cyclic Amenability of T-Lau Product of Banach Algebras

Hossein Javanshiri

Abstract. Associated with two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ and a norm decreasing homomorphism $T : \mathcal{B} \to \mathcal{A}$, there is a certain Banach algebra product $\mathcal{M}_T := \mathcal{A} \times_T \mathcal{B}$, which is a splitting extension of $\mathcal{B}$ by $\mathcal{A}$. In this paper, we investigate approximate cyclic amenability of $\mathcal{M}_T$ which has been introduced and studied by Esslamzadeh and Shojaee in [5]. In particular, apart from the characterization of all cyclic derivations on the Banach algebra $\mathcal{M}_T$, we improve the results of [1, 2] for cyclic amenability of $\mathcal{M}_T$. These results paves the way for obtaining new results for (approximate) cyclic amenability of the Banach algebra $\mathcal{A} \times \mathcal{B}$ equipped with the coordinatewise product algebra and ℓ^1-norm.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary Banach algebras and let $T : \mathcal{B} \to \mathcal{A}$ be an algebra homomorphism with $\|T\| \leq 1$. Then the *T-Lau product* $\mathcal{M}_T := \mathcal{A} \times_T \mathcal{B}$ is the Cartesian product $\mathcal{A} \times \mathcal{B}$ equipped with the multiplication

$$(a, b)(a', b') = \big(aa' + aT(b') + T(b)a', bb'\big)$$

and the norm $\|(a, b)\| = \|a\| + \|b\|$. Then $\mathcal{A} \times_T \mathcal{B}$ is a Banach algebra which contains $\mathcal{A}$ as a closed two-sided ideal. The notation $\mathcal{M}_0$ is used to denote the case where T is zero homomorphism and it is nothing more than the coordinatewise product on $\mathcal{A} \times \mathcal{B}$. Moreover, it should be noticed that the results where presented in this article, can be adapted for $\mathcal{M}_0$ as a new result.

The Banach algebra $\mathcal{M}_T$ was first introduced and studied by Bhatt and Dabhi [2] for the case where $\mathcal{A}$ is a commutative Banach algebra and it is sometimes referred to *Lau products* because in [8] it has been introduced for the special case where $\mathcal{B}$ is the predual of a van Neumann algebra $\mathcal{M}$ for which the identity of $\mathcal{M}$ is a multiplicative linear functional on $\mathcal{B}$. The general definition of these Banach algebras are due to the present author and Nemati [7, 9]. Many basic properties of these Banach algebras such as

Received December 21, 2015; Accepted May 3, 2016.
Communicated by Xiang Fang.
2010 *Mathematics Subject Classification*. Primary: 46H05, 46H25; Secondary: 46H99.
Key words and phrases. Banach algebra, T-Lau product, Cyclic derivation, Approximate cyclic amenability.

the existence of a bounded approximate identity, spectrum, Arens regularity, topological centre, the ideal structure, biflatness, biprojectivity, amenability, approximate amenability and pseudo amenability are investigated in [7, 9].

Recently a notion of amenability has been defined and studied on Banach algebras by Esslamzadeh and Shojaee [5]. In this paper, we are going to investigate this concept on $\mathcal{M}_T$ and their relations with $\mathcal{A}$ and $\mathcal{B}$. The reader will remark that the maps

$$\begin{aligned} q_{\mathcal{B}}\colon \mathcal{M}_T \to \mathcal{B} &\qquad (a,b) \mapsto b, \\ q_{\mathcal{A}}\colon \mathcal{M}_T \to \mathcal{A} &\qquad (a,b) \mapsto a + T(b), \\ \Phi\colon \mathcal{M}_0 \to \mathcal{M}_T &\qquad (a,b) \mapsto (a + T(b), b), \end{aligned}$$

