TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 4, pp. 787–800, August 2016
DOI: 10.11650/tjm.20.2016.6898
This paper is available online at http://journal.tms.org.tw

Anti-invariant Riemannian Submersions: A Lie-theoretical Approach

Peter Gilkey, Mitsuhiro Itoh and JeongHyeong Park*

Abstract. We give a construction which is Lie theoretic of anti-invariant Riemannian submersions from almost Hermitian manifolds, from quaternion manifolds, from para-Hermitian manifolds, from para-quaternion manifolds, and from octonian manifolds. This yields many compact Einstein examples.

1. Introduction

We begin by establishing some notational conventions.

1.1. Riemannian submersions

Let M and N be smooth manifolds of dimension m and n, respectively, and let $\pi\colon M \to N$ be a smooth map. We say that π is a *submersion* if π_* is a surjective map from the tangent space T_PM to the tangent space $T_{\pi P}N$ for every point P of M. Let g_M and g_N be Riemannian metrics on M and N. If $\pi\colon M \to N$ is a submersion, then the *vertical distribution* is the kernel of π_* and the *horizontal distribution* $\mathcal{H}$ is $\mathcal{V}^\perp$. We may then decompose $TM = \mathcal{V} \oplus \mathcal{H}$. We say that π is a *Riemannian submersion* if π_* is an isometry from $\mathcal{H}_P$ to $T_{\pi P}N$ for every point P of M. We refer to O'Neill [12] for further details concerning the geometry of Riemannian submersions. If g_M and g_N are pseudo-Riemannian metrics, we impose in addition the condition that the restriction of g_M to $\mathcal{V}$ is non-degenerate to ensure that $\mathcal{V} \cap \mathcal{H} = \{0\}$. This gives rise to the notation of a *pseudo-Riemannian* submersion.

1.2. Hermitian geometry

An endomorphism J of TM is said to define an *almost complex structure* on M if $J^2 = -\operatorname{id}$, i.e., J gives a complex structure to T_PM for every point P of M. We complexify the tangent bundle and let

$$T^{1,0} := \left\{X \in TM \otimes_{\mathbb{R}} \mathbb{C} : JX = \sqrt{-1}X\right\}.$$

Received October 12, 2015; Accepted March 2, 2016.
Communicated by Sai-Kee Yeung.
2010 *Mathematics Subject Classification.* Primary: 53C15; Secondary: 53B20, 53C43.
Key words and phrases. Riemannian submersion, Anti-invariant almost Hermitian, Anti-invariant quaternion, Anti-invariant para-Hermitian, Anti-invariant para-quaternion, Anti-invariant octonian.
*Corresponding author.

One says J is *integrable* if $T^{1,0}$ is integrable, i.e., X, Y belong to $C^\infty(T^{1,0})$ implies the complex Lie bracket $[X, Y]$ also belongs to $C^\infty(T^{1,0})$. The Newlander-Nirenberg Theorem [11] is the analogue in the complex setting of the Frobenius theorem in the real setting; J is integrable if and only if it arises from an underlying holomorphic structure on N. The Riemannian metric g_M is said to be *almost Hermitian* if $J^* g_M = g_M$, i.e., if $g_M(JX, JY) = g_M(X, Y)$ for all tangent vectors $X, Y \in T_P M$ and all points P of M; the triple (M, g_M, J) is then said to be an *almost Hermitian manifold*; in the pseudo-Riemannian setting one obtains the notion of *almost pseudo-Hermitian* manifold similarly. The notation *Hermitian* or *pseudo-Hermitian* is used if the structure J is integrable.

Let (M, g_M, J) be an almost Hermitian manifold and let π be a Riemannian submersion from (M, g_M) to (N, g_N). Following the seminal work of Sahin [15, 16], one says that π is an *anti-invariant Riemannian submersion from an almost Hermitian manifold* if

$$J\{\mathcal{V}\} \subset \mathcal{H}.$$

If $J\{\mathcal{V}\} = \mathcal{H}$, then π is said to be *Lagrangian*. There have been a number of subsequent papers in this subject extending the work of Sahin [15, 16]; we shall cite just a few representative examples. Lee et al. [9] examined the geometry of anti-invariant Riemannian submersions from a Hermitian and from a Kähler manifold in relation to the Einstein condition and examined when the submersions were Clairant submersions. Ali and Fatima [2] examined the nearly Kähler setting. We also refer to related work of Ali and Fatima [3], of Beri et al. [4], and of Murathan and Küpeli-Erken [10].

1.3. Quaternion geometry

We shall restrict to flat quaternion structures as this is sufficient for our purposes. The *quaternion algebra* $\mathbb{Q} := \mathbb{R}^4 = \operatorname{Span}\{e_0, e_1, e_2, e_3\}$ is defined by the relations:

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

Table 1.1: Multiplicative table of the quaternion algebra $\mathbb{Q}$.

