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Remarks on Normalized Solutions for L^2-Critical Kirchhoff Problems

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Abstract. We study a constraint minimization problem on $S_c = \{u \in H^1(\mathbb{R}^N), |u|_2^2 = c, c \in (0, c^*)\}$ for the following L^2-critical Kirchhoff type functional:

$$\begin{aligned} E_\alpha(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)^2 + \frac{1}{\alpha+2}\int_{\mathbb{R}^N} V(x)|u|^{\alpha+2}\,dx \\ - \frac{N}{2N+8}\int_{\mathbb{R}^2} |u|^{\frac{2N+8}{N}}\,dx, \end{aligned}$$

where $N \le 3$, $a, b > 0$ are constants, $\alpha \in [0, \frac{8}{N})$ and $V(x) \in L^\infty(\mathbb{R}^N)$ is a suitable potential. We prove that the problem has at least one minimizer if $\alpha \in [2, \frac{8}{N})$ and the energy of the minimization problem is negative. Moreover, some non-existence results are obtained when the energy of the problem equals to zero.

1. Introduction

In this paper, we consider the following L^2-critical Kirchhoff equation

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\Delta u + V(x)|u|^\alpha u - |u|^{\frac{8}{N}} u = \lambda u, \quad \lambda \in \mathbb{R}, \tag{1.1}$$

where $N \le 3$, $a, b > 0$ are constants, $\alpha \in [0, \frac{8}{N})$ and $V(x) \in L^\infty(\mathbb{R}^N)$ is a suitable potential.

Kirchhoff type equation was first proposed by Kirchhoff [13] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Some early researches can be found in [3, 15, 19]. The appearance of the nonlocal term $\int_{\mathbb{R}^N} |\nabla u|^2\,dx$ makes the study of equation (1.1) interesting and also leads to some new difficulty from the mathematical point of view. Recently, problem (1.1) has attracted much attention of mathematicians. For example, some researchers seek solutions of (1.1) by fixing the parameter $\lambda \in \mathbb{R}$, see for [1,4,5,9,12,14,18,20,22] and the references therein.

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Also, one can see (1.1) as a nonlinear eigenvalue problem by considering λ as an eigenvalue [23, 24], and some normalized solutions of (1.1) can be obtained by considering the following minimization problem

$$m_\alpha(c) = \inf_{u \in S_c} E_\alpha(u), \tag{1.2}$$

where

$$S_c = \left\{u \in H^1(\mathbb{R}^N), |u|_2^2 = c\right\}$$

and

$$\begin{aligned} E_\alpha(u) = &\frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^2 + \frac{1}{\alpha+2}\int_{\mathbb{R}^N} V(x)\,|u|^{\alpha+2}\, dx \\ &- \frac{N}{2N+8}\int_{\mathbb{R}^2} |u|^{\frac{2N+8}{N}}\, dx. \end{aligned} \tag{1.3}$$

If $V(x) \equiv 0$, by some standard scaling arguments, one can easily prove that there exists $c^* > 0$ (given by (1.5) below) such that $m_\alpha(c) = 0$, $\forall\, c \in (0, c^*]$ and $m_\alpha(c) = -\infty$, $\forall\, c \in (c^*, +\infty)$. This further indicates that (1.2) possesses no minimizer for any $c > 0$. However, the result is quite different for the case of $V(x) \not\equiv 0$. For example, let $\alpha = 2$ and suppose $V(x) \lneqq 0$ satisfies some additional assumptions, it was proved in [24] that (1.2) has at leat one minimizer when $c \in (0, c^*)$ and $a > 0$ is small enough. Also, when $c \in (c^*, \infty)$, then $m_\alpha(c) = -\infty$ and (1.2) cannot be achieved.