are continuous algebra epimorphisms (see [10, page 8] and [3]). From this, we can deduce that $\mathcal{M}_T$ is approximately cyclic amenable if and only if $\mathcal{M}_0$ is approximately cyclic amenable. We should however note that the only reference which investigated the (approximately) cyclic amenability of $\mathcal{M}_0$ in relations to the corresponding ones of $\mathcal{A}$ and $\mathcal{B}$ is [11], where it was shown that if $\mathcal{M}_0$ is approximately cyclic amenable, then so are $\mathcal{A}$ and $\mathcal{B}$ and the converse holds if $\mathcal{A}$ is commutative and $\overline{\mathcal{A}^2} = \mathcal{A}$. Moreover, we would like to mention the result [2, Theorem 4.1, Parts (4) and (5)] by Bhatt and Dabhi which presents an attempt to prove that if $\mathcal{A}$ is commutative, then $\mathcal{M}_T$ is (approximately) cyclic amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ also are. However, the proof contains a gap. Later on, Abtahi and Ghafarpanah [1] investigated that result and correct a gap in the proof but with an extra assumption. In details, they showed that if $\mathcal{M}_T$ is cyclic amenable, then $\mathcal{A}$ and $\mathcal{B}$ are also, and the converse holds when $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras with faithful dual spaces. They used these conditions to complete their proof, see [1, pp. 4–5], whereas it is not shown that the imposed conditions on $\mathcal{A}$ and $\mathcal{B}$ are necessary.

In this paper, we use a brief technique to obtain a more general result. Indeed, we remove the assumption that $\mathcal{B}$ has faithful dual space from [1, Theorem 3.2]. Furthermore, we replace the assumption that $\mathcal{A}$ has faithful dual space with its equivalent condition that $\overline{\mathcal{A}^2} = \mathcal{A}$. To this end, we give a characterization for cyclic derivations on the Banach algebras $\mathcal{A} \times_T \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary Banach algebras and T is a norm decreasing homomorphism from $\mathcal{B}$ into $\mathcal{A}$. We then show that if $\mathcal{A} \times_T \mathcal{B}$ is (approximately) cyclic amenable, then $\mathcal{A}$ and $\mathcal{B}$ are (approximately) cyclic amenable and the converse holds if $\overline{\mathcal{A}^2} = \mathcal{A}$ and in particular, we present an example to show that the condition $\overline{\mathcal{A}^2} = \mathcal{A}$ is necessary. Moreover, we show that $\overline{\mathcal{A}^2} = \mathcal{A}$ if and only if $\mathcal{A}$ has a faithful dual space. Also, our results pave the way to extend Shojaee and Bodaghi's work. In details, we drop the commutativity assumption on $\mathcal{A}$ in [11, Theorem 2.3(i)], and also show that the condition $\overline{\mathcal{A}^2} = \mathcal{A}$ is necessary. Finally, we should remind the reader that our approach is totally different from earlier works on the cyclic amenability of the products of Banach algebras.

2. Preliminaries

Throughout this paper, for any Banach space X, by X^* we denote the dual space of X and we always consider X as naturally embedded into X^{**}. For $x \in X$ and $f \in X^*$, the notation $\langle f, x\rangle$ (and also $\langle x, f\rangle$) is used to denote the natural duality between X and X^*.

Now, let $\mathcal{C}$ be a Banach algebra and let $\mathcal{X}$ be a Banach $\mathcal{C}$-bimodule. Then $\mathcal{X}^*$ is also a Banach $\mathcal{C}$-bimodule by the following module actions:

$$\langle f \cdot c, x\rangle = \langle f, c \cdot x\rangle, \quad \langle c \cdot f, x\rangle = \langle f, x \cdot c\rangle,$$