One says that $x \in \mathbb{Q}$ is an *imaginary quaternion* if $x \in \operatorname{Span}\{e_1, e_2, e_3\}$. A *flat quaternion structure* on a manifold M is a unital action of $\mathbb{Q}$ on TM. If x is a unit length purely imaginary quaternion, then $\xi \to x \cdot \xi$ defines an almost complex structure on M. If g is a Riemannian metric on M, we shall assume in addition that $\|x \cdot \xi\| = \|x\| \cdot \|\xi\|$ for any quaternion x and any tangent vector ξ. Let $\pi\colon (M, g_M) \to (N, g_N)$ be a Riemannian submersion. Then one says π is an *anti-invariant Riemannian submersion from a quaternion manifold* if $x \cdot \mathcal{V} \subset \mathcal{H}$ for any purely imaginary quaternion x. We have assumed that the roles of $\{e_1, e_2, e_3\}$ are globally defined (i.e., the structure is flat); we refer to Alekseevsky and Marchiafava [1] for a discussion of the more general setting. Anti-invariant Riemannian submersion from a quaternion manifold have been studied by K. Park [14].

1.4. Para-Hermitian geometry

Instead of considering almost complex structures, one can consider para-complex structures. Let $\widetilde{\mathbb{C}} := \mathbb{R}^2$ with the para-complex structure $Je_1 = e_2$ and $Je_2 = e_1$. Let (M, g_M) be a pseudo-Riemannian manifold of neutral signature (ℓ, ℓ). If J is an endomorphism of M with $J^2 = \mathrm{Id}$ such that $g_M(JX, JY) = -g_M(X, Y)$ for all $X, Y \in T_P M$ and all points P of M, then the triple (M, g_M, J) is said to be a *para-Hermitian manifold.* Let π be a pseudo-Riemannian submersion from (M, g_M) to (N, g_N) with $J\{\mathcal{V}\} \subset \mathcal{H}$. One then says π is an *anti-invariant Riemannian submersion from para-Hermitian manifold*; π is *Lagrangian para-Hermitian* if $J\{\mathcal{V}\} = \mathcal{H}$; this is examined by Gündüzalp [6].

1.5. Para-quaternion geometry

In place of the quaternion commutation relations given in Table 1.1, one imposes the para-quaternion relations to define the para-quaternions $\widetilde{\mathbb{Q}}$ by setting:

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$+e_0$	$-e_1$
e_3	e_3	e_2	$+e_1$	$+e_0$

Table 1.2: Multiplicative table of the para-quaternions $\widetilde{\mathbb{Q}}$.

If J_1 is Hermitian and if J_2 and J_3 are para-Hermitian, then one obtains the notion of a *para-quaternion manifold.* We refer to Ivanov and Zamkovoy [8] for further details. If $\pi\colon (M, g_M) \to (N, g_N)$ is a pseudo-Riemannian submersion and if $J_i\{\mathcal{V}\} \subset \mathcal{H}$ for $1 \le i \le 3$, then π is said to be an *anti-invariant Riemannian submersion from a para-quaternion manifold.* To the best of our knowledge, there are no papers on such geometries.

1.6. Octonian geometry

The octonians $\mathbb{O}$ arise from a non-associative and non-commutative bilinear multiplication on $\mathbb{R}^8$. If $\{e_0,\ldots,e_7\}$ is the standard basis for $\mathbb{R}^8$, the multiplication is given by the following table (see Wikipedia [17]):

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Table 1.3: Multiplicative table of the octonians $\mathbb{O}$.

The octonians satisfy the identity

$$\|x\cdot y\|=\|x\|\cdot\|y\|\quad\text{for all } x,y\in\mathbb{R}^8.$$

If $x\in\operatorname{Span}\{e_1,\ldots,e_7\}$, then x is said to be a *purely imaginary octonian*. Such an octonian satisfies $x\cdot y\perp y$ for any $y\in\mathbb{R}^8$. Let (M,g) be a Riemannian manifold. A *flat octonian* structure on a Riemannian manifold (M,g) is a unital octonian action on TM such that $\|x\cdot\xi\|=\|x\|\cdot\|\xi\|$ for any octonian x and any tangent vector ξ. If π is a Riemannian submersion from (M,g) to (N,h), then we say that π an *anti-invariant Riemannian submersion from an octonian manifold* if $x\cdot\mathcal{V}\perp\mathcal{V}$ for any purely imaginary octonian $x\in\mathbb{R}^7$. To the best of our knowledge, there are no papers dealing with anti-invariant Riemannian submersion from an octonian manifold.

1.7. A Lie theoretic construction

We now outline a Lie theoretic method that we shall use to construct examples. Let H be a closed and connected subgroup of an even dimensional Lie group G. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the associated Lie algebras, respectively. Let $\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}$ which is invariant under the adjoint action of H and whose restriction to $\mathfrak{h}$ is non-degenerate as well. We use $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ to decompose $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{h}^\perp$ as an orthogonal direct sum. The inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ defines a left-invariant pseudo-Riemannian metric on

G and, since the inner product is invariant under the adjoint action of H, the restriction of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ to $\mathfrak{h}^\perp$ defines a G-invariant pseudo-Riemannian metric on the coset manifold G/H so that the natural projection $\pi\colon G \to G/H$ is a pseudo-Riemannian submersion.