In this paper, we focus on minimization problem (1.2) for the case of $c \in (0, c^*)$ and $\alpha \geq 0$. We try to give some criteria for the existence and non-existence of minimizers for (1.2). In general, one can easily check that $m_\alpha(c) \leq 0$ (see for (2.11) below) if the potential satisfies $\lim_{|x|\to\infty} V(x) = 0$. In what follows, we first show that $m_\alpha(c) < 0$ is critical for the existence of minimizers for (1.2). Furthermore, if $m_\alpha(c) = 0$ and the potential $V(x)$ satisfies some additional assumptions, we show that (1.2) cannot be achieved. Before the statement of the main results, we first recall the following Gagliardo-Nirenberg inequality [21]

$$\int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}\, dx \leq \frac{N+4}{N|Q|_2^{\frac{8}{N}}}\left(\int_{\mathbb{R}^N} |\nabla u(x)|^2\, dx\right)^2\left(\int_{\mathbb{R}^N} |u(x)|^2\, dx\right)^{\frac{4-N}{N}}, \quad u \in H^1(\mathbb{R}^N), \tag{1.4}$$

where $Q(x) = Q(|x|) > 0$ is the unique radial solution of the following field equation

$$-2\Delta Q + \frac{(4-N)}{N}Q = |Q|^{\frac{8}{N}}Q \quad \text{in } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N).$$

Let

$$c^* = \left(\frac{b|Q|^{\frac{8}{N}}}{2}\right)^{\frac{N}{4-N}}. \tag{1.5}$$

The following theorem shows that if $\alpha \geq 2$, then the assumption $m_\alpha(c) < 0$ is sufficient for the achievement of minimization problem (1.2).

Theorem 1.1. *For any fixed $a > 0$, $c \in (0, c^*)$ and $\alpha \in [2, \frac{8}{N})$, assume that*

$$V \in L^\infty(\mathbb{R}^N) \quad \textit{and} \quad \lim_{|x|\to\infty} V(x) = 0. \tag{1.6}$$

If $m_\alpha(c) < 0$, then any minimizing sequence of (1.2) *is compact in $H^1(\mathbb{R}^N)$ and* (1.2) *has at least one minimizer.*

We will mainly use the concentration-compactness principle [16,17] to obtain the compactness of minimizing sequences and further finish the proof of Theorem 1.1. The assumption $m_\alpha(c) < 0$ and $\alpha \in [2, \frac{8}{N})$ guarantee that $m_\alpha(c)$ satisfies the strict sub-additivity inequalities, which are essential to rule out the case of *vanishing* and *dichotomy* for minimizing sequences. Our next theorem tells that the assumption $m_\alpha(c) < 0$ can be verified if $V(x)$ approaches to 0 at infinity in suitable rates.

Theorem 1.2. *For any fixed $c \in (0, c^*)$ and $\alpha \in [2, \frac{8}{N})$, assume that $V(x)$ satisfies* (1.6) *as well as*

$$V(x) \sim -|x|^{-\beta} \quad \textit{for some } \beta > 0 \textit{ as } |x| \to \infty. \tag{1.7}$$

Then $m_\alpha(c) < 0$ if one of the following conditions holds.

(i) *$N = 1, 2, 3$, $\frac{N\alpha}{2} + \beta < 4$ and $a > 0$ is small enough;*

(ii) *$N = 1$ and $\frac{\alpha}{2} + \beta < 2$.*

Theorems 1.1 and 1.2 give some sufficient conditions for the existence of minimizers for (1.2), which generalizes the results in [24], where the case of $\alpha = 2$ and $a > 0$ is small was studied. Especially, when $N = 1$ and $\alpha = 2$, Theorem 1.2(ii) tells that if $0 < \beta < 1$ then (1.2) has at least one minimizer for all $a > 0$, which partly extends the results of Theorem 1.2 in [24].

From above theorems we see that the condition $m_\alpha(c) < 0$ is important to ensure the existence of minimizers for problem (1.2). How about the case of $m_\alpha(c) = 0$? Our following theorem partly answers this question and gives some non-existence results for problem (1.2).

Theorem 1.3. *Let $c \in (0, c^*)$ and suppose $V(x)$ satisfies* (1.6). *If one of the following conditions is satisfied:*

(I) *$V(x) \geq 0$ and $\alpha \in [0, \frac{8}{N})$;*

(II) $\alpha \in [\frac{4}{N}, \frac{8}{N})$ *and the product* $c^{\frac{8-N\alpha+(4-N)\alpha}{8}} \cdot |V(x)|_\infty$ *is small enough.*

Then $m_\alpha(c) = 0$ *and* (1.2) *has no minimizer.*

We finally remark that if $b = 0$ and $\frac{2N+8}{N}$ is replaced with $\frac{2N+4}{N}$ in functional (1.3), then minimization problem (1.2) is also L^2-critical and has attracted recently much attention. For instance, the authors in [6] proved the existence of threshold for the existence of minimizers, moreover, some detailed blow-up phenomena were also studied in [6–8] for different kind of potentials. Moreover, this manuscript is also motivated by [2, 10, 11], where some L^2-minimization problems for Schrödinger-Poisson system and quasi-linear equations were studied.