for all $c \in \mathcal{C}$, $x \in \mathcal{X}$, $f \in \mathcal{X}^*$. A *derivation* from $\mathcal{C}$ into $\mathcal{X}$ is a continuous linear map $D\colon \mathcal{C} \to \mathcal{X}$ such that for every $c_1, c_2 \in \mathcal{C}$, $D(c_1c_2) = D(c_1) \cdot c_2 + c_1 \cdot D(c_2)$. For $x \in \mathcal{X}$, define ad_x from $\mathcal{C}$ into $\mathcal{X}$ by $\mathrm{ad}_x(c) = c \cdot x - x \cdot c$ for all $c \in \mathcal{C}$. It is easy to show that ad_x is a derivation; such derivations are called *inner derivations*. We denote the set of all continuous derivations from $\mathcal{C}$ into $\mathcal{X}$ by $Z^1(\mathcal{C}, \mathcal{X})$. The *first cohomology group* $H^1(\mathcal{C}, \mathcal{X})$ is the quotient of the space of all continuous derivations by the space of all inner derivations, and in many situations triviality of this space is of considerable importance. In particular, $\mathcal{C}$ is called *contractible* (resp. *amenable*) if $H^1(\mathcal{C}, \mathcal{X}) = 0$ (resp. $H^1(\mathcal{C}, \mathcal{X}^*) = 0$) for every Banach $\mathcal{C}$-bimodule $\mathcal{X}$ and $\mathcal{C}$ is called *weakly amenable* if $H^1(\mathcal{C}, \mathcal{C}^*) = 0$. Recall also that a derivation $D\colon \mathcal{C} \to \mathcal{X}$ is called *approximately inner* if there exists a net $(x_\alpha)_\alpha \subseteq \mathcal{X}$ such that $D(c) = \lim_\alpha(c \cdot x_\alpha - x_\alpha \cdot c)$ for all $c \in \mathcal{C}$. Following Grønbæk [6] and Esslamzadeh and Shojaee [5], we say that $\mathcal{C}$ is (*approximately*) *cyclic amenable* if every cyclic derivation $D\colon \mathcal{C} \to \mathcal{C}^*$ is (approximately) inner. Recall that a derivation $D\colon \mathcal{C} \to \mathcal{C}^*$ is *cyclic* if

$$\langle D(c_1), c_2\rangle + \langle c_1, D(c_2)\rangle = 0,$$

for all $c_1, c_2 \in \mathcal{C}$.

Also, recall from [7] that if $\mathcal{A}$ and $\mathcal{B}$ are two Banach algebras and $T\colon \mathcal{B} \to \mathcal{A}$ is an algebra homomorphism with $\|T\| \le 1$, then $\mathcal{M}_T^*$ can be identified with $\mathcal{A}^* \times \mathcal{B}^*$ in the natural way

$$\langle (f, g), (a, b)\rangle = \langle f, a\rangle + \langle g, b\rangle,$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$. The dual norm on $\mathcal{A}^* \times \mathcal{B}^*$ is of course the maximum norm $\|(f, g)\| = \max\{\|f\|, \|g\|\}$. Furthermore, the $\mathcal{M}_T$-bimodule actions on $\mathcal{M}_T^*$ are formulated as follows:

$$(a, b) \cdot (f, g) = ((a + T(b)) \cdot f, T^*(a \cdot f) + b \cdot g)$$

and

$$(f, g) \cdot (a, b) = (f \cdot (a + T(b)), T^*(f \cdot a) + g \cdot b)$$

for all $(a,b) \in \mathcal{M}_T$ and $(f,g) \in \mathcal{A}^* \times \mathcal{B}^*$, where T^* denotes the adjoint of T. Moreover, the homomorphism T allows us to consider $\mathcal{A}$ as a Banach $\mathcal{B}$-bimodule and therefore $\mathcal{A}^*$ is a Banach $\mathcal{B}$-bimodule by the following module actions:

$$\langle f \cdot b, a\rangle = \langle f, T(b)a\rangle \,, \quad \langle b \cdot f, a\rangle = \langle f, aT(b)\rangle$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $f \in \mathcal{A}^*$. In particular, a derivation from $\mathcal{B}$ into $\mathcal{A}^*$ is a continuous linear map $D\colon \mathcal{B} \to \mathcal{A}^*$ such that for every $b_1, b_2 \in \mathcal{B}$

$$D(b_1 b_2) = D(b_1) \cdot T(b_2) + T(b_1) \cdot D(b_2).$$

3. Main results

Throughout this section $\mathcal{A}$ and $\mathcal{B}$ are two Banach algebras, $T\colon \mathcal{B} \to \mathcal{A}$ is an algebra homomorphism with $\|T\| \le 1$, and $\mathcal{M}_T$ denotes the T-Lau product of $\mathcal{A}$ and $\mathcal{B}$. We begin the presentation of the results of this section by the following two results which characterize the set of all cyclic derivations from $\mathcal{M}_T$ into $\mathcal{M}_T$-bimodule $\mathcal{M}_T^*$.