(1) Complex geometry. Assume $\langle\cdot,\cdot\rangle$ is positive definite. Let J be a Hermitian complex structure on $\mathfrak{g}$; J induces a left-invariant Hermitian almost complex structure on G. Assume that $J\{\mathfrak{h}\} \subset \mathfrak{h}^\perp$. Then $\pi\colon G \to G/H$ is an anti-invariant Riemannian submersion from an almost Hermitian manifold; π is Lagrangian if and only if $2\dim\{\mathfrak{h}\} = \dim\{\mathfrak{g}\}$. More generally, if $\langle\cdot,\cdot\rangle$ is only assumed to be a non-degenerate inner product and if the restriction to $\mathfrak{h}$ is assumed to be non-degenerate, then we obtain an anti-invariant Riemannian submersion from an almost pseudo-Hermitian manifold.

(2) Quaternion geometry. Assume $\langle\cdot,\cdot\rangle$ is positive definite. Assume given a Hermitian quaternion structure on $\mathfrak{g}$ such that $x\cdot\mathfrak{h} \subset \mathfrak{h}^\perp$ for any purely imaginary quaternion x. Then $\pi\colon G \to G/H$ is an anti-invariant Riemannian submersion from a quaternion manifold.

(3) Para-complex geometry. Assume $\langle\cdot,\cdot\rangle$ has neutral signature and that $\langle\cdot,\cdot\rangle$ is non-degenerate on $\mathfrak{h}$. Let J be a Hermitian para-complex structure on $\mathfrak{g}$ with $J\{\mathfrak{h}\} \subset \mathfrak{h}^\perp$. Then $\pi\colon G \to G/H$ is an anti-invariant Riemannian submersion from a para-Hermitian manifold; π is Lagrangian if and only if $2\dim\{\mathfrak{h}\} = \dim\{\mathfrak{g}\}$.

(4) Para-quaternion geometry. Assume $\langle\cdot,\cdot\rangle$ has neutral signature and that $\langle\cdot,\cdot\rangle$ is non-degenerate on $\mathfrak{h}$. Assume given a Hermitian para-quaternion structure on $\mathfrak{g}$ such that $x\cdot\mathfrak{h} \subset \mathfrak{h}^\perp$ for any purely imaginary para-quaternion x. Then $\pi\colon G \to G/H$ is an anti-invariant Riemannian submersion from a para-quaternion manifold.

1.8. Outline of the paper

Section 2, we will discuss some examples which arise from the construction given above. We conclude in Section 3 by presenting a different family of examples (including an anti-invariant Riemannian submersion from an octonian manifold) relating to the Hopf fibration where the total space is not a Lie group but which are nevertheless Lie theoretic in nature. It is our hope that having a rich family of examples will inform further investigations in this field.

2. Examples

In this section, we present examples of anti-invariant Riemannian submersions where the total space is a Lie group G and the base space is a homogeneous space upon which G acts transitively by isometries; $H \subset G$ is the isotropy subgroup of the action. Examples 2.1 and 2.2 are flat geometries. Example 2.3 arises from the Hopf fibration $S^1 \to S^3 \to S^2$. In Example 2.4, the total space will be $(S^3)^\nu$. We will take product metrics and if the metric on S^3 is the usual round metric, these examples will be Einstein. In Example 2.5, we take $G = \mathbb{R} \times \mathrm{SL}(2, \mathbb{R})$ to construct negative curvature examples.

2.1. Abelian examples

Example 2.1. Let $G = \mathbb{R}^m$ and let $H = \mathbb{R}^n \oplus 0 \subset G$ for $n < m$. We identify G/H with $0 \oplus \mathbb{R}^{m-n}$ and π with projection on the last $m - n$ coordinates.

(1) Take the standard Euclidean inner product on G to obtain a bi-invariant Riemannian metric so that π is a Riemannian submersion.

  (a) Suppose $m = 2\ell$ and $n = \ell$. Identify $G = \mathbb{C}^\ell$ so that H corresponds to the purely real vectors in $\mathbb{C}^\ell$. We identify $\mathfrak{g}$ with G and $\mathfrak{h}$ with H. Then $\sqrt{-1}\mathfrak{h} \perp \mathfrak{h}$ and we obtain an anti-invariant Riemannian submersion from a Hermitian manifold which is Lagrangian; the almost complex structure corresponds to scalar multiplication by $\sqrt{-1}$ and is integrable.

  (b) Assume $m = 4\ell$ and $n = \ell$. Identify $G = \mathbb{Q}^\ell$ so that H corresponds to the purely real vectors in $\mathbb{Q}^\ell$. Then $x \cdot \mathfrak{h} \perp \mathfrak{h}$ if x is a purely imaginary quaternion and we obtain an anti-invariant Riemannian submersion from a quaternion manifold.

  (c) Assume $m = 8\ell$ and $n = \ell$. Identify $G = \mathbb{O}^\ell$ so that H corresponds to the purely real vectors in $\mathbb{O}^\ell$. Then $x \cdot \mathfrak{h} \perp \mathfrak{h}$ if x is a purely imaginary octonian and we obtain an anti-invariant Riemannian submersion from an octonian manifold.