In this paper, we denote by C the universal positive constant unless specified, $B_r = \{x \in \mathbb{R}^N : |x| < r\}$, and $|u|_p$ denotes the L^p-norm of u. The weak convergence and strong convergence are denoted by "$\rightharpoonup$" and "$\rightarrow$", respectively.

2. Proofs of Theorems 1.1 to 1.3

The main purpose of this section is to prove Theorem 1.1 to Theorem 1.3, and we first establish the following lemma.

Lemma 2.1. *For any fixed* $a > 0$, $c \in (0, c^*)$ *and* $\alpha \in [2, \frac{8}{N})$, *if* $m_\alpha(c) < 0$, *then*

$$m_\alpha(c) < m_\alpha(c_0) + m_\alpha(c - c_0), \quad \text{for any } c_0 \in (0, c). \tag{2.1}$$

Proof. Let $\{u_n\}$ be any minimizing sequence of $m_\alpha(c)$. Since $\alpha \in [2, \frac{8}{N})$, it then follows from the Hölder inequality that

$$\int_{\mathbb{R}^N} |u_n|^{2+\alpha}\, dx \le \left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}}\, dx\right)^{\frac{\alpha N}{8}} \left(\int_{\mathbb{R}^N} |u_n|^2\, dx\right)^{\frac{8-\alpha N}{8}}. \tag{2.2}$$

This together with (1.4) implies that

$$\begin{aligned} E_\alpha(u_n) \ge{}& \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx + \frac{b}{4}\left[1 - \left(\frac{c}{c^*}\right)^{\frac{4-N}{N}}\right]\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx\right)^2 \\ &+ \frac{1}{\alpha+2}\int_{\mathbb{R}^N} V(x)|u_n|^{\alpha+2}\, dx \\ \ge{}& \frac{b}{4}\left[1 - \left(\frac{c}{c^*}\right)^{\frac{4-N}{N}}\right]\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx\right)^2 - \frac{|V|_\infty}{\alpha+2}\int_{\mathbb{R}^N} |u_n|^{\alpha+2}\, dx \\ \ge{}& \frac{b}{4}\left[1 - \left(\frac{c}{c^*}\right)^{\frac{4-N}{N}}\right]\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx\right)^2 - \frac{c^{\frac{8-\alpha N}{8}}|V|_\infty}{\alpha+2}\left(\int_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}}\right)^{\frac{\alpha N}{8}} \end{aligned} \tag{2.3}$$

$$\ge \frac{b}{4}\left[1 - \left(\frac{c}{c^*}\right)^{\frac{4-N}{N}}\right]\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx\right)^2 - C\left(\int_{\mathbb{R}^N} |\nabla u_n|^2\right)^{\frac{\alpha N}{4}}. \tag{2.4}$$

Since $c \in (0, c^*)$ and $\alpha \in [2, \frac{8}{N})$, we can deduce from above that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, which indicates that $m_\alpha(c) > -\infty$. This together with the assumption of $m_\alpha(c) < 0$ gives that

$$-\infty < m_\alpha(c) < 0 \quad \forall\, c \in (0, c^*). \tag{2.5}$$

Moreover, one can deduce from (2.3) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n|^{\alpha+2}\, dx < 0 \tag{2.6}$$

Let $\widetilde{u}_n = \sqrt{\theta} u_n$ with $\theta > 1$, then $\widetilde{u}_n \in S_{\theta c}$. Noting that $\alpha \in [2, \frac{8}{N})$, we can obtain from (2.5) and (2.6) that