Proposition 3.1. *A continuous linear map $D\colon \mathcal{M}_T \to \mathcal{M}_T^*$ is a cyclic derivation if and only if there exists $d_1 \in Z^1(\mathcal{A}, \mathcal{A}^*)$, $d_2 \in Z^1(\mathcal{B}, \mathcal{B}^*)$, $d_3 \in Z^1(\mathcal{B}, \mathcal{A}^*)$ and a bounded linear map $S\colon \mathcal{A} \to \mathcal{B}^*$, such that for each $a, a' \in \mathcal{A}$ and $b \in \mathcal{B}$*

(i) $D((a,b)) = (d_1(a) + d_3(b), S(a) + d_2(b))$,

(ii) $d_1(T(b)a) = d_3(b) \cdot a + T(b) \cdot d_1(a)$,

(iii) $d_1(aT(b)) = a \cdot d_3(b) + d_1(a) \cdot T(b)$,

(iv) $S(T(b)a) = T^*(d_3(b) \cdot a) + b \cdot S(a)$,

(v) $S(aT(b)) = T^*(a \cdot d_3(b)) + S(a) \cdot b$,

(vi) $S(aa') = T^*(d_1(aa'))$,

(vii) *d_1 and d_2 are cyclic derivations,*

(viii) *$S^* \circ \Phi + d_3 = 0$, where Φ is the usual injection from $\mathcal{B}$ into $\mathcal{B}^{**}$.*

Proof. We only need to prove the "only if" part of this proposition, which is the essential part of it. To this end, assume that D is a cyclic derivation from $\mathcal{M}_T$ into $\mathcal{M}_T^*$. Then there exist bounded linear mappings $D_1\colon \mathcal{M}_T \to \mathcal{A}^*$ and $D_2\colon \mathcal{M}_T \to \mathcal{B}^*$ such that $D = (D_1, D_2)$. Then there exist $d_1 \in Z^1(\mathcal{A}, \mathcal{A}^*)$, $d_2 \in Z^1(\mathcal{B}, \mathcal{B}^*)$, $d_3 \in Z^1(\mathcal{B}, \mathcal{A}^*)$ and a bounded linear map $S\colon \mathcal{A} \to \mathcal{B}^*$ which satisfies in conditions (i)–(vi) follows from [7, Theorem 3.1] for $n = 0$. Recall from the proof of [7, Theorem 3.1] that for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$

$$d_1(a) = D_1((a,0)), \quad d_2(b) = D_2((0,b)), \quad d_3(b) = D_1((0,b)) \quad \text{and} \quad S(a) = D_2((a,0)).$$

Moreover, since D is a cyclic derivation, for every $(a,b),(a',b')$ in $\mathcal{M}_T$ we have

$$\big\langle D((a,b)),(a',b')\big\rangle + \big\langle (a,b), D((a',b'))\big\rangle = 0. \tag{3.1}$$

It follows that for all $a, a' \in \mathcal{A}$

$$\big\langle d_1(a), a'\big\rangle + \big\langle a, d_1(a')\big\rangle = \big\langle D((a,0)),(a',0)\big\rangle + \big\langle (a,0), D((a',0))\big\rangle = 0, \tag{3.2}$$

and all $b, b' \in \mathcal{B}$

$$\big\langle d_2(b), b'\big\rangle + \big\langle b, d_2(b')\big\rangle = \big\langle D((0,b)),(0,b')\big\rangle + \big\langle (0,b), D((0,b'))\big\rangle = 0. \tag{3.3}$$

Hence d_1 and d_2 are cyclic derivations. On the other hand, by the equalities (3.1)–(3.3), we conclude that

$$\big\langle S(a), b'\big\rangle + \big\langle S(a'), b\big\rangle + \big\langle d_3(b), a'\big\rangle + \big\langle d_3(b'), a\big\rangle = 0, \tag{3.4}$$

for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. Choosing $b = 0$ in (3.4), we have

$$\big\langle (S^* \circ \Phi + d_3)(b'), a'\big\rangle = 0, \quad a' \in \mathcal{A} \text{ and } b' \in \mathcal{B}.$$

It follows that $S^* \circ \Phi + d_3 = 0$, and this completes the proof. □

Now, we examine approximately inner derivation from $\mathcal{M}_T$ into $\mathcal{M}_T^*$. It should be noticed that, the following result which is stated for approximately inner derivation is also true for inner derivation.