(2) Let $m = 2\ell$ and $n = \ell$. Identify G with $\widetilde{\mathbb{C}}^\ell$ so that $H = \mathbb{R}^\ell$ corresponds to the purely real para-complex vectors. More specifically, we take a basis $\{e_i, f_i\}$ for $\mathbb{R}^{2\ell}$ where $H = \mathrm{Span}\{e_i\}$. Set

$$\langle e_i, e_i \rangle = 1, \quad \langle f_i, f_i \rangle = -1, \quad \langle e_i, f_j \rangle = 0, \quad \widetilde{J} e_i = f_i, \quad \widetilde{J} f_i = e_i.$$

We obtain an anti-invariant Riemannian from a para-Hermitian manifold which is Lagrangian. By taking the inner product $\langle e_i, e_i \rangle = -\langle f_i, f_i \rangle = \epsilon_i$ for $\epsilon_i = \pm 1$, we can ensure that the base has arbitrary signature.

(3) Let $m = 4\ell$ and $n = \ell$. Identify $G = \widetilde{\mathbb{Q}}^\ell$ so that $H = \mathbb{R}^\ell$ corresponds to the purely real vectors in G. We obtain an anti-invariant Riemannian submersion from a para-quaternion manifold.

The total space $G = \mathbb{R}^m$ is non-compact in Example 2.1. We compactify by dividing by an integer lattice.

Example 2.2. Let $\mathbb{Z}^k$ be the integer lattice in $\mathbb{R}^k$ and let $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$ be the k-dimensional torus $S^1 \times \cdots \times S^1$ with the flat product metric. Let $G = \mathbb{T}^m$ and let $H = \mathbb{T}^n$. We can repeat the construction of Example 2.1 to obtain examples which are compact.

2.2. The Hopf fibration

Examples 2.1 and 2.2 are flat. We can use the Hopf fibration to construct examples which are not flat. We identify $\mathbb{R}^4$ with the quaternions $\mathbb{Q}$; this identifies S^3 with the unit quaternions and gives S^3 a Lie group structure. Let

$$e_1(x) = x \cdot i, \quad e_2(x) = x \cdot j, \quad e_3(x) = x \cdot k. \tag{2.1}$$

This is then a basis for the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on S^3 and

$$[e_1, e_2] = -2e_3, \quad [e_2, e_3] = -2e_1, \quad [e_3, e_1] = -2e_2.$$

Every 1-dimensional Lie subalgebra of S^3 corresponds to a 1-dimensional compact Abelian subgroup S^1 of S^3. Let $G = S^1 \times S^3$ and let e_0 generate the Lie algebra of S^1 so that $\mathfrak{g} = \text{Span}\{e_0, e_1, e_2, e_3\}$. If $\epsilon \neq 0$, define

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j = 2, \\ 1 & \text{if } i = j = 3, \\ \epsilon & \text{if } i = j = 0, \\ \epsilon & \text{if } i = j = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

These metrics are among the metrics first introduced by Hitchin [7] in his study of harmonic spinors and are Kaluza-Klein metrics. Let

$$\begin{aligned} &Je_0 = e_1, && Je_1 = -e_0, && Je_2 = e_3, && Je_3 = -e_2, \\ &\check{J}e_0 = e_2, && \check{J}e_2 = -e_0, && \check{J}e_3 = e_1, && \check{J}e_1 = -e_3, \\ &\widetilde{J}e_0 = e_2, && \widetilde{J}e_2 = e_0, && \widetilde{J}e_1 = -e_3, && \widetilde{J}e_3 = -e_1. \end{aligned} \tag{2.3}$$

Then J is an integrable Hermitian complex structure on $S^1 \times S^3$ for any $\epsilon \neq 0$. If $\epsilon = 1$, then $\{1, J, \check{J}, J\check{J}\}$ is a Hermitian quaternion structure on $S^1 \times S^3$. If $\epsilon = -1$, then $\widetilde{J}$ is a para-Hermitian para-complex structure and $\{1, J, \widetilde{J}, J\widetilde{J}\}$ is a Hermitian para-quaternion structure on $S^1 \times S^3$. If $\epsilon = 1$, then $\langle \cdot, \cdot \rangle$ is bi-invariant. If $\epsilon \neq 1$, then $\langle \cdot, \cdot \rangle$ is right invariant under the 2-dimensional Lie subgroup H with $\mathfrak{h} = \text{Span}\{e_0, e_1\}$ but is not bi-invariant.

Example 2.3. Let $G = S^1 \times S^3$. Let $\mathfrak{h}$ be the Lie sub-algebra of a closed subgroup H of G. Adopt the notation of (2.2) and (2.3).

(1) Let $\mathfrak{h} = \mathrm{Span}\{e_0, e_1\}$.

   (a) Let $\epsilon = 1$. Then $G/H = S^2$ is a round sphere of a suitably chosen radius in $\mathbb{R}^3$ and has constant sectional curvature. We use $\check{J}$ to obtain an anti-invariant Riemannian submersion from a Hermitian manifold which is Lagrangian. The fibers of the submersion are minimal and the horizontal distribution is not integrable (see Park [13]).