$$\begin{aligned}
m_\alpha(\theta c) &\le \lim_{n\to\infty} E_\alpha(\widetilde{u}_n) = \lim_{n\to\infty} E_\alpha(\sqrt{\theta} u_n) \\
&= \theta^2 \lim_{n\to\infty} \Bigg[\frac{a}{2\theta} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \right)^2 \\
&\qquad + \frac{\theta^{\frac{\alpha-2}{2}}}{\alpha+2} \int_{\mathbb{R}^N} V(x)|u|^{\alpha+2}\, dx - \frac{N\theta^{\frac{4-N}{N}}}{2N+8} \int_{\mathbb{R}^2} |u|^{\frac{2N+8}{N}}\, dx \Bigg] \\
&\le \theta^2 \lim_{n\to\infty} \Bigg[\frac{a}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \right)^2 \\
&\qquad + \frac{1}{\alpha+2} \int_{\mathbb{R}^N} V(x)|u|^{\alpha+2}\, dx - \frac{N}{2N+8} \int_{\mathbb{R}^2} |u|^{\frac{2N+8}{N}}\, dx \Bigg] \\
&= \theta^2 \lim_{n\to\infty} E_\alpha(u_n) = \theta^2 m_\alpha(c) < \theta m_\alpha(c).
\end{aligned}$$

Then (2.1) follows from Lemma II.1 of [16]. □

Based on the above lemma, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. For any $c \in (0, c^*)$, let $\{u_n\}$ be a minimizing sequence of $m_\alpha(c)$. Similar to the proof of (2.4), one can easily check that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We are going to prove the compactness of $\{u_n\}$ by using the concentration-compactness principle in [16, 17]. For this purpose, let

$$\rho_n = |u_n|^2, \quad n \in \mathbb{N}.$$

We first rule out the possibility of *vanishing*: indeed if vanishing occurs, it then follows from [16, Lemma I.1] that there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{y+B_R} |u_n|^2 = 0, \quad \text{for all } R < \infty.$$

It then follows from Lemma I.1 in [17] that

$$u_n \to 0 \quad \text{strongly in } L^p(\mathbb{R}^N),\ p \in [2, 2^*).$$

Thus

$$\left|\int_{\mathbb{R}^N} V(x)|u_n|^{\alpha+2}\,dx\right| \le V_\infty \int_{\mathbb{R}^N} |u_n|^{\alpha+2}\,dx \xrightarrow{n} 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}}\,dx \xrightarrow{n} 0.$$

This implies that

$$m_\alpha(c) = \lim_{n\to\infty} E_\alpha(u_n) \ge 0,$$

which however contradicts the assumption of $m_\alpha(c) < 0$. Hence, *vanishing* cannot occur.

We now prove that *dichotomy* does not occur. Otherwise, one can prove, similar to [16, Lemma III.1], that there exist $c_0 \in (0, c)$ and $H^1(\mathbb{R}^N)$-bounded sequences $\{u_n^1\}$, $\{u_n^2\}$ such that for any $\varepsilon > 0$,

$$\begin{cases} \left|u_n - u_n^1 - u_n^2\right|_p \le \delta_p(\varepsilon) \xrightarrow{\varepsilon\to 0} 0 \quad \text{for } p \in [2, 2^*); \\ \left|\int_{\mathbb{R}^N} |u_n^1|^2\,dx - c_0\right| \le \varepsilon, \quad \left|\int_{\mathbb{R}^N} |u_n^2|^2\,dx - (c-c_0)\right| \le \varepsilon; \\ \operatorname{dist}(\operatorname{supp} u_n^1, \operatorname{supp} u_n^2) \xrightarrow{n} \infty; \\ \lim_{n\to\infty} \int_{\mathbb{R}^N} \left[|\nabla u_n|^2 - |\nabla u_n^1|^2 - |\nabla u_n^2|^2\right] dx \ge 0. \end{cases}$$

This implies that

$$\begin{aligned} m_\alpha(c) = \lim_{n\to\infty} E_\alpha(u_n) &\ge \lim_{n\to\infty} \left[E_\alpha(u_n^1) + E_\alpha(u_n^2)\right] + \delta(\varepsilon) \\ &\ge m_\alpha(c_0) + m_\alpha(c-c_0) + \delta(\varepsilon), \end{aligned}$$

where $\delta(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$. By taking $\varepsilon \to 0$ we obtain that

$$m_\alpha(c) \ge m_\alpha(c_0) + m_\alpha(c-c_0),$$

this contradicts (2.1). Hence, *dichotomy* does not occur.