Corollary 3.2. *A cyclic derivation D from $\mathcal{M}_T$ into $\mathcal{M}_T^*$ is approximately inner derivation and $D = \lim_\alpha \operatorname{ad}_{(f_\alpha, g_\alpha)}$ for some nets $(f_\alpha)_\alpha \subseteq \mathcal{A}^*$ and $(g_\alpha)_\alpha \subseteq \mathcal{B}^*$ if and only if*

$$D((a,b)) = (d_1(a) + d_3(b), S(a) + d_2(b)),$$

where $d_1(a) = \lim_\alpha \operatorname{ad}_{f_\alpha}(a)$, $d_2(b) = \lim_\alpha \operatorname{ad}_{g_\alpha}(b)$, $d_3(b) = \lim_\alpha (T(b)\cdot f_\alpha - f_\alpha \cdot T(b))$ and $S(a) = \lim_\alpha T^*(\operatorname{ad}_{f_\alpha}(a))$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Proof. The proof of the backward implication involves nothing more than routine calculations which are left to the reader. We give the proof of the direct implication only. To this end, suppose that D is a cyclic derivation such that $D = \lim_\alpha \operatorname{ad}_{(f_\alpha, g_\alpha)}$ for some nets $(f_\alpha)_\alpha \subseteq \mathcal{A}^*$ and $(g_\alpha)_\alpha \subseteq \mathcal{B}^*$. Suppose also that a and b are arbitrary elements of

$\mathcal{A}$ and $\mathcal{B}$, respectively. By Proposition 3.1, there exist $d_1 \in Z^1(\mathcal{A},\mathcal{A}^*)$, $d_2 \in Z^1(\mathcal{B},\mathcal{B}^*)$, $d_3 \in Z^1(\mathcal{B},\mathcal{A}^*)$ and a bounded linear map $S\colon \mathcal{A} \to \mathcal{B}^*$ such that

$$D((a,b)) = (d_1(a) + d_3(b), S(a) + d_2(b)) .$$

It follows that

$$\begin{aligned}(d_1(a), S(a)) &= D((a,0)) \\ &= \lim_\alpha \operatorname{ad}_{(f_\alpha,g_\alpha)}(a,0) \\ &= \lim_\alpha (\operatorname{ad}_{f_\alpha}(a), T^*(\operatorname{ad}_{f_\alpha}(a))) .\end{aligned}$$

Hence, $d_1(a) = \lim_\alpha \operatorname{ad}_{f_\alpha}(a)$ and $S(a) = \lim_\alpha T^*(\operatorname{ad}_{f_\alpha}(a))$. Similarly, $d_2(b) = \lim_\alpha \operatorname{ad}_{g_\alpha}(b)$ and $d_3(b) = \lim_\alpha (T(b)\cdot f_\alpha - f_\alpha \cdot T(b))$, and this completes the proof. □

The following result which is stated for approximate cyclic amenability is also true for cyclic amenability and it is a characterization of (approximate) cyclic amenability of $\mathcal{M}_T$ in relation to the corresponding ones of $\mathcal{A}$ and $\mathcal{B}$. Here, it is worthwhile to mention that the (approximately) cyclic amenability of the Banach algebra $\mathcal{A}$ does not imply that $\overline{\mathcal{A}^2} = \mathcal{A}$, see [6].

Theorem 3.3. *If $\mathcal{M}_T$ is approximately cyclic amenable, then $\mathcal{A}$ and $\mathcal{B}$ are approximately cyclic amenable. The converse holds if $\overline{\mathcal{A}^2} = \mathcal{A}$.*