   (b) Let $\epsilon = -1$. Then $G/H = S^2$. We use $\widetilde{J}$ to obtain an anti-invariant Riemannian submersion from a para-Hermitian manifold which is Lagrangian.

(2) If $\mathfrak{h} = \mathrm{Span}\{e_0\}$, set $B = S^3$. If $\mathfrak{h} = \mathrm{Span}\{e_1\}$, set $B = S^1 \times S^2$.

   (a) Let $\epsilon \neq 0$ be arbitrary. Then J defines an anti-invariant Riemannian submersion from the Hermitian manifold G to B.

   (b) Let $\epsilon = +1$. We use J and $\check{J}$ to identify $\mathfrak{g} = \mathbb{Q}$ with the quaternions to define an anti-invariant Riemannian submersion from the quaternion manifold G to B.

   (c) Let $\epsilon = -1$. We use $\widetilde{J}$ to define an anti-invariant Riemannian submersion from the Hermitian manifold G to B.

   (d) Let $\epsilon = -1$. We use J and $\widetilde{J}$ to identify $\mathfrak{g} = \mathbb{Q}$ with the para-quaternions and obtain anti-invariant Riemannian submersion from para-quaternion manifold G to B.

2.3. Einstein geometry

Example 2.4. Let g_{S^3} be the standard round metric on S^3 defined by $\langle e_i, e_j\rangle = \delta_{ij}$. Let $G = (S^3)^\nu = S^3 \times \cdots \times S^3$. We take a product metric on G where the metric on each factor is $\pm g_{S^3}$; thus this metric is bi-invariant. Let H be a closed subgroup of G and let $\pi\colon G \to G/H$ be the associated Riemannian submersion.

(1) Let $G = S^3 \times S^3$ and $\mathfrak{g} = \mathrm{Span}\{e_1, e_2, e_3, f_1, f_2, f_3\}$.

   (a) Let $g_G = g_{S^3} \oplus g_{S^3}$ be the standard bi-invariant Einstein metric on $S^3 \times S^3$.

      (i) Let $\mathfrak{h} = \mathrm{Span}\{e_2, f_2\}$. Let $Je_1 = f_1$, $Jf_1 = -e_1$, $Je_2 = e_3$, $Je_3 = -e_2$, $Jf_2 = f_3$, $Jf_3 = -f_2$. This almost complex structure is integrable and using J yields an anti-invariant Riemannian submersion from the Hermitian manifold $S^3 \times S^3$ to $S^2 \times S^2$.

      (ii) Let $\mathfrak{h} = \mathrm{Span}\{e_1, e_2, e_3\}$. Let $Je_i = f_i$ and $Jf_i = -e_i$ where $1 \le i \le 3$. This almost complex structure is not integrable. Using J yields an anti-invariant

Riemannian submersion from the almost Hermitian manifold $S^3 \times S^3$ to S^3.

(b) Let $g_G = g_{S^3} \oplus -g_{S^3}$ be the standard bi-invariant neutral signature metric on $S^3 \times S^3$.

(i) Let $\mathfrak{h} = \operatorname{Span}\{e_1, f_1\}$. Let $\widetilde{J}e_1 = f_2$, $\widetilde{J}f_2 = e_1$, $\widetilde{J}f_1 = e_2$, $\widetilde{J}e_2 = f_1$, $\widetilde{J}e_3 = f_3$, and $\widetilde{J}f_3 = e_3$. Using $\widetilde{J}$ yields an anti-invariant Riemannian submersion from the para-Hermitian manifold $S^3 \times S^3$ to $S^2 \times S^2$.

(ii) Let $\mathfrak{h} = \operatorname{Span}\{e_1, e_2, e_3\}$. Let $\widetilde{J}e_i = f_i$ and $\widetilde{J}f_i = e_i$. Using $\widetilde{J}$ yields an anti-invariant Riemannian submersion from the para Hermitian manifold $S^3 \times S^3$ to S^3.

(2) Let $G = (S^3)^4$, let $g_G = g_{S^3} \oplus g_{S^3} \oplus g_{S^3} \oplus g_{S^3}$, and let $\dim\{H\} \leq 3$.

(a) Identify $\mathfrak{g}$ with $\mathbb{Q}^3$ in such a way that $\mathfrak{h}$ is real and the action of $\mathbb{Q}$ is Hermitian. Then π is an anti-invariant Riemannian submersion from a quaternion manifold.

(b) Identify $\mathfrak{g}$ with $\widetilde{\mathbb{Q}}^3$ in such a way that $\mathfrak{h}$ is real and the action of $\widetilde{\mathbb{Q}}$ is para-Hermitian. Then π is an anti-invariant Riemannian submersion from a para-quaternion manifold.