We now have the compactness of the minimizing sequence $\{u_n\}$ in the following sense: $\forall\, \varepsilon > 0$, there exist $\{y_n\} \subset \mathbb{R}^N$ and $R_\varepsilon > 0$ such that

$$\int_{B_{R_\varepsilon}(y_n)} |u_n|^2\,dx \ge c - \varepsilon.$$

If $|y_n| \xrightarrow{n} \infty$, since $\lim_{|x|\to\infty} V(x) = 0$, we then have

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{B_{R_\varepsilon}(y_n)} V(x)|u_n|^{\alpha+2} = 0. \tag{2.7}$$

Moreover, from (2.2) we also obtain that

$$\begin{aligned} \lim_{n\to\infty} \left|\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(y_n)} V(x)|u_n|^{\alpha+2}\right| &\le C\left(\int_{\mathbb{R}^N\setminus B_{R_\varepsilon}(y_n)} |u_n|^2\,dx\right)^{\frac{8-\alpha N}{8}} \\ &\le C\varepsilon^{\frac{8-\alpha N}{8}} \xrightarrow{\varepsilon\to 0} 0. \end{aligned} \tag{2.8}$$

By passing $\varepsilon \to 0$, it then follows from (2.7) and (2.8) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n|^{\alpha+2} = 0.$$

Since $c \in (0, c^*)$, it then follows from (2.3) that

$$m_\alpha(c) = \lim_{n\to\infty} E_\alpha(u_n) \geq 0,$$

which also leads to a contradiction.

So, $\{y_n\}$ is bounded and there exists $u \in H^1(\mathbb{R}^N)$ such that

$$u_n \to u \quad \text{strongly in } L^2(\mathbb{R}^N).$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)|u_n|^{\alpha+2}\,dx &= \int_{\mathbb{R}^N} V(x)|u|^{\alpha+2}\,dx,\\ \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{\frac{2N+8}{N}}\,dx &= \int_{\mathbb{R}^N} |u|^{\frac{2N+8}{N}}\,dx\end{aligned}$$

and

$$m_\alpha(c) \leq E_\alpha(u) \leq \lim_{n\to\infty} E_\alpha(u_n) = m_\alpha(c).$$

The above two inequalities indicate that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx = \int_{\mathbb{R}^N} |\nabla u|^2\,dx.$$

Therefore, $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$ and u is a minimizer of $m_\alpha(c)$. $\square$

In the following, we are going to prove Theorem 1.2, which shows that the condition $m_\alpha(c) < 0$ can be verified under suitable assumptions.

Proof of Theorem 1.2. Choose $\varphi(x) \in C_c^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} \varphi^2\,dx = c$ and $\operatorname{supp}\varphi \in B_1(x_0)$ with fixed $|x_0| = 2$. Set

$$\varphi_\tau(x) = \tau^{\frac{N}{2}}\varphi(\tau x), \quad \tau > 0.$$

Then

$$\int_{\mathbb{R}^N} |\nabla\varphi_\tau|^2\,dx = \tau^2 \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx, \quad \int_{\mathbb{R}^N} |\varphi_\tau|^{2+\frac{8}{N}}\,dx = \tau^4 \int_{\mathbb{R}^N} |\varphi|^{2+\frac{8}{N}}\,dx, \tag{2.9}$$

and it follows from (1.7) that

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)|\varphi_\tau|^{\alpha+2}\,dx &= \tau^{\frac{N\alpha}{2}} \int_{B_1(x_0)} V\left(\frac{x}{\tau}\right)|\varphi|^{\alpha+2}\,dx\\ &\leq -C\tau^{\frac{N\alpha}{2}+\beta} \quad \text{as } \tau \to 0^+.\end{aligned}$$

Therefore,

$$
\begin{aligned}
E_\alpha(\varphi_\tau) \le{}& \frac{a\tau^2}{2}\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx \\
&+\underbrace{\frac{b\tau^4}{4}\left(\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx\right)^2-\frac{N\tau^4}{2N+8}\int_{\mathbb{R}^2}|\varphi|^{\frac{2N+8}{N}}\,dx-C\tau^{\frac{N\alpha}{2}+\beta}}_{I_\tau}.
\end{aligned}
\tag{2.10}
$$

(i) If $N\le 3$ and $\frac{N\alpha}{2}+\beta<4$, we can choose $\tau_0>0$ small enough such that $I_{\tau_0}<0$. Furthermore, if we take $0<a<-\frac{2I_{\tau_0}}{\tau_0^2\int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx}$, then

$$m_\alpha(c)\le E_\alpha(\varphi_{\tau_0})<0,$$

and (i) is proved.