Proof. Suppose firstly that $\mathcal{M}_T$ is approximately cyclic amenable. Let also $d_1\colon \mathcal{A} \to \mathcal{A}^*$ and $d_2\colon \mathcal{B} \to \mathcal{B}^*$ be two cyclic derivations. By Proposition 3.1, we conclude that the map $D\colon \mathcal{M}_T \to \mathcal{M}_T^*$ defined by

$$D((a,b)) = (d_1(a) + d_3(b), S(a) + d_2(b)) , \quad (a,b) \in \mathcal{M}_T,$$

is a cyclic derivation, where $d_3 = d_1 \circ T$ and $S = T^* \circ d_1$. Therefore, since $\mathcal{M}_T$ is approximately cyclic amenable, there exists a net $((f_\alpha, g_\alpha))_\alpha \subseteq \mathcal{M}_T^*$ such that $D = \lim_\alpha \operatorname{ad}_{(f_\alpha,g_\alpha)}$. From this, by Corollary 3.2, we can deduce that $d_1 = \lim_\alpha \operatorname{ad}_{f_\alpha}$ and $d_2 = \lim_\alpha \operatorname{ad}_{g_\alpha}$. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are approximately cyclic amenable.

Conversely, suppose that $D\colon \mathcal{M}_T \to \mathcal{M}_T^*$ is a cyclic derivation. Hence, Proposition 3.1 guarantees that there exist $d_3 \in Z^1(\mathcal{B},\mathcal{A}^*)$, a bounded linear map $S\colon \mathcal{A} \to \mathcal{B}^*$ and two cyclic derivations $d_1 \in Z^1(\mathcal{A},\mathcal{A}^*)$, $d_2 \in Z^1(\mathcal{B},\mathcal{B}^*)$, such that $S^*(b) + d_3(b) = 0$ and

$$D((a,b)) = (d_1(a) + d_3(b), S(a) + d_2(b)) ,$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. By another application of Proposition 3.1, we have

$$S(aa') = T^*(d_1(aa')),$$

for all $a, a' \in \mathcal{A}$. This together with the fact that $\overline{\mathcal{A}^2} = \mathcal{A}$ implies $S = T^* \circ d_1$. Moreover, since $\mathcal{A}$ and $\mathcal{B}$ are approximately cyclic amenable, then there exist $(f_\alpha) \subseteq \mathcal{A}^*$ and $(g_\alpha) \subseteq \mathcal{B}^*$ such that $d_1 = \lim_\alpha \operatorname{ad}_{f_\alpha}$ and $d_2 = \lim_\alpha \operatorname{ad}_{g_\alpha}$. Hence, for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have

$$S(a) = \lim_\alpha T^*(\operatorname{ad}_{f_\alpha}(a)),$$

and

$$\begin{aligned}
\langle d_3(b), a\rangle &= -\langle S(a), b\rangle \\
&= -\lim_\alpha \langle \operatorname{ad}_{f_\alpha}(a), T(b)\rangle \\
&= -\lim_\alpha \langle a \cdot f_\alpha - f_\alpha \cdot a, T(b)\rangle \\
&= -\lim_\alpha [\langle f_\alpha, T(b)a\rangle - \langle f_\alpha, aT(b)\rangle] \\
&= \lim_\alpha [\langle T(b) \cdot f_\alpha, a\rangle - \langle f_\alpha \cdot T(b), a\rangle] \\
&= \left\langle \lim_\alpha (T(b) \cdot f_\alpha - f_\alpha \cdot T(b)), a \right\rangle.
\end{aligned}$$

Therefore, $D = \lim_\alpha \operatorname{ad}_{(f_\alpha, g_\alpha)}$ by Corollary 3.2. □

Recall from [6, Example 2.5] that if $0 \neq \mathcal{A}$ is a Banach algebra with zero algebra product, then $\mathcal{A}$ is cyclicly amenable if and only if $\dim \mathcal{A} = 1$. Moreover, recall also from [5] that, for commutative Banach algebras, the two notions of cyclic amenability and approximate cyclic amenability coincide.

Now, we are in position to show that the condition $\overline{\mathcal{A}^2} = \mathcal{A}$ in Theorem 3.3 can not be omitted.

Example 3.4. Let $\mathcal{A} = \mathbb{C}$ with zero algebra product. Then by [6, Example 2.5] $\mathcal{A}$ is approximately cyclic amenable. In particular, $\overline{\mathcal{A}^2} \subsetneqq \mathcal{A}$. Now by choosing $T = 0$, we have $\mathcal{M}_0 = \mathcal{A} \times_0 \mathcal{A}$ which is a Banach algebra with zero algebra product. But, since $\mathcal{M}_0$ is a commutative Banach algebra with dimension 2, it is not an approximately cyclic amenable Banach algebra.