(3) Let $G = (S^3)^8$, let $g_G = g_{S^3} \oplus \cdots \oplus g_{S^3}$, and let $\dim\{H\} \leq 7$. Identify $\mathfrak{g}$ with $\mathbb{O}^3$ in such a way that $\mathfrak{h}$ is real and the action of $\mathbb{O}$ is Hermitian. Then π is an anti-invariant Riemannian submersion from an octonian manifold

2.4. Negative curvature

Our previous examples have, for the most part, involved the Lie group S^3 and the Hopf fibration $S^3 \to S^2$. We now turn to the negative curvature dual. Let $\mathbb{H}^2(0,2)$ be the hyperbolic plane with a Riemannian metric of constant sectional curvature $-\frac{1}{4}$ and let $\mathbb{H}^2(1,1)$ be the Lorentzian analogue. We recall some facts about the Lie group $\mathrm{SL}(2,\mathbb{R})$ and refer to Section 6.8 of Gilkey, Park, and Vázquez-Lorenzo [5]—there are, of course, many excellent references. $\mathrm{SL}(2,\mathbb{R})$ is a 3-dimensional Lie group and the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is the vector space of trace free 2×2 real matrices. The canonical basis for $\mathfrak{sl}(2,\mathbb{R})$ is

$$f_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The bracket relations then take the form

$$[f_1, f_2] = 2f_3, \quad [f_2, f_3] = -2f_1, \quad [f_3, f_1] = 2f_2.$$

The Lie algebra $\mathfrak{s}^3$ of S^3 is the Lie algebra of the special unitary group $SU(2)$ in positive definite signature and the Lie algebra of $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ is the Lie algebra of the special unitary group $SU(1,1)$ in indefinite signature; the two are related by complexification. Let $\mathrm{ad}(\xi)\colon \eta \to [\xi,\eta]$ be the adjoint action and let $K(\xi,\eta) := \mathrm{Tr}\{\mathrm{ad}(\xi)\,\mathrm{ad}(\eta)\}$ be the Killing form. One then has

$$K(f_i,f_j) = \begin{cases} -8 & \text{if } i=j=1,\\ +8 & \text{if } i=j=2,\\ +8 & \text{if } i=j=3,\\ 0 & \text{otherwise.}\end{cases}$$

There is no bi-invariant Riemannian metric on $\mathrm{SL}(2,\mathbb{R})$. However, $\frac{1}{8}K$ is a bi-invariant Lorentzian metric on $\mathrm{SL}(2,\mathbb{R})$. Let

$$\sigma_1(x) := \begin{pmatrix} \cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{pmatrix},\ \sigma_2(x) := \begin{pmatrix} \cosh(x) & \sinh(x)\\ \sinh(x) & \cosh(x)\end{pmatrix},\ \sigma_3(x) := \begin{pmatrix} e^x & 0\\ 0 & e^{-x}\end{pmatrix}.$$

These define closed Abelian Lie sub-groups H_i of $\mathrm{SL}(2,\mathbb{R})$ whose associated Lie-algebras are spanned by f_i. The natural coset spaces $\mathrm{SL}(2,\mathbb{R})/H_i$ have constant negative sectional curvature $-\frac{1}{4}$ from the above reason and may be identified with $\mathbb{H}^2(0,2)$ if $i=1$ and $\mathbb{H}^2(1,1)$ if $i=2,3$.

Let $G=\mathbb{R}\times\mathrm{SL}(2,\mathbb{R})$. Let f_0 correspond to the Abelian factor. Define a bi-invariant neutral signature metric on G by setting:

$$\langle f_i,f_j\rangle = \begin{cases} -1 & \text{if } i=j=0,\\ -1 & \text{if } i=j=1,\\ +1 & \text{if } i=j=2,\\ +1 & \text{if } i=j=3,\\ 0 & \text{otherwise.}\end{cases}$$

In analogy with (2.2), we set:

$$\begin{array}{llll} Jf_0=f_1, & Jf_1=-f_0, & Jf_2=f_3, & Jf_3=-f_2,\\ \widetilde{J}f_0=f_2, & \widetilde{J}f_2=f_0, & \widetilde{J}f_1=-f_3, & \widetilde{J}f_3=-f_1.\end{array}$$

Then J is a Hermitian complex structure on G and $\widetilde{J}$ is a Hermitian para-complex structure on G; J and $\widetilde{J}$ generate a Hermitian para-complex structure on G.

Example 2.5. Let $G=\mathbb{R}\times\mathrm{SL}(2,\mathbb{R})$, let H be a closed subgroup of G, and let π be the natural projection from G to G/H.

(1) If $\mathfrak{h} = \text{Span}\{f_0\}$, then π is an anti-invariant Riemannian submersion from the Hermitian manifold, from the para-Hermitian manifold, and from the para-quaternion manifold G to $\text{SL}(2,\mathbb{R})$.

(2) If $\mathfrak{h} = \text{Span}\{f_1\}$, then π is an anti-invariant Riemannian submersion from the Hermitian manifold, from the para-Hermitian manifold, and from the para-quaternion manifold G to $\mathbb{R} \times \mathbb{H}^2(0,2)$.

(3) If $\mathfrak{h} = \text{Span}\{f_2\}$, then π is an anti-invariant Riemannian submersion from the Hermitian manifold, from the para-Hermitian manifold, and from the para-quaternion manifold G to $\mathbb{R} \times \mathbb{H}^2(1,1)$.