(ii) If $N=1$ and $\frac{\alpha}{2}+\beta<2$, one can deduce from (2.10) that

$$m_\alpha(c)\le E_\alpha(\varphi_\tau)<0$$

if $\tau>0$ is small, and (ii) also holds. □

In the end of this section, we intend to prove Theorem 1.3, which addresses some non-existence results for problem (1.2).

Proof of Theorem 1.3. Let φ_τ be that as in the proof of Theorem 1.2. In view of (1.6), we have

$$\left|\int_{\mathbb{R}^N}V(x)|\varphi_\tau|^{\alpha+2}\,dx\right|=\tau^{\frac{N\alpha}{2}}\left|\int_{B_1(x_0)}V\left(\frac{x}{\tau}\right)|\varphi|^{\alpha+2}\,dx\right|\to 0\quad\text{as }\tau\to 0.$$

This together with (2.9) yields that

$$m_\alpha(c)\le E_\alpha(\varphi_\tau)\to 0\quad\text{as }\tau\to 0. \tag{2.11}$$

Case (I). If $V(x)\ge 0$ and $c\in(0,c^*)$. It follows from (1.4) that for any $u\in S_c$,

$$
\begin{aligned}
E_\alpha(u)\ge{}& \frac{a}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx+\frac{b}{4}\left[1-\left(\frac{c}{c^*}\right)^{\frac{4-N}{N}}\right]\left(\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)^2\\
&+\frac{1}{\alpha+2}\int_{\mathbb{R}^N}V(x)|u|^{\alpha+2}\,dx.
\end{aligned}
\tag{2.12}
$$

Thus,

$$E_\alpha(u)>0\quad\text{for any fixed }u\in S_c. \tag{2.13}$$

We then further have $m_\alpha(c)=\inf_{u\in S_c}E_\alpha(u)\ge 0$, which together with (2.11) gives that

$$m_\alpha(c)=0.$$

This indicates that (1.2) has no minimizer. Otherwise, if there exists $u_0 \in S_c$ being a minimizer of (1.2), then

$$E_\alpha(u_0) = m_\alpha(c) = 0,$$

which however contradicts (2.13).

Case (II). From (1.4) and (2.2), we see that for any $u \in S_c$,

$$\left| \int_{\mathbb{R}^N} V(x)|u|^{\alpha+2}\, dx \right| \leq \left(\frac{N+4}{N|Q|_2^{\frac{8}{N}}} \right)^{\frac{N\alpha}{8}} |V|_\infty c^{\frac{8-N\alpha+(4-N)\alpha}{8}} \left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx \right)^{\frac{N\alpha}{4}}. \tag{2.14}$$

When $\alpha \in [\frac{4}{N}, \frac{8}{N})$, i.e., $\frac{N\alpha}{4} \in [1,2)$, one can easily check that if $c^{\frac{8-N\alpha+(4-N)\alpha}{8}}|V|_\infty$ is small, then

$$\frac{1}{\alpha+2} \left(\frac{N+4}{N|Q|_2^{\frac{8}{N}}} \right)^{\frac{N\alpha}{8}} |V|_\infty c^{\frac{8-N\alpha+(4-N)\alpha}{8}} t^{\frac{N\alpha}{4}} < \frac{a}{2}t + \frac{b}{4}\left[1 - \left(\frac{c}{c^*}\right)^{\frac{4-N}{N}}\right] t^2$$

for any $t > 0$. This together with inequalities (2.12) and (2.14) implies that

$$E_\alpha(u) > 0 \quad \text{for any } u \in S_c.$$

One can similarly to Case (I) deduce that $m_\alpha(c) = 0$ and (1.2) possesses no minimizer. □

References

[1] A. Arosio and S. Panizzi, *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc. **348** (1996), no. 1, 305–330.
http://dx.doi.org/10.1090/S0002-9947-96-01532-2

[2] J. Bellazzini, L. Jeanjean and T. Luo, *Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 2, 303–339. http://dx.doi.org/10.1112/plms/pds072

[3] S. Bernstein, *Sur une classe d'équations fonctionnelles aux dérivées partielles*, Bull. Acad. Sci. URRS.Sér. Math. **4** (1940), 17–26.