As usual, we say that a Banach algebra $\mathcal{C}$ has a left (resp. right) faithful dual space if for each nonzero $f \in \mathcal{C}^*$, there exists $c \in \mathcal{C}$ such that $c \cdot f \neq 0$ (resp. $f \cdot c \neq 0$). Moreover, $\mathcal{C}$ has a faithful dual space if $\mathcal{C}$ has both a left and a right faithful dual space.

Remark 3.5. Let $\mathcal{C}$ be a Banach algebra. Then $\mathcal{C}$ has a faithful dual space if and only if $\overline{\mathcal{C}^2} = \mathcal{C}$. Indeed, the backward implication is trivial, and in order to prove the direct implication, we suppose on the contrary that $\overline{\mathcal{C}^2} \subsetneqq \mathcal{C}$, then by Hahn Banach Theorem there exists $0 \neq f$ in $\mathcal{C}^*$ such that $f|_{\overline{\mathcal{C}^2}} = 0$. It follows that $c \cdot f = 0$ and $f \cdot c = 0$ for all $c \in \mathcal{C}$ which is a contradiction.

Finally, as an application of Theorem 3.3 with the aid of Remark 3.5 one can obtain the following improvement of [1, Theorems 3.2] and [11, Theorem 2.3(i)].

Theorem 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras such that $\mathcal{A}$ has a faithful dual space. If $\mathcal{A}$ and $\mathcal{B}$ are cyclic amenable, then $\mathcal{M}_T$ is cyclic amenable.*

Acknowledgments

The author is grateful to the referee for bringing [11] to our attention, the careful reading of the paper and the detailed criticism.

References

[1] F. Abtahi and A. Ghafarpanah, *A note on cyclic amenability of the Lau product of Banach algebras defined by a Banach algebra morphism*, Bull. Aust. Math. Soc. **92** (2015), no. 2, 282–289. `http://dx.doi.org/10.1017/s0004972715000544`

[2] S. J. Bhatt and P. A. Dabhi, *Arens regularity and amenability of Lau product of Banach algebras defined by a Banach algebra morphism*, Bull. Aust. Math. Soc. **87** (2013), no. 2, 195–206. `http://dx.doi.org/10.1017/s000497271200055x`

[3] Y. Choi, *Triviality of the generalized Lau product associated to a Banach algebra homomorphism*, Preprint in Bulletin of the Australian Mathematical Society. `http://dx.doi.org/10.1017/S0004972715001823`

[4] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs, New Series **24**, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 2000.

[5] G. H. Esslamzadeh and B. Shojaee, *Approximate weak amenability of Banach algebras*, Bull. Belg. Math. Soc. Simon Stevin. **18** (2011), no. 3, 415–429.

[6] N. Grønbæk, *Weak and cyclic amenability for non-commutative Banach algebras*, Proc. Edinb. Math. Soc. (2) **35** (1992), no. 2, 315–328. `http://dx.doi.org/10.1017/s0013091500005587`

[7] H. Javanshiri and M. Nemati, *On a certain product of Banach algebras and some of its properties*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **15** (2014), no. 3, 219–227.

[8] A. T. M. Lau, *Analysis on a class of Banach algebras with applications to Harmonic analysis on locally compact groups and semigroups*, Fund. Math. **118** (1983), no. 3, 161–175.

[9] M. Nemati and H. Javanshiri, *Some homological and cohomological notions on T-Lau product of Banach algebras*, Banach J. Math. Anal. **9** (2015), no. 2, 183–195. http://dx.doi.org/10.15352/bjma/09-2-13

[10] ______, *The multiplier algebra and BSE–functions for certain product of Banach algebras*, Preprint, see arXiv:1509.00895.

[11] B. Shojaee and A. Bodaghi, *A generalization of cyclic amenability of Banach algebras*, Math. Slovaca **65** (2015), no. 3, 633–644. http://dx.doi.org/10.1515/ms-2015-0044

Hossein Javanshiri
Department of Mathematical sciences, Yazd University, P.O. Box: 89195-741, Yazd, Iran
E-mail address: h.javanshiri@yazd.ac.ir