(4) If $\mathfrak{h} = \text{Span}\{f_0, f_2\}$, then $\pi\colon G \to G/H$ is an anti-invariant Riemannian submersion from the Hermitian manifold G to $\mathbb{H}^2(1,1)$.

3. Examples where the total space is not a Lie group

In this section, we present examples where the total space is not a Lie group. We identify $\mathbb{R}^{4k}$ with $\mathbb{C}^{2k}$ to define an action of S^1 on S^{4k-1}; the quotient S^{4k-1}/S^1 is complex projective space $\mathbb{CP}^{2k-1}$ with a Fubini-Study metric of constant positive holomorphic sectional curvature. We identify $\mathbb{R}^{4\ell}$ with $\mathbb{Q}^\ell$ to define an action of S^3 on $S^{4\ell-1}$; the quotient $S^{4\ell-1}/S^3$ is quaternionic projective space $\mathbb{QP}^{\ell-1}$. Instead of taking the Euclidean inner product on $\mathbb{R}^{2k}$, we could take an indefinite signature metric. Let

$$\begin{aligned} \langle x, y\rangle &:= -x^1y^1 - x^2y^2 + x^3y^3 + \cdots + x^{2\ell}y^{2\ell}, \\ \langle\langle x, y\rangle\rangle &:= -x^1y^1 - \cdots - x^4y^4 + x^5y^5 + \cdots + x^{4\ell}y^{4\ell}, \\ \widetilde{S}^{2k-1} &:= \langle x, y\rangle = -1, \\ \check{S}^{4k-1} &:= \Big\{x \in \mathbb{R}^{4\ell} : \langle\langle x, x\rangle\rangle = -1\Big\}. \end{aligned} \tag{3.1}$$

The pseudo-spheres $\widetilde{S}^{2k-1}$ and $\check{S}^{4k-1}$ inherit indefinite signature metrics of constant sectional curvature. The quotient $\widetilde{S}^{2k-1}/S^1$ is the negative curvature dual of $\mathbb{CP}^{k-1}$ and the quotient $\check{S}^{4k-1}/S^3$ is the negative curvature of $\mathbb{QP}^{k-1}$.

Let $m = 2k$ and $N = S^{2k-1}$ or $N = \widetilde{S}^{2k-1}$, or let $m = 4k$ and $N = \check{S}^{4k-1}$. There is an orthogonal direct sum decomposition $T(\mathbb{R}^m)|_N = \nu \oplus T(N)$ where ν is the normal bundle. Let $M = S^1 \times N$ with the product metric. Since ν is a trivial line bundle, we have a natural isometry $\Xi : TM \approx M \times \mathbb{R}^n$. Let ∂_θ be the natural unit tangent vector field on S^1. Let $x = (\theta, \Theta) \in M$. Then $\Xi(\theta, \Theta)\nu = \Theta$ and $\Xi(TN) = \Theta^\perp$.

Example 3.1. Let $\ell \geq 3$. Adopt the notation of (3.1). Let J be complex multiplication by i on $TM = M \times \mathbb{C}^\ell$. Let $H = S^1$. Since $\dim\{\mathcal{V}\} = 1$, $J\mathcal{V} \perp \mathcal{V}$.

(1) Let $M = S^1 \times S^{2\ell-1}$. Let H act on S^1 by complex multiplication and trivially on $S^{2\ell-1}$. Then $\pi\colon M \to S^{2\ell-1}$ is an anti-invariant Riemannian submersion from a Hermitian manifold.

(2) Let $M = S^1 \times S^{2\ell-1}$. Let H act on trivially on S^1 and by complex multiplication on $S^{2\ell-1}$. Then $\pi\colon M \to S^1 \times \mathbb{CP}^{\ell-1}$ is an anti-invariant Riemannian submersion from a Hermitian manifold.

(3) Let $M = S^1 \times \widetilde{S}^{2\ell-1}$. Let H act on S^1 by complex multiplication and trivially on $\widetilde{S}^{2\ell-1}$. Then $\pi\colon M \to \widetilde{S}^{2\ell-1}$ is an anti-invariant Riemannian submersion from a Hermitian manifold.

(4) Let $M = S^1 \times \widetilde{S}^{2\ell-1}$. Let H act on trivially on S^1 and by complex multiplication on $\widetilde{S}^{2\ell-1}$. Then $\pi\colon M \to S^1 \times \widetilde{\mathbb{CP}}^{\ell-1}$ is an anti-invariant Riemannian submersion from a Hermitian manifold.

We have taken $\ell \geq 3$ since the case $\ell = 2$ recovers the Hopf fibration $S^3 \to S^2$ or $S^3 \to \mathbb{H}^2$.

Example 3.2. Adopt the notation of (3.1). Let $\ell \geq 2$. Let $H = S^1 \times S^1$. Let J be quaternion multiplication on $TM = M \times \mathbb{Q}^\ell$.