[4] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differential Equations **6** (2001), no. 6, 701–730.

[5] P. D'Ancona and S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math. **108** (1992), no. 2, 247–262.
http://dx.doi.org/10.1007/bf02100605

[6] Y. Guo and R. Seiringer, *On the mass concentration for Bose-Einstein condensates with attractive interactions*, Lett. Math. Phys. **104** (2014), no. 2, 141–156. http://dx.doi.org/10.1007/s11005-013-0667-9

[7] Y. Guo, Z. Wang, X. Zeng and H.-S. Zhou, *Properties of ground states of attractive Gross-Pitaevskii equations with multi-well potentials*, arXiv:1502.01839.

[8] Y. Guo, X. Zeng and H.-S. Zhou, *Energy estimates and symmetry breaking in attractive Bose-Einstein condensates with ring-shaped potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire. http://dx.doi.org/10.1016/j.anihpc.2015.01.005

[9] X. He and W. Zou, *Existence and concentration behavior of positive solutions for a Kirchhoff equation in* $\mathbb{R}^3$, J. Differential Equations **252** (2012), no. 2, 1813–1834. http://dx.doi.org/10.1016/j.jde.2011.08.035

[10] L. Jeanjean and T. Luo, *Sharp nonexistence results of prescribed* L^2*-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations*, Z. Angew. Math. Phys. **64** (2013), no. 4, 937–954. http://dx.doi.org/10.1007/s00033-012-0272-2

[11] L. Jeanjean, T. Luo and Z.-Q. Wang, *Multiple normalized solutions for quasi-linear Schrödinger equations*, J. Differential Equations **259** (2015), no. 8, 3894–3928. http://dx.doi.org/10.1016/j.jde.2015.05.008

[12] J. Jin and X. Wu, *Infinitely many radial solutions for Kirchhoff-type problems in* $\mathbb{R}^N$, J. Math. Anal. Appl. **369** (2010), no. 2, 564–574. http://dx.doi.org/10.1016/j.jmaa.2010.03.059

[13] G. Kirchhoff, *Mechanik, Teubner, Leipzig*, 1883.

[14] G. Li and H. Ye, *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in* $\mathbb{R}^3$, J. Differential Equations **257** (2014), no. 2, 566–600. http://dx.doi.org/10.1016/j.jde.2014.04.011

[15] J.-L. Lions, *On some questions in boundary value problems of mathematical physics*, in *Contemporary Developments in Continuum Mechanics and Partial Differential Equations* (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), 284–346, North-Holland Math. Stud., **30**, North-Holland, Amsterdam-New York, 1978. http://dx.doi.org/10.1016/s0304-0208(08)70870-3

[16] P.-L. Lions, *The concentration-compactness principle in the calculus of variations, The locally compact case, I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 109–145.

[17] ———, *The concentration-compactness principle in the calculus of variations, The locally compact case, II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 223–283.

[18] W. Liu and X. He, *Multiplicity of high energy solutions for superlinear Kirchhoff equations*, J. Appl. Math. Comput. **39** (2012), no. 1-2, 473–487. http://dx.doi.org/10.1007/s12190-012-0536-1

[19] S. I. Pohožaev, *A certain class of quasilinear hyperbolic equations*, Mat. Sb. (N.S.) **96(138)** (1975), 152–168.

[20] J. Wang, L. Tian, J. Xu and F. Zhang, *Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth*, J. Differential Equations **253** (2012), no. 7, 2314–2351. http://dx.doi.org/10.1016/j.jde.2012.05.023

[21] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1983), no. 4, 567–576. http://dx.doi.org/10.1007/bf01208265

[22] X. Wu, *Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in $\mathbb{R}^N$*, Nonlinear Anal. Real World Appl. **12** (2011), no. 2, 1278–1287. http://dx.doi.org/10.1016/j.nonrwa.2010.09.023

[23] H. Ye, *The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations*, Math. Methods Appl. Sci. **38** (2015), no. 13, 2663–2679. http://dx.doi.org/10.1002/mma.3247

[24] ———, *The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations*, Z. Angew. Math. Phys. **66** (2015), no. 4, 1483–1497. http://dx.doi.org/10.1007/s00033-014-0474-x

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