(1) Use the product action to let H act on the first and on the second factor of $M = S^1 \times S^{4\ell-1}$. Let π be the associated Riemannian submersion from M to $\mathbb{CP}^{2\ell-1}$. Then $\mathcal{V}(\theta, \Theta) = \operatorname{Span}\{\Theta, i \cdot \Theta\}$. Since $j \cdot \mathcal{V} \perp \mathcal{V}$, π is an anti-invariant Riemannian submersion from a Hermitian manifold.

(2) Use the product action to let H act on the first and on the second factor of $M = S^1 \times \check{S}^{4\ell-1}$. Let π be the associated Riemannian submersion from M to $\widetilde{\mathbb{CP}}^{2\ell-1}$. Then $\mathcal{V}(\theta, \Theta) = \operatorname{Span}\{\Theta, i \cdot \Theta\}$. Since $j \cdot \mathcal{V} \perp \mathcal{V}$, π is an anti-invariant Riemannian submersion from a Hermitian manifold.

We have taken $\ell \geq 2$ since the case $\ell = 1$ recovers the Hopf fibration $S^3 \to S^2$ or $S^3 \to \mathbb{H}^2$.

Acknowledgments

Research of the authors was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2014053413). It is a pleasant task to thank the referee for helpful suggestions concerning the presentation of this material.

References

[1] D. V. Alekseevsky and S. Marchiafava, *Almost complex submanifolds of quaternionic manifolds*, Steps in differential geometry (Debrecen, 2000), 23–28, Inst. Math. Inform., Debrecen, 2001.

[2] S. Ali and T. Fatima, *Anti-invariant Riemannian submersions from nearly Kaehler manifolds*, Filomat **27** (2013), no. 7, 1219–1235. http://dx.doi.org/10.2298/fil1307219a

[3] ______, *Generic Riemannian submersions*, Tamkang J. Math. **44** (2013), no. 4, 395–409. http://dx.doi.org/10.5556/j.tkjm.44.2013.1211

[4] A. Beri, I. Küpeli Erken and C. Murathan, *Anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds*, Turkish J. Math. **40** (2016), 540–552. http://dx.doi.org/10.3906/mat-1504-47

[5] P. Gilkey, J. Park and R. Vázquez-Lorenzo, *Aspects of Differential Geometry II*, Synthesis Lectures on Mathematics and Statistics **16**, Morgan & Claypool, Williston, VT, 2015. http://dx.doi.org/10.2200/s00645ed1v01y201505mas016

[6] Y. Gündüzalp, *Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds*, J. Funct. Spaces Appl. **2013**, Art. ID 720623, 7 pp. http://dx.doi.org/10.1155/2013/720623

[7] N. Hitchin, *Harmonic spinors*, Advances in Math. **14** (1974), no. 1, 1–55. http://dx.doi.org/10.1016/0001-8708(74)90021-8

[8] S. Ivanov and S. Zamkovoy, *Parahermitian and paraquaternionic manifolds*, Differential Geom. Appl. **23** (2005), no. 2, 205–234. http://dx.doi.org/10.1016/j.difgeo.2005.06.002

[9] J. Lee, J. Park, B. Sahin and D.-Y. Song, *Einstein conditions for the base space of anti-invariant Riemannian submersions and Clairaut submersions*, Taiwanese J. Math. **19**, no. 4, 1145–1160. http://dx.doi.org/10.11650/tjm.19.2015.5283

[10] C. Murathan and I. Küpeli Erken, *Anti-invariant Riemannian submersions from cosymplectic manifolds onto Riemannian manifolds*, Filomat **29** (2015), no. 7, 1429–1444. http://dx.doi.org/10.2298/fil1507429m

[11] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), no. 3, 391–404. http://dx.doi.org/10.2307/1970051

[12] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), no. 4, 458–469. http://dx.doi.org/10.1307/mmj/1028999604

[13] J. H. Park, *The Laplace-Beltrami operator and Rimanian submersion with minimal and not totally geodesic fibers*, Bull. Korean Math. Soc. **27** (1990), 39–47.

[14] K. Park, *H-anti-invariant submersions from almost quaternionic Hermitian manifolds*, arXiv:1507.04473v1 [math.DG].

[15] B. Sahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds*, Cent. Eur. J. Math. **8** (2010), no. 3, 437–447. http://dx.doi.org/10.2478/s11533-010-0023-6

[16] ______, *Riemannian submersions from almost Hermitian manifolds*, Taiwanese J. Math. **17** (2013), no. 2, 629–659. http://dx.doi.org/10.11650/tjm.17.2013.2191

[17] Wikipedia, https://en.wikipedia.org/wiki/octonian.

Peter Gilkey
Mathematics Department, University of Oregon, Eugene OR 97403, USA
E-mail address: gilkey@uoregon.edu

Mitsuhiro Itoh
University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8577 Japan
E-mail address: itohm@sakura.cc.tsukuba.ac.jp

JeongHyeong Park
Department of Mathematics, Sungkyunkwan University, 2066 Seobu-ro, Jangan-gu, Suwon, 16419, Korea
and
School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro Dongdaemun-gu, Seoul 02455, Korea
E-mail address: parkj@skku.edu